MATH 360 –Homework 1
Due Wednesday, September 6, 2017

To discuss in recitation:

1. Prove Lemma 4, using the proof of Lemma 2 as a guide.

---

2. Prove that the first few terms of a sequence do not affect convergence (or the limit). More formally, show that if there exists an N such that $a_n = b_n$ for all $n \geqslant N$, then $a_n \to L$ as $n \to \infty$ implies $b_n \to L$ as $n \to \infty$.

---

3. Show by means of an example that if $a_n \to L$ and $a_n > M$ for all n, it does not necessarily follow that $L > M$ (i.e., taking the limit can destroy strict inequality).

Does it follow that $L \geqslant M$? Proof or counterexample.

---

4. (a) Show from the definition that if $a_n \to L$, then $ca_n \to cL$ (as $n \to \infty$).

(b) Show from the definition (don't use Lemma 2) that if $a_n \to L$ as $n \to \infty$, then $a_n^2 \to L^2$.

---

5. Use Theorem 8 to prove: If $f\colon [a, b] \to \mathbb{R}$ is continuous and $f(a) \geqslant C \geqslant f(b)$, then there exists $c \in [a, b]$ such that $f(c) = C$.

To be handed in on September 6:

1. (a) Suppose $a_n \geqslant c_n \geqslant b_n$ for all n. Show that, if $a_n \to L$ and $b_n \to L$ as $n \to \infty$ then also $c_n \to L$ as $n \to \infty$.

(b) Now suppose $|a_n| \geqslant |c_n| \geqslant |b_n|$ for all n and that $a_n \to L$ and $b_n \to L$ as $n \to \infty$. Does it follow that $c_n \to L$ as $n \to \infty$? Proof or counterexample.

---

2. Show that any real polynomial of odd degree has at least one root. Is the result true for polynomials of even degree? Proof or counterexample.

---

3. (a) Suppose that $g\colon [0, 1] \to [0, 1]$ is a continuous function. Show that there exists a $c \in [0, 1]$ with $g(c) = c$ (i.e., every continuous map of $[0, 1]$ to itself has a "fixed point"). (*Hint*: consider $f(x) = g(x) - x$)

(b) Give an example of a bijective (one-to-one and onto) function $h\colon (0, 1) \to (0, 1)$ such that $h(x) \neq x$ for all $x \in (0, 1)$.

This exercise shows that there is an essential difference between open and closed intervals.

---

4. (a) Suppose $g\colon (A, B) \to \mathbb{R}$ is a differentiable function such that $g'(x) \geqslant 0$ for all $x \in (A, B)$. For $a, b \in (A, B)$ with $b > a$, show that $g(b) - g(a) \geqslant 0$.

(b) Prove the converse of the result it part (a) (you might want to prove that $g'(x) \geqslant -\varepsilon$ for all $\varepsilon > 0$).

---

5. Assume that $a_n \geqslant 1$ for all n, and that $a_n + a_n^{-1}$ tends to a limit as $n \to \infty$. Show that a_n also tends to a limit.

By giving an example, show that the result is false if we replace the condition $a_n \geqslant 1$ with $a_n \geqslant k$ where $0 < k < 1$.
