Fourier transforms

Motivation and definition

Up to now, we've been expressing functions on finite intervals (usually the interval $0 \leqslant x \leqslant L$ or $-L \leqslant x \leqslant L$) as Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

and

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.$$

We also occasionally thought about the complex exponential version of Fourier series: Since $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$, or equivalently

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e{-i\theta}}{2i},$$

we can rewrite the above series as:

$$\begin{aligned}
f(x) &= a_0 e^{0ix} + \sum_{n=1}^{\infty} a_n \frac{e^{n\pi ix/L} + e^{-n\pi ix/L}}{2} + b_n \frac{e^{n\pi ix/L} - e^{-n\pi ix/L}}{2i} \\
&= a_0 e^{0ix} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{n\pi ix/L} + \frac{a_n + ib_n}{2} e^{-n\pi ix/L} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{-n\pi ix/L}
\end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{2}(a_n + ib_n) & \text{for } n > 0 \\ a_0 & \text{for } n = 0 \\ \frac{1}{2}(a_{-n} - ib_{-n}) & \text{for } n < 0 \end{cases}$$

Using the formulas for a_n and b_n given above, we see that, for $n > 0$.

$$\begin{aligned}
c_n &= \frac{1}{2}(a_n + ib_n) \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx + \frac{i}{2L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right]\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{n\pi ix/L}\,dx.
\end{aligned}$$

If $n < 0$ we have

$$\begin{aligned}
c_n &= \frac{1}{2}(a_{-n} - ib_{-n}) \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\cos\left(-\frac{n\pi x}{L}\right)\,dx - \frac{i}{2L}\int_{-L}^{L} f(x)\sin\left(-\frac{n\pi x}{L}\right)\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) + i\sin\left(\frac{n\pi x}{L}\right)\right]\,dx \\
&= \frac{1}{2L}\int_{-L}^{L} f(x)e^{n\pi ix/L}\,dx
\end{aligned}$$

because cosine is an even function and sine is odd. So the same formula works for all the coefficients (even c_0) in this case and we have

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-n\pi ix/L} \quad \text{where} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{n\pi ix/L}\,dx.$$

What we want to do here is let L tend to infinity, so we can consider problems on the whole real line. To see what happens to our Fourier series formulas when we do this, we introduce two new variables: $\omega = n\pi/L$ and $\Delta\omega = \pi/L$. Then our complex Fourier series formulas become

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i\omega x} \quad \text{where} \quad c_n = \frac{\Delta\omega}{2\pi}\int_{-L}^{L} f(x)e^{i\omega x}\,dx$$

and the n in the formula for c_n is hiding in the variable ω. Now, if we let $\widetilde{c_\omega} = c_n/\Delta\omega$, we can rewrite these as

$$f(x) = \sum_{n=-\infty}^{\infty} \widetilde{c_\omega} e^{-i\omega x}\Delta\omega \quad \text{where} \quad \widetilde{c_\omega} = \frac{1}{2\pi}\int_{-L}^{L} f(x)e^{i\omega x}\,dx.$$

The variable $\omega = n\pi/L$ takes on more and more values which are closer and closer together as $L \to \infty$, so $\widetilde{c_\omega}$ begins to feel like a function of the variable ω defined for all real ω. Likewise, the sum on the left looks an awful lot like a Riemann sum approximating an integral. What happens in the limit as $L \to \infty$ is:

$$f(x) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}\,d\omega \quad \text{where} \quad c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ix\omega}\,dx.$$

The formula on the right defines the function $c(\omega)$ as the *Fourier transform* of $f(x)$, and the formula on the left defines $f(x)$ as the *inverse Fourier transform* of $c(\omega)$. These formulas hold true (and the inverse Fourier transform of the Fourier transform of $f(x)$ is $f(x)$ — the so-called *Fourier inversion formula*) for reasonable functions $f(x)$ that decay to zero as $|x| \to \infty$ in such a way so that $|f(x)|$ and/or $|f(x)|^2$ has a finite integral over the whole real line.

There are many standard notations for Fourier transforms (and alternative definitions with the minus sign in the Fourier transform rather than in the inverse, and with the 2π factor in different places, so watch out if you're looking in books other than our textbook!), including

$$\widehat{f}(\omega) = F(\omega) = \mathcal{F}[f(x)](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ix\omega}\,dx$$

and

$$\check{F}(x) = f(x) = \mathcal{F}^{-1}[F(\omega)](x) = \int_{-\infty}^{\infty} F(\omega)e^{-ix\omega}\,d\omega.$$

Properties and examples.

The Fourier transform is an operation that maps a function of x, say $f(x)$ to a function of ω, namely $\mathcal{F}[f](\omega) = \check{f}(\omega)$. It is clearly a *linear* operator, so for functions $f(x)$ and $g(x)$ and constants α and β we have

$$\mathcal{F}\left[\alpha f(x) + \beta g(x)\right] = \alpha\mathcal{F}\left[f(x)\right] + \beta\mathcal{F}\left[g(x)\right].$$

Some other properties of the Fourier transform are

1. **Translation** (or shifting): $\mathcal{F}\left[f(x-a)\right](\omega) = e^{i\omega a}\mathcal{F}\left[f(x)\right](\omega)$. And in the other direction, $\mathcal{F}\left[e^{iax}f(x)\right](\omega) = \mathcal{F}\left[f(x)\right](\omega + a)$.
2. **Scaling**: $\mathcal{F}\left[\frac{1}{a}f\left(\frac{x}{a}\right)\right](\omega) = \mathcal{F}\left[f(x)\right](a\omega)$, and likewise $\mathcal{F}\left[f(ax)\right](\omega) = \frac{1}{a}\mathcal{F}\left[f(x)\right]\left(\frac{\omega}{a}\right)$.
3. **Operational property** (derivatives): $\mathcal{F}\left[f'(x)\right](\omega) = -i\omega\mathcal{F}\left[f(x)\right](\omega)$, and $\mathcal{F}\left[xf(x)\right](\omega) = -i\frac{d}{d\omega}\left(\mathcal{F}\left[f(x)\right](\omega)\right)$.

The operational property is of essential importance for the study of differential equations, since it shows that the Fourier transform converts derivatives to multiplication

– so it converts calculus to algebra (or might reduce a partial differential equation to an ordinary one).

Here are the proofs of the first of each of the three pairs of formulas to give a sense of how to work with Fourier transforms, and leave the other three as exercises. For the first shifting rule, we make the substitution $y = x - a$ (so $dy = dx$ and $x = y + a$) to calculate

$$\begin{aligned}
\mathcal{F}\left[f(x-a)\right](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-a)e^{i\omega x}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{i\omega y}e^{i\omega a}\,dx \\
&= e^{i\omega a}\mathcal{F}\left[f(x)\right](\omega)
\end{aligned}$$

For the first scaling rule, we make the substitution $y = x/a$ (so $dx = a\,dy$) and get

$$\begin{aligned}
\mathcal{F}\left[\frac{1}{a}f\left(\frac{x}{a}\right)\right](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a}f\left(\frac{x}{a}\right)e^{i\omega x}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{ia\omega y}\,dy \\
&= \mathcal{F}\left[f(x)\right](a\omega)
\end{aligned}$$

For the operational property we first point out that since the Fourier transforms of both $f'(x)$ and $f(x)$ exist, we must have that $f(x) \to 0$ and $f'(x) \to 0$ as $x \to \pm\infty$. Therefore the endpoint terms will vanish when we integrate by parts (with $u = e^{i\omega x}$ and $dv = f'(x)\,dx$, so $du = i\omega e^{i\omega x}$ and $v = f(x)$):

$$\begin{aligned}
\mathcal{F}\left[f'(x)\right](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f'(x)e^{i\omega x}\,dx \\
&= \frac{1}{2\pi}e^{i\omega x}f(x)\Big|_{x=-\infty}^{x=\infty} - \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega f(x)e^{i\omega x}\,dx \\
&= 0 - \frac{i\omega}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx \\
&= -i\omega\mathcal{F}\left[f(x)\right](\omega)
\end{aligned}$$

Let's calculate a few basic examples of Fourier transforms:

Example 1. Let $S_a(x)$ be the function defined by

$$S_a(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathcal{F}\left[S_a(x)\right](\omega) = \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega x}\,dx = \frac{e^{i\omega a} - e^{-i\omega a}}{2\pi i\omega} = \frac{\sin a\omega}{\pi\omega}.$$

Example 2. Let $u(x) = e^{-ax^2/2}$, so the graph of $u(x)$ is a "Gaussian" or "bell-shaped curve". Then $u(x)$ satisfies the differential equation $u' + axu = 0$. We can use this fact and the properties of the Fourier transform to calculate $\widehat{u}$ as follows: Take the Fourier transform of the differential equation and use linearity and both parts of property (3) above to get

$$\begin{aligned} 0 = \mathcal{F}\left[u' + axu\right](\omega) &= \mathcal{F}\left[u'\right] + a\mathcal{F}\left[xu\right] \\ &= -i\omega\mathcal{F}\left[u\right] - ai\frac{d\mathcal{F}\left[u\right]}{d\omega} \end{aligned}$$

Therefore $\mathcal{F}\left[u\right]$ satisfies the differential equation

$$\mathcal{F}\left[u\right]' + \frac{1}{a}\omega\mathcal{F}\left[u\right] = 0$$

the solution of which is

$$\mathcal{F}\left[u\right] = Ce^{-\omega^2/(2a)}.$$

The constant C is the value of $\mathcal{F}\left[u\right](0)$, i.e.,

$$C = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ax^2/2}\,dx = \frac{1}{2\pi}\sqrt{\frac{2}{a}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{1}{\sqrt{2\pi a}},$$

using the substitution $y = \sqrt{\frac{a}{2}}\,x$ and the familiar (or at least accessible) fact that $\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}$. Therefore

$$\mathcal{F}\left[e^{-ax^2/2}\right](\omega) = \frac{1}{\sqrt{2\pi a}}e^{-\omega^2/(2a)},$$

so the original Gaussian is transformed into a different one.

An interesting observation is what happens for $a = 1$: Then we have

$$\mathcal{F}\left[e^{-x^2/2}\right] = \frac{1}{\sqrt{2\pi}}e^{-\omega^2/2},$$

so the specific Gaussian $e^{-x^2/2}$ is an *eigenfunction of the Fourier transform with eigenvalue* $1/\sqrt{2\pi}$.

Example 3. Let $f(x) = e^{-a|x|}$, so

$$f(x) = \begin{cases} e^{-ax} & \text{if } x \geqslant 0 \\ e^{ax} & \text{if } x < 0 \end{cases}.$$

Then

$$\begin{aligned}
\mathcal{F}\left[e^{-a|x|}\right](\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|x|}e^{i\omega x}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ax}e^{i\omega x}\,dx + \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\omega}e^{i\omega x}\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(a+i\omega)x}\,dx + \frac{1}{2\pi}\int_{0}^{\infty} e^{(-a+i\omega)x}\,dx \\
&= \frac{1}{2\pi}\left.\frac{e^{(a+i\omega)x}}{a+i\omega}\right|_{x=-\infty}^{x=0} + \frac{1}{2\pi}\left.\frac{e^{(-a+i\omega)x}}{-a+i\omega}\right|_{x=0}^{x=\infty} \\
&= \frac{1}{2\pi}\left(\frac{1}{a+i\omega}+\frac{1}{a-i\omega}\right) \\
&= \frac{a}{\pi(a^2+\omega^2)}
\end{aligned}$$

(the limiting values of the exponentials at $\pm\infty$ are zero because $e^{-|a|x}$ goes to zero as x goes to $\pm\infty$ and $e^{i\omega x}$ stays bounded).

An observation. Because the formulas for the Fourier transform and the inverse Fourier transform are so similar, we can get inverse transform formulas from the direct ones and vice versa. In particular, note that if we let $y = -x$ then

$$\mathcal{F}\left[f(x)\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(-y)e^{-i\omega y}\,dy = \frac{1}{2\pi}\mathcal{F}^{-1}\left[f(-y)\right](\omega)$$

Likewise

$$\mathcal{F}^{-1}\left[F(\omega)\right](x) = \int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega = \int_{-\infty}^{\infty} F(-\alpha)e^{i\alpha x}\,d\alpha = 2\pi\mathcal{F}\left[F(-\alpha)\right](x)$$

So if we know a Fourier transform formula or an inverse Fourier transform formula, we can get another one for free by "reversing the inverse". For example, since

$$\mathcal{F}\left[S_a(x)\right](\omega) = \frac{\sin a\omega}{\pi\omega},$$

we immediately have that

$$\mathcal{F}^{-1}\left[\frac{\sin a\omega}{\pi\omega}\right](x) = S_a(x).$$

From either of the formulas above and the fact that $\sin x/x$ is an even function, we have

$$\mathcal{F}\left[\frac{\sin ax}{\pi x}\right](\omega) = \frac{1}{2\pi}S_a(\omega),$$

or

$$\mathcal{F}\left[\frac{\sin ax}{ax}\right] = \frac{1}{2a}S_a(\omega).$$

Similarly, since we know that

$$\mathcal{F}\left[e^{-a|x|}\right] = \frac{a}{\pi(a^2+\omega^2)}$$

for $a > 0$, we can immediately write

$$\mathcal{F}^{-1}\left[\frac{a}{\pi(a^2+\omega^2)}\right] = e^{-a|x|}.$$

And since

$$\frac{a}{\pi(a^2+\omega^2)}$$

is an even function of ω, we have

$$\mathcal{F}\left[\frac{a}{\pi(a^2+x^2)}\right] = \frac{1}{2\pi}e^{-a|\omega|},$$

or

$$\mathcal{F}\left[\frac{1}{a^2+x^2}\right] = \frac{1}{2a}e^{-a|\omega|}.$$

Convolutions.

We need one more Fourier transform formula, and it involves an operation on functions that might seem new to you. It is called *convolution* and it starts with two functions, $f(x)$ and $g(x)$ and produces a new one, denoted $f * g$ or $(f * g)(x)$ (or sometimes just $f * g(x)$), defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy.$$

If you think about it, convolution is like multiplication of polynomials or series, wherein

$$\left(\sum a_n\right)\left(\sum b_n\right) = \sum c_n \quad \text{where} \quad c_n = \sum a_m b_{n-m}.$$

This motivates the definition of convolution as an operation that might have has its Fourier transform the product of the transforms of the individual functions, and it does (up to a nuisance factor of 2π), but more on that in a moment.

First some basic properties of convolutions:

1. Convolution is linear in each of the two functions. In other words, if f, g and h are functions and α and β are constants, then

$$(\alpha f + \beta g) * h = \alpha f * h + \beta g * h \quad \text{and} \quad f * (\alpha g + \beta h) = \alpha f * g + \beta f * h.$$

2. Convolution is commutative: $f*g = g*f$. To prove this, we make the substitution $z = x - y$ in the integral (so we think of x as being constant while we're doing the integral, and $y = x - z$ and $dy = dz$) and get

$$\begin{aligned}(f * g)(x) &= \int_{-\infty}^{\infty} f(y)g(x-y)\,dy \\ &= -\int_{\infty}^{-\infty} f(x-z)g(z)\,dz \\ &= \int_{-\infty}^{\infty} g(z)f(x-z)\,dz \\ &= (g * f)(x)\end{aligned}$$

3. Now for the Fourier transform:

$$\mathcal{F}\left[f * g\right](\omega) = 2\pi \mathcal{F}\left[f\right](\omega)\mathcal{F}\left[g\right](\omega) \quad \text{or} \quad (\widehat{f * g})(\omega) = 2\pi \widehat{f}(\omega)\widehat{g}(\omega).$$

To see this we use the substitution $z = x - y$ again, and break up $e^{i\omega x}$ as $e^{i\omega(x-y+y)} = e^{i\omega y}\, e^{i\omega(x-y)}$ to get:

$$\begin{aligned}(\widehat{f * g})(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(y)g(x-y)\,dy\right) e^{i\omega x}\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(x-y)e^{i\omega x}\,dx\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)e^{i\omega y}\, g(x-y)e^{i\omega(x-y)}\,dx\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(x-y)e^{i\omega(x-y)}\,dx\right) f(y)e^{i\omega y}\,dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(z)e^{i\omega z}\,dz\right) f(y)e^{i\omega y}\,dy \\ &= 2\pi\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} g(z)e^{i\omega z}\,dz\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y)e^{i\omega y}\,dy\right) \\ &= 2\pi\widehat{f}(\omega)\widehat{g}(\omega)\end{aligned}$$

Likewise, the transform of the product of two functions is the convolution of the transforms, except without the factor of 2π, namely $\mathcal{F}^{-1}\left[F * G\right](x) = \check{F}(x)\check{G}(x)$.

One last thing: the convolution product of two functions has an interesting property relative to derivatives:

$$\frac{d}{dx}(f * g) = \frac{df}{dx} * g = f * \frac{dg}{dx}$$

so when you take derivatives of convolution of two functions, you get to stick the derivative on whichever of the two functions is more convenient (or differentiable) — this is an easy consequence of the commutativity of convolution, and it has the powerful consequence that the convolution of two functions has the better of the differentiability properties of the two individual functions. So if f is discontinuous but g is smooth, then $f * g$ will be smooth.

The heat equation on the whole line.

Now we're ready to use all of this for something! We seek to solve the heat equation

$$u_t = ku_{xx}$$

for $t > 0$ and $-\infty < x < \infty$, with initial conditions $u(x, 0) = f(x)$ and assuming u decays to zero at $x = -\infty$ and $x = \infty$.

We'll start by taking the Fourier transform of both sides of the differential equation in the x-variable. By this we mean

$$\widehat{u}(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, t)e^{ix\omega}\, dx,$$

so $\widehat{u}$ satisfies

$$\frac{\partial \widehat{u}}{\partial t} = -k\omega^2\widehat{u}$$

and

$$\widehat{u}(\omega, 0) = \widehat{f}(\omega).$$

This is an ordinary differential equation where the independent variable is t and ω should be treated as a constant. The general solution of the differential equation is

$$\widehat{u}(\omega, t) = c(\omega)e^{-k\omega^2 t}.$$

Putting $t = 0$ shows that $c(\omega) = \widehat{f})\omega)$. Therefore Fourier transform of the solution of our problem is

$$\widehat{u}(\omega, t) = \widehat{f}(\omega)e^{-k\omega^2 t}.$$

We can now recover $u(x, t)$ by taking the inverse Fourier transform of both sides, using the rule that the Fourier transform of a convolution is 2π times the product of the individual Fourier transforms, so

$$u(x, t) = \frac{1}{2\pi} f(x) * \mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right]$$

and we have to calculate the inverse Fourier transform of $e^{-k\omega^2 t}$. But we have the rule for Gaussians:

$$\mathcal{F}\left[e^{-ax^2/2}\right] = \frac{1}{\sqrt{2\pi a}} e^{-\omega^2/(2a)},$$

or

$$\mathcal{F}^{-1}\left[e^{-\omega^2/(2a)}\right] = \sqrt{2\pi a}\, e^{-ax^2/2}.$$

Since we want to calculate the inverse Fourier transform of $e^{-k\omega^2 t}$, we should set $1/(2a) = kt$, or $a = 1/(2kt)$. Then we get

$$\mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right] = \sqrt{\frac{\pi}{kt}}\, e^{-x^2/(4kt)},$$

and so

$$u(x,t) = \frac{1}{2\pi} f(x) * \left(\sqrt{\frac{\pi}{kt}}\, e^{-x^2/(4kt)}\right) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y) e^{-(x-y)^2/(4kt)}\, dy.$$

The function

$$G(x,y,t) = \frac{1}{\sqrt{4\pi kt}} e^{-(x-y)^2/(4kt)}$$

is called the *fundamental solution of the heat equation*. It is the solution to the initial value for the heat equation for the situation where the initial conditions are such that a single unit of heat energy is introduced at the point y at time $t = 0$. So the above formula for $u(x,t)$ says that we can solve the heat equation for arbitrary initial conditions $u(x,t)$ by integrating together all the contributions to the temperature at time $t = 0$ at all points, as described by the differential $f(y)\, dy$.

The wave equation on the whole line.

Next, let's look at the initial-value problem for the wave equation on the whole line. We'll solve the wave equation

$$u_{tt} = c^2 u_{xx}$$

together with initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x).$$

As we did with the heat equation, we'll take the Fourier transform of both sides of the differential equation in the x-variable. So once again, let

$$\widehat{u}(\omega,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t) e^{i\omega x}\, dx$$

so $\widehat{u}$ satisfies

$$\frac{\partial^2 \widehat{u}}{\partial t^2} = -c^2\omega^2\widehat{u}$$

and

$$\widehat{u}(\omega,0) = \widehat{f}(\omega) \quad \text{and} \quad \widehat{u}_t(\omega,0) = \widehat{g}(\omega).$$

We will again treat this as an ordinary differential equation in t with ω treated as a constant. The general solution of this equation is

$$u^(\omega,t) = c_1(\omega)\cos\omega ct + c_2(\omega)\sin\omega ct,$$

and the initial conditions imply

$$u(\omega,0) = c_1(\omega) = \widehat{f}(\omega) \quad \text{and} \quad u_t(\omega,0) = c\omega c_2(\omega) = \widehat{g}(\omega).$$

Therefore, $c_1(\omega) = \widehat{f}(\omega)$ and $c_2(\omega) = \widehat{g}(\omega)/(c\omega)$ and we have obtained the Fourier transform of the solution:

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)\cos\omega ct + \frac{\widehat{g}(\omega)}{c\omega}\sin\omega ct.$$

Therefore

$$u(x,t) = \mathcal{F}^{-1}\left[\widehat{f}(\omega)\cos\omega ct\right] + \mathcal{F}^{-1}\left[\frac{\widehat{g}(\omega)}{c\omega}\sin\omega ct\right].$$

We'll take the two terms one at a time. For the first, we write $\cos\omega ct$ in complex exponential form, so we're trying to compute

$$\begin{aligned}
\mathcal{F}^{-1}\left[\widehat{f}(\omega)\frac{e^{i\omega ct}+e^{-i\omega ct}}{2}\right] &= \frac{1}{2}\int_{-\infty}^{\infty}\widehat{f}(\omega)[e^{i\omega ct}+e^{-i\omega ct}]e^{i\omega x}\,d\omega \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{i\omega(x+ct)}\,dx + \int_{-\infty}^{\infty}\widehat{f}(\omega)e^{i\omega(x-ct)}\,dx\right) \\
&= \frac{1}{2}[f(x+ct)+f(x-ct)]
\end{aligned}$$

using the definition of the inverse Fourier transform.

For the second term, we proceed differently. Recalling that $\widehat{S_a}(\omega) = \sin a\omega/(\pi\omega)$ and that the inverse Fourier transform of a product of two functions is their convolution (divided by 2π), we have

$$\begin{aligned}
\mathcal{F}^{-1}\left[\widehat{g}(\omega)\frac{\sin\omega ct}{c\omega}\right] &= \frac{1}{2\pi}g(x) * \left(\frac{\pi}{c}S_{ct}(x)\right) \\
&= \frac{1}{2c}\int_{-\infty}^{\infty} g(x-y)S_{ct}(y)\,dy \\
&= \frac{1}{2c}\int_{-ct}^{ct} g(x-y)\,dy \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du
\end{aligned}$$

where in the last step we made the substitution $u = x-y$ (so $y = x-u$ and $dy = -du$). We put both terms together to get the solution to our initial-value problem for the wave equation:

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du.$$

This is called *d"Alembert's solution of the wave equation*, and clearly shows that signals propagate with speed c, since the value of the solution at a point x and time t depends only on the initial position and velocity values in the interval $[x-ct, x+ct]$, and conversely that the initial values at a point x influence the solution at time t only within the interval $[x-ct, x+ct]$.

Linear algebraic properties of the Fourier transform: Parseval's theorem and Hermite functions

We remarked earlier that the Fourier transform is a linear transformation from functions of x to functions of ω defined on the whole line (which can be integrated etc.). We also found that the function $e^{-x^2/2}$ is an eigenfunction of the Fourier transform with eigenvalue $1/\sqrt{2\pi}$. We explore some further consequences of these observations in this section.

First, we can compare the inner product of the Fourier transforms of two functions with the inner product of the functions themselves (remember, in the *complex* (or Hermitian) inner product, we have to take the complex conjugate of the second factor, and we write z^* for the complex conjugate of z):

$$\begin{aligned}
\langle f\,,\,g\rangle &= \int_{-\infty}^{\infty} f(x)(g(x))^*\,dx \\
&= \int_{-\infty}^{\infty} f(x)\left(\int_{-\infty}^{\infty}\widehat{g}(\omega)e^{-i\omega x}\,d\omega\right)^*\,dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{i\omega x}(\widehat{g}(\omega))^*\,dx\,d\omega \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx\right)(\widehat{g}(\omega))^*\,d\omega \\
&= 2\pi\int_{-\infty}^{\infty}\widehat{f}(\omega)(\widehat{g}(\omega))^*\,d\omega \\
&= 2\pi\left\langle \widehat{f}\,,\,\widehat{g}\right\rangle.
\end{aligned}$$

In particular, if $f = g$ we have that $\|f\|^2 = \langle f\,,\,f\rangle = 2\pi\left\langle \widehat{f}\,,\,\widehat{f}\right\rangle = 2\pi\|\widehat{f}\|^2$ or

$$\|\widehat{f}\|^2 = \frac{1}{2\pi}\|f\|^2.$$

This is called *Parseval's equality* or Parseval's theorem — it says that the Fourier transforms shrinks the norms of all functions by a factor of $1/\sqrt{2\pi}$ (so it wasn't an accident that we found an eigenfunction with that eigenvalue!), and it has a number of interesting consequences.

As a simple example. since we know that (recalling that $S_a(x)$ is the step function for the interval $[-a, a]$,

$$\mathcal{F}\left[S_a(x)\right](\omega) = \frac{\sin a\omega}{\pi\omega}$$

we can conclude that

$$\left\|\frac{\sin a\omega}{\pi\omega}\right\|^2 = \frac{1}{2\pi}\|S_a(x)\|^2.$$

The right side is easy to compute, it's simply

$$\frac{1}{2\pi}\int_{-a}^{a} 1\,dx = \frac{a}{\pi}.$$

But we learn something interesting by comparing this to the left side:

$$\int_{-\infty}^{\infty} \frac{\sin^2 a\omega}{\pi^2\omega^2}\,d\omega = \frac{a}{\pi}$$

or

$$\int_{-\infty}^{\infty} \frac{\sin^2 a\omega}{\omega^2}\,d\omega = \pi a,$$

which is certainly something we didn't know before.

Now, back to the linear algebra. We ask a curious question, related to the observation we made on page 6: What happens if you take the Fourier transform of the Fourier transform of a function? Let's see:

$$\mathcal{F}\left[\mathcal{F}\left[f(x)\right]\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(z)e^{i\omega z}\,dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(z)e^{-i(-\omega)z}\,dz = \frac{1}{2\pi}f(-\omega),$$

since the last integral is the inverse Fourier transform evaluated at $-\omega$. But this shows that doing the Fourier transform twice to a function gives back its "reverse" multiplied by $1/(2\pi)$. So if we do the Fourier transform *four times* to a function we should get $1/(4\pi^2)$ times the reverse of the reverse, which is the original function. In other words,

$$\mathcal{F}^4[f(x)] = \frac{1}{4\pi^2}f(x).$$

We could write this as an equation about the Fourier transform operator:

$$\mathcal{F}^4 - \frac{1}{4\pi^2}I = 0,$$

where I is the identity operator. Knowing this, we can see that if f is *any* eigenfunction of $\mathcal{F}$ with eigenvalue λ, then

$$\mathcal{F}^4[f] - \frac{1}{4\pi^2} I[f] = \left(\lambda^4 - \frac{1}{4\pi^2}\right) f = 0,$$

which means that *all* of the eigenvalues of the Fourier transform must satisfy the equation

$$\lambda^4 - \frac{1}{4\pi^2} = 0,$$

so the only possible eigenvalues of $\mathcal{F}$ are

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{i}{\sqrt{2\pi}}, \quad \frac{-1}{\sqrt{2\pi}} \quad \text{and} \quad \frac{-i}{\sqrt{2\pi}}.$$

We already have that the Gaussian $e^{-x^2/2}$ is an eigenfunction of $\mathcal{F}$ with eigenvalue $1/\sqrt{2\pi}$. Now we'll find eigenfunctions for the other eigenvalues.

We have two Fourier transform rules that look quite similar:

$$\widehat{\frac{df}{dx}}(\omega) = -i\omega \widehat{f}(\omega) \quad \text{and} \quad \widehat{xf(x)}(\omega) = -i\frac{d\widehat{f}}{d\omega}.$$

So the Fourier transforms of each of the operations "multiply by the variable" and "take the derivative" on the x-side is the other operation on the ω-side (well, there are factors of i to keep track of, and we will). Because of this, the sum and difference of these two operators have a special relationship with the Fourier transform operator:

$$\mathcal{F}\left[\frac{df}{dx} + xf(x)\right] = -i\omega\mathcal{F}[f](\omega) - i\frac{d\mathcal{F}[f]}{d\omega} = -i\left(\frac{d\mathcal{F}[f]}{d\omega} + \omega\mathcal{F}[f](\omega)\right)$$

and

$$\mathcal{F}\left[\frac{df}{dx} - xf(x)\right] = -i\omega\mathcal{F}[f](\omega) + i\frac{d\mathcal{F}[f]}{d\omega} = i\left(\frac{d\mathcal{F}[f]}{d\omega} - \omega\mathcal{F}[f](\omega)\right).$$

These equations show that if $f(x)$ is an eigenfunction of the Fourier transform operator with eigenvalue λ then $f'(x) + xf(x)$ will be an eigenfunction of the Fourier transform with eigenvalue $-i\lambda$ (unless $f'(x) + xf(x) = 0$), and $f'(x) - xf(x)$ will be an eigenfunction of the Fourier transform with eigenvalue $i\lambda$ (unless $f'(x) - xf(x) = 0$).

Let's try this with the eigenfunction we know, namely $f(x) = e^{-x^2/2}$. Unfortunately, $f' + xf = 0$, but

$$f'(x) - xf(x) = -xe^{-x^2/2} - xe^{-x^2/2} = -2xe^{-x^2/2},$$

so $-2xe^{-x^2/2}$ is an eigenfunction of the Fourier transform with eigenvalue $i/\sqrt{2\pi}$:

$$\mathcal{F}\left[2xe^{-x^2/2}\right](\omega) = \frac{i}{\sqrt{2\pi}}(-2\omega e^{-\omega^2/2}).$$

And we can keep going: If we set $f_1(x) = -2xe^{-x^2/2}$, then

$$f_1' + xf_1 = -2e^{-x^2/2} = -2f(x),$$

so we get nothing new here. But

$$f_1' - xf_1 = (4x^2 - 2)e^{-x^2/2},$$

so we call $f_2(x) = (4x^2 - 2)e^{-x^2/2}$ and we have

$$\mathcal{F}\left[(4x^2 - 2)e^{-x^2/2}\right] = \frac{-1}{\sqrt{2\pi}}(4\omega^2 - 2)e^{-\omega^2/2}.$$

We can continue in this way and obtain an infinite sequence of eigenfunctions of the Fourier transform, starting from $f_0(x) = e^{-x^2/2}$, where

$$f_{n+1}(x) = f_n'(x) - xf_n(x),$$

and the eigenvalue of f_n will be $i^n/\sqrt{2\pi}$. There will be more to say about this in the homework!