Math 241: Solving the heat equation

D. DeTurck

University of Pennsylvania

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1D heat equation with Dirichlet boundary conditions

We derived the one-dimensional heat equation

$$u_t = k u_{xx}$$

and found that it's reasonable to expect to be able to solve for $u(x,t)$ (with $x \in [a,b]$ and $t > 0$) provided we impose initial conditions:

$$u(x,0) = f(x)$$

for $x \in [a,b]$ and boundary conditions such as

$$u(a,t) = p(t)\,, \quad u(b,t) = q(t)$$

for $t > 0$.
We showed that this problem has at most one solution, now it's time to show that a solution exists.

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- Since the heat equation is linear (and homogeneous), a linear combination of two (or more) solutions is again a solution. So if u_1, u_2,... are solutions of $u_t = ku_{xx}$, then so is

$$c_1 u_1 + c_2 u_2 + \cdots$$

for any choice of constants $c_1, c_2, \ldots$. (Likewise, if $u_\lambda(x,t)$ is a solution of the heat equation that depends (in a reasonable way) on a parameter λ, then for any (reasonable) function $f(\lambda)$ the function

$$U(x,t) = \int_{\lambda_1}^{\lambda_2} f(\lambda) u_\lambda(x,t)\, d\lambda$$

is also a solution.

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- Then we'll consider problems with zero initial conditions but non-zero boundary values.
- We can add these two kinds of solutions together to get solutions of general problems, where both the initial and boundary values are non-zero.

Symmetry

Some more observations:

- If $u(x, t)$ is a solution, then so is $u(a \pm x, b + t)$ for any constants a and b.
  Note the $\pm$ with the x but only $+$ with t — you can't "reverse time" with the heat equation. This shows that the heat equation respects (or reflects) the second law of thermodynamics (you can't unstir the cream from your coffee).

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  Note the $\pm$ with the x but only $+$ with t — you can't "reverse time" with the heat equation. This shows that the heat equation respects (or reflects) the second law of thermodynamics (you can't unstir the cream from your coffee).
- If $u(x,t)$ is a solution then so is $u(a^2 t, at)$ for any constant a. We'll use this observation later to solve the heat equation in a surprising way, but for now we'll just store it in our memory bank.

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- Next, taking our cue from the initial-value problem, suppose $u(x, 0) = p_0(x)$ for some polynomial $p_0(x)$, and try to construct a solution of the form

$$u(x, t) = p_0(x) + tp_1(x) + t^2 p_2(x) + \cdots$$

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- We have

$$u_t = p_1(x) + 2tp_2(x) + 3t^2 p_3(x) + \cdots$$

and

$$u_{xx} = p_0''(x) + tp_1''(x) + t^2 p_2''(x) + \cdots$$

Polynomial solutions

- So the heat equation tells us:

$$p_1 = kp_0'', \quad p_2 = \frac{k}{2}p_1'' = \frac{k^2}{2}p_0'''',$$

$$p_3 = \frac{k}{3}p_2'' = \frac{k^3}{3!}p_0^{(6)}, \ \ldots, \ p_n = \frac{k^n}{n!}p_0^{(2n)}$$

- This process will stop if p_0 is a polynomial, and we'll get a polynomial solution of the heat equation whose x-degree is twice its t-degree:

$$u(x,t) = p_0(x) + \frac{kt}{1!}p_0'' + \frac{k^2t^2}{2!}p_0'''' + \cdots + \frac{k^nt^n}{n!}p_0^{(2n)} + \cdots .$$

But. . .

The trouble with polynomial solutions, or even with extending the idea of polynomials to power series in two variables (ick!), is that it would be very difficult if not impossible to figure out how to choose the coefficients of the polynomial p_0 so that the boundary values, even simple ones, would be matched at both ends.

There are also tricky convergence questions, etc for power series, and we don't want to get overwhelmed with these.

Separation of variables

- A more fruitful strategy is to look for *separated solutions* of the heat equation, in other words, solutions of the form $u(x,t) = X(x)T(t)$.

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$$X\frac{dT}{dt} = k\frac{d^2X}{dx^2}T.$$

- Divide both sides by kXT and get

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}.$$

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- The power of this method comes in the application of boundary conditions, which we turn to next.

Boundary conditions

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- As an example, let's suppose the x-interval goes from 0 to L, and we have homogeneous Dirichlet conditions: $u(0,t) = u(L,t) = 0$ for all t.

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- So let $\lambda = -\alpha^2$. The general solution of $X'' + \alpha^2 X = 0$ is

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- The easiest way to satisfy the boundary conditions on u is to insist that $X(0) = X(L) = 0$. What does this imply?

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- This gives a condition on α: $\sin \alpha L = 0$, or $\alpha L = n\pi$ for some integer n.
- Since $\sin(-x) = -\sin x$, we need only consider positive integers n. Thus

$$\alpha = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

and

$$X = c_2 \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

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- The general solution of these are
$$T = ce^{-n^2 k\pi^2 t/L^2}, \qquad n = 1, 2, 3, \ldots.$$
- Putting this together with our X solutions, we get solutions $u(x, t)$ of the form:
$$u(x, t) = b_n e^{-n^2 k\pi^2 t/L^2} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Superposition, again

- We can put these solutions together to get solutions of the form

$$u(x,t) = b_1 e^{-k\pi^2 t/L^2} \sin \frac{\pi x}{L} + b_2 e^{-4\pi^2 t/L^2} \sin \frac{2\pi x}{L} + \cdots \\ + b_N e^{-N^2 k\pi^2 t/L^2} \sin \frac{N\pi x}{L}.$$

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$$+ b_N e^{-N^2 k\pi^2 t/L^2} \sin\frac{N\pi x}{L}.$$

- This function $u(x,t)$ automatically satisfies the boundary conditions $u(0,t) = u(L,t) = 0$, since all of the pieces do.
- And it would be great for satisfying initial conditions given in trigonometric form — for example

$$u(x,0) = 3\sin\frac{\pi x}{L} - 2\sin\frac{3\pi x}{L} + \sin\frac{6\pi x}{L}.$$

Taking things to the limit

- But what if the initial condition isn't trigonometric? Could we consider adding together an infinite number of pieces? As in

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2 k \pi^2 t / L^2} \sin \frac{n\pi x}{L}$$

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- So the question is, which functions $f(x)$ (for x in the interval $[0, L]$) can be expressed as an infinite series of sines?
- The (somewhat surprising) answer is, ALL OF THEM!
- Let's see how this might work in practice (and we'll take up the question of proving this claim later).

The *Fourier ansatz*

- Just for something concrete, let's suppose we want to solve the problem

$$u_t = \frac{1}{5} u_{xx}, \quad u(0,t) = u(3,t) = 0, \quad u(x,0) = 3x - x^2$$

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for $t > 0$ and $0 \le x \le 3$.
- We'll *assume* that we can express u as
$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/45} \sin\frac{n\pi x}{3}$$
(this is the *ansatz*) and see if we can figure out what the constants b_n should be — we know that the boundary conditions are automatically satisfied, and perhaps we can choose the b_n's so that
$$3x - x^2 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3}.$$

Integrals rather than derivatives

- We're trying to find b_n's so that

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- By contrast with Taylor series, where you find the coefficients by integration rather than differentiation.
- We'll use two basic facts:
- If $n \neq m$ then

$$\int_0^3 \sin \frac{n\pi x}{3} \sin \frac{m\pi x}{3} \, dx = 0.$$

- If $n = m$ then

$$\int_0^3 \sin \frac{n\pi x}{3} \sin \frac{m\pi x}{3} \, dx = \int_0^3 \sin^2 \frac{n\pi x}{3} \, dx = \frac{3}{2}.$$

Finding the coefficients

- We're still trying to find b_n's so that

$$3x - x^2 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}.$$

- Motivated by the facts on the previous slide, we multiply both sides by $\sin \frac{m\pi x}{3}$ and integrate both sides from 0 to 3:

$$\begin{aligned}\int_0^3 (3x - x^2) \sin \frac{m\pi x}{3}\, dx &= \int_0^3 \left(\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} \right) \sin \frac{m\pi x}{3}\, dx \\ &= \sum_{n=1}^{\infty} b_n \int_0^3 \sin \frac{n\pi x}{3} \sin \frac{m\pi x}{3}\, dx \\ &= \frac{3b_m}{2}\end{aligned}$$

Integration by parts

- It's an exercise in integration by parts to show that

$$\int_0^3 (3x - x^2) \sin \frac{m\pi x}{3}\, dx = \frac{54}{m^3\pi^3}\left(1 - (-1)^m\right)$$

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- Therefore,

$$b_m = \frac{36}{m^3\pi^3} \left(1 - (-1)^m\right) = \begin{cases} 0 & m \ \textit{even} \\ \dfrac{72}{m^3\pi^3} & m \ \textit{odd} \end{cases}$$

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- So we arrive at a candidate for the solution:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{72}{(2n+1)^3\pi^3} e^{-(2n+1)^2\pi^2 t/45} \sin \frac{(2n+1)\pi x}{3}$$

Validating the solution

- The series

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converges for $t \geq 0$, and certainly satisfies the boundary conditions. What about the initial condition $u(x,0) = 3x - x^2$?

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- Well,

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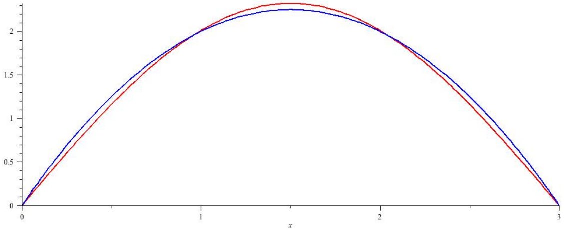
  .

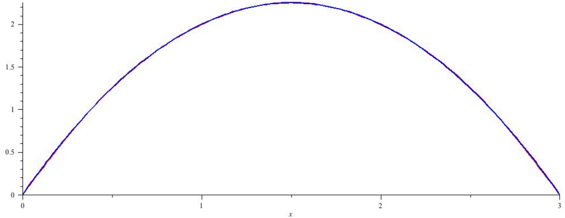
Graphical evidence

Red graph: $3x - x^3$, Blue graph: sum

One term:

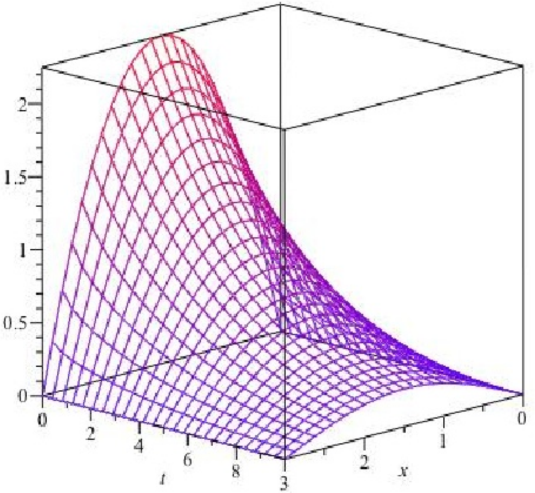


Three terms:



Plotting the solution

Here is a plot of the sum of of the first three terms of the solution:



Another example

- Now, let's look at a problem with insulated ends:

$$u_t = \frac{1}{5} u_{xx}, \quad u_x(0,t) = u_x(3,t) = 0, \quad u(x,0) = 3x - x^2$$

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  for $t > 0$ and $0 \leq x \leq 3$.
- This time, with the boundary conditions in mind, we'll *assume* that we can express u as

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-n^2\pi^2 t/45} \cos \frac{n\pi x}{3}$$

  (and see if we can figure out what the constants a_n should be — the boundary conditions are automatically satisfied, and we will choose the a_n's so that

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Useful integrals

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whereas if $n = m = 0$ we get $\int_0^3 1^2\, dx = 3$.

Finding the coefficients

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$$3x - x^2 = \sum_{n=0}^{\infty} b_n \cos \frac{n\pi x}{3}.$$

- Motivated by the facts on the previous slide, we multiply both sides by $\cos \frac{m\pi x}{3}$ and integrate both sides from 0 to 3. For $m > 0$ we get:

$$\begin{aligned}
\int_0^3 (3x - x^2) \cos \frac{m\pi x}{3}\, dx &= \int_0^3 \left(\sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{3} \right) \sin \frac{m\pi x}{3}\, dx \\
&= \sum_{n=0}^{\infty} a_n \int_0^3 \cos \frac{n\pi x}{3} \cos \frac{m\pi x}{3}\, dx \\
&= \frac{3a_m}{2}
\end{aligned}$$

and we get $3a_0$ for $m = 0$.

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$$\int_0^3 (3x - x^2)\cos\frac{m\pi x}{3}\,dx = \begin{cases} 0 & m\ odd \\ \frac{9}{2} & m = 0 \\ -\frac{27}{m^2\pi^2} & m > 0.\ even \end{cases}$$

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- So we arrive at a candidate for the solution:

$$u(x,t) = \frac{3}{2} - \sum_{n=1}^{\infty} \frac{36}{(2n)^2\pi^2} e^{-(2n)^2\pi^2 t/45} \cos\frac{2n\pi x}{3}$$

Validating the solution

- The series

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- Well,

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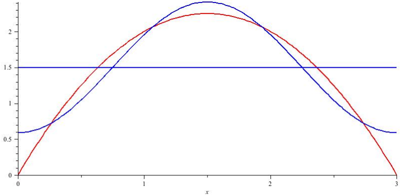
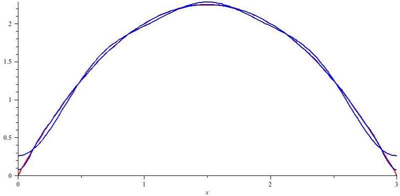
Graphical evidence

Red graph: $3x - x^3$, Blue graph: sum

One term, two terms:

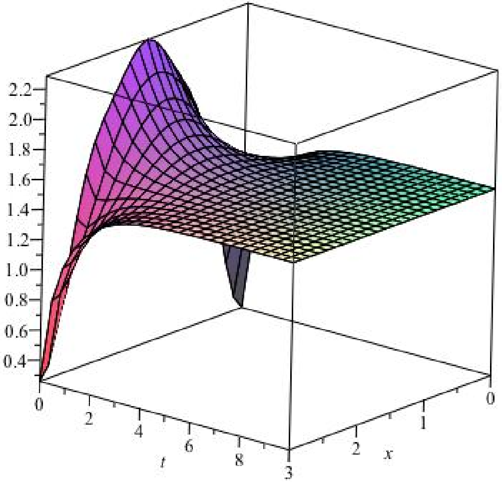


Four terms, thirteen terms:



Plotting the solution

Here is a plot of the sum of of the first three terms of the solution:



Other boundary conditions

Questions for discussion:

- How would you handle boundary conditions $u_x(0,t) = 0$, $u(3,t) = 0$?
- What about $u(0,t) = 0$, $u_x(3,t) = 0$?
- What about something like $u(0,t) = 0$, $u_x(3,t) + u(3,t) = 0$?