Solution outlines for Stovall midterm 2

1. Use the trig identity to replace $\sin x \cos x$ with $\frac{1}{2}\sin 2x$. Then integrate by parts with $u = \frac{1}{2}x$ and $dv = \sin 2x\,dx$ and get

$$\begin{aligned}\int x \sin x \cos x\,dx &= \frac{1}{2}\int x \sin 2x\,dx = -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C\\ &= -\frac{1}{4}x(1-2\sin^2 x) + \frac{1}{8}\sin 2x + C = \frac{1}{2}x\sin^2 x - \frac{1}{4}x + \frac{1}{8}\sin 2x + C\end{aligned}$$

(B)

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2. First do long division (to make the fraction proper) and then partial fractions:

$$\begin{aligned}\int \frac{x^3+1}{x^3-x}\,dx &= \int 1 - \frac{x+1}{x(x-1)(x+1)}\,dx = \int 1 - \frac{1}{x(x-1)}\,dx\\ &= \int 1 - \frac{1}{x} + \frac{1}{x-1}\,dx = x - \ln|x| + \ln|x-1| + C\end{aligned}$$

(F)

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3. Trig substitution: From the form $4-v^2$ we expect to use the identity $1-\sin^2\theta = \cos^2\theta$. Multiply this by 4 to get $4-4\sin^2\theta = 4\cos^2\theta$. This leads us to want to replace v^2 by $4\sin^2\theta$, leading to the substitution $v = 2\sin\theta$ (and so $dv = 2\cos\theta\,d\theta$). This gives us

$$\begin{aligned}\int \frac{8\,dv}{v^2\sqrt{4-v^2}} &= \int \frac{16\cos\theta\,d\theta}{4\sin^2\theta\,2\cos\theta} = \int 2\csc^2\theta\,d\theta\\ &= -2\cot\theta + C = -2\,\frac{\sqrt{4-v^2}}{v} + C\end{aligned}$$

(since $\sin\theta = v/2$ and using a triangle).
(A)

4.

$$\int \sec^2 x\,\sin^3 x\,dx = \int \frac{\sin^3 x}{\cos^2 x}\,dx = \int \frac{\cos^2 x - 1}{\cos^2 x}(-\sin x)\,dx$$

Making the substitution $u = \cos x$ (so $du = -\sin x\, dx$) transforms this integral into

$$\int \frac{u^2 - 1}{u^2}\, du = \int 1 - \frac{1}{u^2}\, du = u + \frac{1}{u} + C = \cos x + \sec x + C.$$

This is (E), since

$$\frac{\sin^2 x}{\cos x} + 2\cos x = \frac{1 - \cos^2 x}{\cos x} + 2\cos x = \sec x - \cos x + 2\cos x = \cos x + \sec x.$$

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5. This is a separable equation, since $e^{-x-y-2} = e^{-y}e^{-x-2}$. So rewrite it as $e^y\, dy = e^{-x-2}\, dx$ and integrate both sides to obtain

$$e^y = -e^{-x-2} + C$$

Now use $y(0) = -2$ to get the equation $e^{-2} = -e^{-2} + C$, which implies $C = 2e^{-2}$. Therefore

$$e^y = -e^{-x-2} + 2e^{-2}$$

or

$$y = \ln(-e^{-x-2} + 2e^{-2})$$

(F)

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6. According to the formula in the box at the bottom of page 498 of the textbook, we can bound the error for Simpson's rule by

$$|E| < \frac{M(b-a)^5}{180n^4}$$

where $M = \max |f''''|$ on $[a, b]$, $n =$ the number of intervals, and the integral is $\int_a^b f(x)\, dx$.

So for $\int_1^2 f(x)\, dx$ where $M \leq 3$, we have $E < \frac{3(1)^5}{180n^4} = \frac{1}{60n^4}$.

If we want $E < 10^{-5}$ we need $\frac{1}{60n^4} < 10^{-5}$, in other words $\frac{10^5}{60} < n^4$ In other words

$$n > \frac{10}{\sqrt[4]{6}} \approx 6.38$$

We need n bigger than this and *even*, so $n = 8$.
(D)

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7. Integrate by parts with $u = x$ and $dv = e^{3x}\,dx$:

$$\int_{-\infty}^{0} xe^{3x}\,dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\Big|_{-\infty}^{0} = -\frac{1}{9}$$

since the limits of e^{3x} and the limit of xe^{3x} are both zero as $x \to -\infty$.
(H)

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8. For this to be a probability distribution, we need $C > 0$ and

$$1 = \int_0^5 Cx\sqrt{25 - x^2}\,dx = -\frac{C}{2}\int_{25}^{0}\sqrt{u}\,du = -\frac{C}{2}\,\frac{2}{3}\,u^{3/2}\Big|_{25}^{0} = \frac{125}{3}\,C$$

(using the substitution $u = 25 - x^2$). Therefore $C = \dfrac{3}{125}$.
(C)