Solution outlines for Stovall midterm 3

1. $y' - 2y = x$ is a linear equation with $P = -2x$ and $Q = x$, so $\int P = -x^2$. Therefore

$$y = e^{-\int P} \int Q\, e^{\int P} = e^{x^2} \int x e^{-x^2}\, dx = e^{x^2} \left(-\frac{1}{2} e^{-x^2} + C \right) = -\frac{1}{2} + C e^{x^2}$$

(D)

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2. Since $\lim_{n\to\infty} \dfrac{n^2}{1+n^3} = 0$, we have $\lim_{n\to\infty} \dfrac{(-1)^n n^2}{1+n^3} = 0$ as well
(A)

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3. Can use the integral test (with the substitution $u = 1 + x^2$):

$$\int_1^\infty \frac{x}{\sqrt{1+x^2}}, dx = \frac{1}{2} \int_2^\infty u^{-1/2}\, du = \sqrt{u}\Big|_2^\infty = \infty.$$

Since the integral diverges (and is infinite), the series diverges and the sum is infinite.
(B)

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4. For sufficiently large n, we know that $\ln n < n^{0.1}$ Therefore

$$\sum \left(\frac{\ln n}{n} \right)^2 < \sum \frac{(n^{0.1})^2}{n^2} = \sum \frac{1}{n^{1.8}}$$

which is a convergent p-series with $p = 1.8 > 1$. So the original series converges by the direct comparison test.

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5. I. $\sum \dfrac{3 + \cos n}{3^n} < \sum \dfrac{4}{3^n}$, which is a convergent geometric series with $r = \frac{1}{3} < 1$. So the original series converges by direct comparison.

II. Use the ratio test (and divide the numerator and denominator by $6^n = 2^n \cdot 3^n$):

$$\lim_{n\to\infty} \frac{1+2^{n+1}}{1+3^{n+1}} \cdot \frac{1+3^n}{1+2^n} = \lim_{n\to\infty} \frac{(1+2^{n+1})\frac{1}{2^n}(1+3^n)\frac{1}{3^n}}{(1+3^{n+1})\frac{1}{3^n}(1+2^n)\frac{1}{2^n}} = \lim_{n\to\infty} \frac{\left(\frac{1}{2^n}+2\right)\left(\frac{1}{3^n}+1\right)}{\left(\frac{1}{3^n}+3\right)\left(\frac{1}{2}^n+1\right)} = \frac{2}{3} < 1$$

therefore the series converges.

So both series converge.

(D)

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6. I. This series is not alternating (so can't be conditionally convergent). Ratio test:

$$\lim_{n\to\infty} \frac{(n+3)(n+2)}{10^{n+1}} \cdot \frac{10^n}{(n+2)(n+1)} = \frac{1}{10} < 1$$

so this series converges absolutely.

II. Since $\dfrac{n}{n^2+4}$ is decreasing and its limit as $n \to \infty$ is zero, the series converges by the alternating series test. But the integral:

$$\int_1^\infty \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4)\Big|_1^\infty = \infty$$

so the series $\sum \dfrac{n}{n^2+4}$ of absolute values diverges and so the original series converges conditionally.

(E)

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7. First use the ratio test:

$$\lim_{n\to\infty} \left| \frac{3^{n+1}x^{n+1}}{(n+2)^2} \cdot \frac{(n+1)^2}{3^n x^n} \right| = 3|x|$$

For the ratio test to guarantee convergence we need $3|x| < 1$, so $-\frac{1}{3} < x < \frac{1}{3}$.

Next check the endpoints: At $x = \frac{1}{3}$ the series becomes $\sum \dfrac{1}{(n+1)^2} < \sum \dfrac{1}{n^2}$ (a convergent p-series), so the series converges at $x = \frac{1}{3}$. Likewise at $x = -\frac{1}{3}$, the same test shows the series converges absolutely.

So the interval of convergence is $[-\frac{1}{3}, \frac{1}{3}]$ and the radius of convergence is $r = \frac{1}{3}$

(B)

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8. Since $\cos Z = \displaystyle\sum_{n=0}^{\infty} \frac{-1)^n Z^{2n}}{(2n)!}$ with infinite radius of convergence, we can replace Z by x^3 to get

$$\cos x^3 = \sum_{n=0}^{\infty} \frac{-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{-1)^n x^{6n}}{(2n)!}$$

again with infinite radius of convergence.

(C)