1.1 Differential equations

In this chapter we are going to study differential equations, with particular emphasis on how to solve them with computers. We assume that the reader has previously met differential equations, and so we're going to review the most basic facts about them rather quickly.

A *differential equation* is an equation in an unknown function, say $y(x)$, where the equation contains various derivatives of y and various known functions of x. The problem is to "find" the unknown function. The *order* of a differential equation is the order of the highest derivative that appears in it.

Here's an easy equation of first order:

$$y'(x) = 0. \tag{1.1.1}$$

The unknown function is $y(x) = \text{constant}$, and so we have solved the given equation (1.1.1).

The next one is a little harder:

$$y'(x) = 2y(x). \tag{1.1.2}$$

A solution will, now doubt, arrive after a bit of thought, namely $y(x) = e^{2x}$. But, if $y(x)$ is a solution of (1.1.2), then so is $10y(x)$, or $49.6y(x)$, or in fact $cy(x)$ for any constant c. Hence $y = ce^{2x}$ is a solution of (1.1.2). Are there any other solutions? No there aren't, because if y is any function that satisfies (1.1.2) then

$$(ye^{-2x})' = e^{-2x}(y' - 2y) = 0,$$

and so ye^{-2x} must be a constant, C.

In general, we can expect that if a differential equation is of the first order, then the most general solution will involve one arbitrary constant C. This is not always the case, since we can write down differential equations that have no solutions at all. We would have, for instance, a fairly hard time (why?) finding a real function $y(x)$ for which

$$(y')^2 = -y^2 - 2. \tag{1.1.3}$$

There are certain special kinds of differential equations that can always be solved, and it's often important to be able to recognize them. Among there are the "first-order linear" equations

$$y'(x) + a(x)y(x) = 0, \tag{1.1.4}$$

where $a(x)$ is a given function of x.

Before we describe the solution of these equations, let's discuss the word *linear*. To say that an equation is linear is to say that if we have any two solutions $y_1(x)$ and $y_2(x)$ of the equation, then $c_1y_1(x) + c_2y_2(x)$ is also a solution of the equation, where c_1 and c_2 are any two constants (in other words, the set of solutions forms a vector space).

Equation (1.1.1) is linear, in fact, $y_1(x) = 7$ and $y_2(x) = 23$ are both solutions, and so is $7c_1 + 23c_2$. Less trivially, the equation

$$y''(x) + y(x) = 0 \tag{1.1.5}$$

is linear. The linearity of (1.1.5) can be checked right from the equation itself, without knowing what the solutions are (do it!). For an example, though, we might note that $y = \sin x$ is a solution of (1.1.5), that $y = \cos x$ is another solution of (1.1.5), and finally, by linearity, that the function $y = c_1 \sin x + c_2 \cos x$ is a solution, whatever the constants c_1 and c_2. Now let's consider an instance of the first order linear equation (1.1.4):

$$y'(x) + xy(x) = 0. \tag{1.1.6}$$

So we're looking for a function whose derivative is $-x$ times the function. Evidently $y = e^{-x^2/2}$ will do, and the general solution is $y(x) = ce^{-x^2/2}$.

If instead of (1.1.6) we had

$$y'(x) + x^2 y(x) = 0, \tag{1.1.7}$$

then we would have found the general solution $ce^{-x^3/3}$.

As a last example, take

$$y'(x) + (\cos x)\, y(x) = 0.$$

The right medicine is now $y(x) = e^{\sin x}$. In the next paragraph we'll give the general rule of which the above are three examples. The reader might like to put down the book at this point and try to formulate the rule for solving (1.1.4) before going on to read about it.

Ready? What we need is to choose some antiderivative $A(x)$ of $a(x)$, and then the solution is $y(x) = ce^{-A(x)}$.

Since that was so easy, next let's put a more interesting right hand side into (1.1.4), by considering the equation

$$y'(x) + a(x)y(x) = b(x) \tag{1.1.9}$$

where now $b(x)$ is also a given function of x (Is (1.1.9) a linear equation? Are you sure?).

To solve (1.1.9), once again choose some antiderivative $A(x)$ of $a(x)$, and then note that we can rewrite (1.1.9) in the equivalent form

$$e^{-A(x)} \frac{d}{dx}\left(e^{A(x)} y(x)\right) = b(x). \tag{1.1.10}$$

Now if we multiply through by $e^{A(x)}$ we see that

$$\frac{d}{dx}\left(e^{A(x)} y(x)\right) = b(x)e^{A(x)}$$

and so, if we integrate both sides,

$$e^{A(x)} y(x) = \int^x b(t)e^{A(t)}\, dt + \text{const.}\,,$$

where on the right side, we mean any antiderivative of the function under the integral sign. Consequently

$$y(x) = e^{-A(x)}\left(\int^x b(t)e^{A(t)}\, dt + \text{const.}\right). \tag{1.1.11}$$

As an example, consider the equation

$$y' + \frac{y}{x} = x + 1\,. \tag{1.1.12}$$

We find that $A(x) = \log x$, then from (1.1.11) we get

$$\begin{aligned} y(x) &= \frac{1}{x}\left(\int^x (t+1)t\, dt + C\right) \\ &= \frac{x^2}{3} + \frac{x}{2} + \frac{C}{x}. \end{aligned}$$

We may be doing a disservice to the reader by beginning with this discussion of certain types of differential equations that can be solved analytically, because it would be erroneous to assume that most, or even many, such equations can be dealt with by these techniques. Indeed, the reason for the importance of the numerical methods that are the main subject of this chapter is precisely that most equations that arise in "real" problems are quite intractable by analytical means, and so the computer is the only hope.

Despite the above disclaimer, in the next section we will study yet another important family of differential equations that can be handled analytically, namely linear equations with constant coefficients.

Exercises 1.1

1. Find the general solution of each of the following equations:
   (a) $y' = 2\cos x$
   (b) $y' + \frac{2}{x}y = 0$
   (c) $y' + xy = 3$
   (d) $y' + \frac{1}{x}y = x + 5$
   (e) $2yy' = x + 1$
2. Show that the equation (1.1.3) has no real solutions.
3. Go to your computer or terminal and familiarize yourself with the equipment, the operating system, and the specific software you will be using. Then write a program that will calculate and print the sum of the squares of the integers $1, 2, \ldots, 100$. Run this program.
4. For each part of problem 1, find the solution for which $y(1) = 1$.

1.2 Linear equations with constant coefficients

One particularly pleasant, and important, type of linear differential equation is the variety with constant coefficients, such as

$$y'' + 3y' + 2y = 0\,. \tag{1.2.1}$$

It turns out that what we have to do to solve these equations is to try a solution of a certain form, and we will then find that all of the solutions indeed are of that form.

Let's see if the function $y(x) = e^{\alpha x}$ is a solution of (1.2.1). If we substitute in (1.2.1), and then cancel the common factor $e^{\alpha x}$, we are left with the quadratic equation

$$\alpha^2 + 3\alpha + 2 = 0\,, \tag{1.2.2}$$

whose solutions are $\alpha = -2$ and $\alpha = -1$. Hence for those two values of α our trial function $y(x) = e^{\alpha x}$ is indeed a solution of (1.2.1). In other words, e^{-2x} is a solution, e^{-x} is a solution, and since the equation is linear,

$$y(x) = c_1 e^{-2x} + c_2 e^{-x} \tag{1.2.3}$$

is also a solution, where c_1 and c_2 are arbitrary constants. Finally, (1.2.3) must be the most general solution since it has the "right" number of arbitrary constants, namely two.

Trying a solution in the form of an exponential is always the correct first step in solving linear equations with constant coefficients. Various complications can develop, however, as illustrated by the equation

$$y'' + 4y' + 4y = 0\,. \tag{1.2.4}$$

Again, let's see if there is a solution of the form $y = e^{\alpha x}$. This time, substitution into (1.2.4) and cancellation of the factor $e^{\alpha x}$ leads to the quadratic equation

$$\alpha^2 + 4\alpha + 4 = 0\,, \tag{1.2.5}$$

whose two roots are identical, both being -2. Hence e^{-2x} is a solution, and of course so is c_1e^{-2x}, but we don't yet have the general solution because there is, so far, only one arbitrary constant. The difficulty, of course, is caused by the fact that the roots of (1.2.5) are not distinct.

In this case, it turns out that xe^{-2x} is another solution of the differential equation (1.2.4) (verify this), and so the general solution is $(c_1 + c_2x)e^{-2x}$.

Suppose that we begin with an equation of third order, and that all three roots turn out to be the same. For instance, to solve the equation

$$y''' + 3y'' + 3y' + y = 0 \tag{1.2.6}$$

we would try $y = e^{\alpha x}$, and we would then be facing the cubic equation

$$\alpha^3 + 3\alpha^2 + 3\alpha + 1 = 0\,, \tag{1.2.7}$$

whose "three" roots are all equal to -1. Now, not only is e^{-x} a solution, but so are xe^{-x} and x^2e^{-x}.

To see why this procedure works in general, suppose we have a linear differential equation with constant coeficcients, say

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_ny = 0 \tag{1.2.8}$$

If we try to find a solution of the usual exponential form $y = e^{\alpha x}$, then after substitution into (1.2.8) and cancellation of the common factor $e^{\alpha x}$, we would find the polynomial equation

$$\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + \cdots + a_n = 0\,. \tag{1.2.9}$$

The polynomial on the left side is called the *characteristic polynomial* of the given differential equation. Suppose now that a certain number $\alpha = \alpha^*$ is a root of (1.2.9) of multiplicity p. To say that α^* is a root of multiplicity p of the equation is to say that $(\alpha - \alpha^*)^p$ is a factor of the characteristic polynomial. Now look at the left side of the given differential equation (1.2.8). We can write it in the form

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_n)y = 0\,, \tag{1.2.10}$$

in which D is the differential operator d/dx. In the parentheses in (1.2.10) we see the polynomial $\varphi(D)$, where φ is exactly the characteristic polynomial in (1.2.9).

Since $\varphi(\alpha)$ has the factor $(\alpha - \alpha^*)^p$, it follows that $\varphi(D)$ has the factor $(D - \alpha^*)^p$, so the left side of (1.2.10) can be written in the form

$$g(D)(D - \alpha^*)^py = 0\,, \tag{1.2.11}$$

where g is a polynomial of degree $n-p$. Now it's quite easy to see that $y = x^ke^{\alpha^*x}$ satisfies (1.2.11) (and therefore (1.2.8) also) for each $k = 0, 1, \ldots, p-1$. Indeed, if we substitute this function y into (1.2.11), we see that it is enough to show that

$$(D - \alpha^*)^p(x^ke^{\alpha^*x}) = 0 \qquad k = 0, 1, \ldots, p-1\,. \tag{1.2.12}$$

However, $(D-\alpha^*)(x^ke^{-\alpha^*x}) = kx^{k-1}e^{\alpha^*x}$, and if we apply $(D-\alpha^*)$ again, $(D-\alpha^*)^2(x^ke^{-\alpha^*x}) = k(k-1)x^{k-2}e^{\alpha^*x}$, etc. Now since $k < p$ it is clear that $(D - \alpha^*)^p(x^ke^{-\alpha^*x}) = 0$, as claimed.

To summarize, then, if we encounter a root α^* of the characteristic equation, of multiplicity p, then corresponding to α^* we can find exactly p linearly independent solutions of the differential equation, namely

$$e^{\alpha^* x},\ xe^{\alpha^* x},\ x^2e^{\alpha^* x}, \ldots,\ x^{p-1}e^{\alpha^* x}.$$

Another way to state it is to say that the portion of the general solution of the given differential equation that corresponds to a root α^* of the characteristic polynomial equation is $Q(x)e^{\alpha^* x}$, where $Q(x)$ is an arbitrary polynomial whose degree is one less than the multiplicity of the root α^*.

One last mild complication may arise from roots of the characteristic equation that are not real numbers. These don't really require any special attention, but they do present a few options. For instance, to solve $y'' + 4y = 0$, we find the characteristic equation $\alpha^2 + 4 = 0$, and the complex roots $\pm 2i$. Hence the general solution is obtained by the usual rule as

$$y(x) = c_1e^{2ix} + c_2e^{-2ix}.$$

This is a perfectly acceptable form of the solution, but we could make it look a bit prettier by using deMoivre's theorem, which says that

$$\begin{aligned} e^{2ix} &= \cos 2x + i \sin 2x \\ e^{-2ix} &= \cos 2x - i \sin 2x. \end{aligned}$$

Then our general solution would look like

$$y(x) = (c_1 + c_2)\cos 2x + (ic_1 - ic_2)\sin 2x.$$

But c_1 and c_2 are just arbitrary constants, hence so are $c_1 + c_2$ and $ic_1 - ic_2$, so we might as well rename them c_1 and c_2, in which case solution would take the form

$$y(x) = c_1 \cos 2x + c_2 \sin 2x.$$

Here's an example that shows the various possibilities:

$$y^{(8)} - 5y^{(7)} + 17y^{(6)} - 997y^{(5)} + 110y^{(4)} - 531y^{(3)} + 765y^{(2)} - 567y' + 162y = 0. \tag{1.2.13}$$

The equation was cooked up to have a characteristic polynomial that can be factored as

$$(\alpha - 2)(\alpha^2 + 9)^2(\alpha - 1)^3.$$

Hence the roots of the characteristic equation are 2 (simple), $3i$ (multiplicity 2), $-3i$ (multiplicity 2), and 1 (multiplicity 3).

Corresponding to the root 2, the general solution will contain the term c_1e^{2x}. Corresponding to the double root at $3i$ we have terms $(c_2 + c_3x)e^{3ix}$ in the solution. From the double root at $-3i$ we get a contribution $(c_4 + c_5x)e^{-3ix}$, and finally from the triple root at 1 we get $(c_6 + c_7x + c_8x^2)e^x$. The general solution is the sum of these eight terms. Alternatively, we might have taken the four terms that come from $3i$ in the form

$$(c_2 + c_3x)\cos 3x + (c_4 + c_5x)\sin 3x.$$

Exercises 1.2

1. Obtain the general solutions of each of the following differential equations:

   (a) $y'' + 5y' + 6y = 0$

(b) $y'' - 8y' + 7y = 0$

(c) $(D+3)^2 y = 0$

(d) $(D^2+16)^2 y = 0$

(e) $(D+3)^3(D^2-25)^2(D+2)^3 y = 0$

2. Find a curve $y = f(x)$ that passes through the origin with unit slope, and which satisfies $(D+4)(D-1)y = 0$.

1.3 Difference equations

Whereas a differential equation is an equation in an unknown function, a *difference equation* is an equation in an unknown *sequence*. For example, suppose we know that a certain sequence of numbers $y_0, y_1, y_2, \ldots$ satisfies the following conditions:

$$y_{n+2} + 5y_{n+1} + 6y_n = 0 \qquad n = 0, 1, 2, \ldots \tag{1.3.1}$$

and furthermore that $y_0 = 1$ and $y_1 = 3$.

Evidently, we can compute as many of the y_n's as we need from (1.3.1), thus we would get $y_2 = -21$, $y_3 = 87$, $y_4 = -309$ and so forth. The entire sequence of y_n's is determined by the difference equation (1.3.1) together with the two starting values.

Such equations are encountered when differential equations are solved on computers. Naturally, the computer can provide the values of the unknown function only at a discrete set of points. These values are computed by replacing the given differential equations by a difference equation that approximates it, and then calculating successive approximate values of the desired function from the difference equation.

Can we somehow "solve" a difference equation by obtaining a formula for the values of the solution sequence? The answer is that we can, as long as the difference equation is linear and has constant coefficients, as in (1.3.1). Just as in the case of differential equations with constant coefficients, the correct strategy for solving them is to try a solution of the right form. In the previous section, the right form to try was $y(x) = e^{\alpha x}$. Now the winning combination is $y = \alpha^n$, where α is a constant.

In fact, let's substitute α^n for y_n in (1.3.1) to see what happens. The left side becomes

$$\alpha^{n+2} + 5\alpha^{n+1} + 6\alpha^n = \alpha^n(\alpha^2 + 5\alpha + 6) = 0.$$

Just as we were able to cancel the common factor $e^{\alpha x}$ in the differential equation case, so here we can cancel the α^n, and we're left with the quadratic equation

$$\alpha^2 + 5\alpha + 6 = 0. \tag{1.3.2}$$

The two roots of this *characteristic equation* are $\alpha = -2$ and $\alpha = -3$. Therefore the sequence $(-2)^n$ satisfies (1.3.1) and so does $(-3)^n$. Since the difference equation is linear, it follows that

$$y_n = c_1(-2)^n + c_2(-3)^n \tag{1.3.3}$$

is also a solution, whatever the values of the constants c_1 and c_2.

Now it is evident from (1.3.1) itself that the numbers y_n are uniquely determined if we prescribe the values of just two of them. Hence, it is very clear that when we have a solution that contains two arbitrary constants we have the most general solution.

When we take account of the given data $y_0 = 1$ and $y_1 = 3$, we get the two equations

$$\left\{ \begin{array}{rcl} 1 & = & c_1 + c_2 \\ 3 & = & (-2)c_1 + (-3)c_2 \end{array} \right. \tag{1.3.4}$$

from which $c_1 = 6$ and $c_2 = -5$. Finally, we use these values of c_1 and c_2 in (1.3.3) to get

$$y_n = 6(-2)^n - 5(-3)^n \qquad n = 0, 1, 2, \ldots. \tag{1.3.5}$$

Equation (1.3.5) is the desired formula that represents the unique solution of the given difference equation together with the prescribed starting values.

Let's step back a few paces to get a better view of the solution. Notice that the formula (1.3.5) expresses the solution as a linear combination of nth powers of the roots of the associated characteristic equation (1.3.2). When n is very large, is the number y_n a large number or a small one? Evidently the powers of -3 overwhelm those of -2, and so the sequence will behave roughly like a constant times powers of -3. This means that we should expect the members of the sequence to alternate in sign and to grow rapidly in magnitude.

So much for the equation (1.3.1). Now let's look at the general case, in the form of a linear difference equation of order p:

$$y_{n+p} + a_1 y_{n+p-1} + a_2 y_{n+p-2} + \cdots + a_p y_n = 0. \tag{1.3.6}$$

We try a solution of the form $y_n = \alpha^n$, and after substituting and canceling, we get the characteristic equation

$$\alpha^p + a_1 \alpha^{p-1} + a_2 \alpha^{p-2} + \cdots + a_p = 0. \tag{1.3.7}$$

This is a polynomial equation of degree p, and so it has p roots, counting multiplicities, somewhere in the complex plane.

Let α^* be one of these p roots. If α^* is simple (*i.e.*, has muiltiplicity 1) then the part of the general solution that corresponds to α^* is $c(\alpha^*)^n$. If, however, α^* is a root of multiplicity $k > 1$ then we must multiply the solution $c(\alpha^*)^n$ by an arbitrary polynomial in n, of degree $k-1$, just as in the corresponding case for differential equations we used an arbitrary polynomial in x of degree $k-1$.

We illustrate this, as well as the case of complex roots, by considering the following difference equation of order five:

$$y_{n+5} - 5y_{n+4} + 9y_{n+3} - 9y_{n+2} + 8y_{n+1} - 4y_n = 0. \tag{1.3.8}$$

This example is rigged so that the characteristic equation can be factored as

$$(\alpha^2 + 1)(\alpha - 2)^2(\alpha - 1) = 0 \tag{1.3.9}$$

from which the roots are obviously i, $-i$, 2 (multiplicity 2), 1.

Corresponding to the roots i, $-i$, the portion of the general solution is $c_1 i^n + c_2(-i)^n$. Since

$$i^n = e^{in\pi/2} = \cos\left(\frac{n\pi}{2}\right) + i \sin\left(\frac{n\pi}{2}\right)$$

and similarly for $(-i)^n$, we can also take this part of the general solution in the form

$$c_1 \cos\left(\frac{n\pi}{2}\right) + c_2 \sin\left(\frac{n\pi}{2}\right). \tag{1.3.10}$$

The double root $\alpha = 2$ contributes $(c_3 + c_4 n)2^n$, and the simple root $\alpha = 1$ adds c_5 to the general solution, which in its full glory is

$$y_n = c_1 \cos\left(\frac{n\pi}{2}\right) + c_2 \sin\left(\frac{n\pi}{2}\right) + (c_3 + c_4 n)2^n + c_5. \tag{1.3.11}$$

The five constants would be determined by prescribing five initial values, say y_0, y_1, y_2, y_3 and y_4, as we would expect for the equation (1.3.8).

Exercises 1.3

1. Obtain the general solution of each of the following difference equations:
   (a) $y_{n+1} = 3y_n$
   (b) $y_{n+1} = 3y_n + 2$
   (c) $y_{n+2} - 2y_{n+1} + y_n = 0$
   (d) $y_{n+2} - 8y_{n+1} + 12y_n = 0$
   (e) $y_{n+2} - 6y_{n+1} + 9y_n = 1$
   (f) $y_{n+2} + y_n = 0$
2. Find the solution of the given difference equation that takes the prescribed initial values:
   (a) $y_{n+2} = 2y_{n+1} + y_n$; $y_0 = 0$; $y_1 = 1$
   (b) $y_{n+1} = \alpha y_n + \beta$; $y_0 = 1$
   (c) $y_{n+4} + y_n = 0$; $y_0 = 1$; $y_1 = -1$; $y_2 = 1$; $y_3 = -1$
   (d) $y_{n+2} - 5y_{n+1} + 6y_n = 0$; $y_0 = 1$; $y_1 = 2$
3. (a) For each of the difference equations in problems 1 and 2, evaluate
   $$\lim_{n\to\infty} \frac{y_{n+1}}{y_n}$$
   if it exists.
   (b) Formulate and prove a general theorem about the existence of, and value of the limit in part (a) for a linear difference equation with constant coefficients.
   (c) Reverse the process: given a polynomial equation, find its root of largest absolute value by computing from a certain difference equation and evaluating the ratios of consecutive terms.
   (d) Write a computer program to implement the method in part (c). Use it to calculate the largest root of the equation
   $$x^8 = x^7 + x^6 + x^5 + \cdots + 1.$$

1.4 Computing with difference equations

This is, after all, a book about computing, so let's begin with computing from difference equations since they will give us a chance to discuss some important questions that concern the design of computer programs. For a sample difference equation we'll use

$$y_{n+3} = y_{n+2} + 5y_{n+1} + 3y_n \tag{1.4.1}$$

together with the starting values $y_0 = y_1 = y_2 = 1$. The reader might want, just for practice, to find an explicit formula for this sequence by the methods of the previous section.

Suppose we want to compute a large number of these y's in order to verify some property that they have, for instance to check that

$$\lim_{n\to\infty} \frac{y_{n+1}}{y_n} = 3$$

as it must, since 3 is the root of largest absolute value of the characteristic equation.

As a first approach, we might declare y to be a linear array of some size large enough to accommodate the expected length of the calculation. Then the rest is easy. For each n, we would calculate the next y_{n+1} from (1.4.1), we would divide it by its predecessor y_n to get a new ratio. If the new ratio agrees sufficiently well with the previous ratio we announce that the computation has terminated and print the new ratio as our answer. Otherwise, we move the new ratio to the location of the old ratio, increase n and try again.

If we were to write this out as formal procedure (algorithm) it might look like:

$y_0 \leftarrow 1$; $y_1 \leftarrow 1$; $y_2 \leftarrow 1$; $n \leftarrow 2$;
newrat $\leftarrow -10$; oldrat $\leftarrow 1$;
while $|\text{newrat} - \text{oldrat}| \geq 0.000001$ **do**
  oldrat $\leftarrow$ newrat; $n \leftarrow n + 1$;
  $y_n \leftarrow y_{n-1} + 5 * y_{n-2} + 3 * y_{n-3}$;
  newrat $\leftarrow y_n / y_{n-1}$
endwhile
print newrat; **Halt**.

Well use the leftarrow "$\leftarrow$ to mean that we are to compute the quantity on the right, if necessary, and then store it in the place named on the left. It can be read as "is replaced by or "is assigned". Also, the block that begins with "**while**" and ends with "**endwhile**" represents a group of instructions that are to be executed repeatedly until the condition that follows "**while**" becomes false, at which point the line following "**endwhile**" is executed.

The procedure just described is fast, but it uses lots of storage. If, for instance, such a program needed to calculate 79 y's before convergence occurred, then it would have used 79 locations of array storage. In fact, the problem above doesnt need that many locations because convergence happens a lot sooner. Suppose you wanted to find out how much sooner, given only a programmable pocked calculator with ten or twenty memory locations. Then you might appreciate a calculation procedure that needs just four locations to hold all necessary y's.

Thats fairly easy to accomplish, though. At any given moment in the program, what we need to find the next y are just the previous three y's. So why not save only those three? Well use the previous three to calculate the next one, and stow it for a moment in a fourth location. Then well compute the new ratio and compare it with the old. If theyre not close enough, we move each one of the three newest ys back one step into the places where we store the latest three y's and repeat the process. Formally, it might be:

$y \leftarrow 1$; ym1 $\leftarrow 1$; ym2 $\leftarrow 1$;
newrat $\leftarrow -10$; oldrat $\leftarrow 1$;
while $|\text{newrat} - \text{oldrat}| \geq 0.000001$ **do**
  ym3 $\leftarrow$ ym2; ym2 $\leftarrow$ ym1; ym1 $\leftarrow y$;
  oldrat $\leftarrow$ newrat;
  $y \leftarrow \text{ym1} + 5 * \text{ym2} + 3 * \text{ym3}$;
  newrat $\leftarrow y/\text{ym1}$ **endwhile**;
print newrat; **Halt**.

The calculation can now be done in exactly six memory locations (y, ym1, ym2, ym3, oldrat, newrat) no matter how many y's have to be calculated, so you can undertake it on your pocket calculator with complete confidence. The price that we pay for the memory saving is that we must move the data around a bit more.

One should not think that such programming methods are only for pocket calculators. As we progress through the numerical solution of differential equations we will see situations in which each of the quantities that appears in the difference equation will itself be an array (!), and that very large numbers, perhaps thousands, of these arrays will need to be computed. Even large computers might quake at the thought of using the first method above, rather than the second, or doing the calculation. Fortunately, it will almost never be necessary to save in memory all of the computed values simultaneously. Normally, they will be computed, and then printed or plotted, and never needed except in the calculation of their immediate successors.

Exercises 1.4

1. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined by the recurrence formula $F_{n+2} = F_{n+1} + F_n$ for $n = 0, 1, 2, \ldots$ together with the starting values $F_0 = 1$, $F_1 = 1$.

   (a) Write out the first ten Fibonacci numbers.

   (b) Derive an explicit formula for the nth Fibonacci number F_n.

   (c) Evaluate your formula for $n = 0, 1, 2, 3, 4$.

   (d) Prove directly from your formula that the Fibonacci numbers are integers (This is perfectly obvious from their definition, but is not so obvious from the formula!).

   (e) Evaluate
   $$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

   (f) Suppose we change the starring value $F_1 = 1$ to something else, say $F_1 = \alpha$. Show that there is exactly one value of α for which the sequence F_n approaches zero as $n \to \infty$, and determine that value of α.

   (g) Write a computer program that will compute Fibonacci numbers and print out the limit in part (e) above, correct to six decimal places.

   (h) Write a computer program that will compute the first 40 members of the modified Fibonacci sequence, in which $F_1 = \alpha$, the number that you found in (f) above. Do these computed numbers seem to be approaching zero? Explain carefully what you see and why it happens.

   (i) Modify the program of part (h) to run in higher (or double) precision arithmetic. Does it change any of your answers?

2. Find the most general solution of each of the following difference equations:

   (a) $y_{n+1} - 2y_n + y_{n-1} = 0$

   (b) $y_{n+1} = 2y_n$

   (c) $y_{n+2} + y_n = 0$

   (d) $y_{n+2} + 3y_{n+1} + 3y_n + y_{n-1} = 0$