Solution outlines for Stovall midterm 1

1. With pyramid oriented as in the picture, the volume of a slice at x of width dx is the area of a square with side $\dfrac{Lx}{h}$ times thickness dx. So the total volume is

$$V = \int_0^h \left(\frac{Lx}{h}\right)^2 dx = \frac{L^2}{h^2}\frac{x^3}{3}\bigg|_0^h = \frac{L^2h^3}{3h^2} = \frac{L^2h}{3}.$$

(H)

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2. Rotate a vertical section at x around x-axis and obtain a disk with radius e^x and thickness dx, so volume of the disk is $\pi e^{2x}\,dx$. So the total volume is

$$V = \int_0^2 \pi e^{2x}\,dx = \frac{\pi}{2}e^{2x}\bigg|_0^2 = \frac{\pi}{2}(e^4 - 1)$$

(B)

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3. There is a discontinuity, but you can find the volume in pieces. The left half (from 0 to 1) is just a cylinder with height 1 and radius $\frac{1}{2}$, so its volume is $\pi r^2 h = \pi/4$.

For the right half, use vertical sections and get disks with radius $x^2 - 2x + 2$ and width dx so the volume of the right half is

$$\begin{aligned}
V_2 &= \pi\int_1^2 (x^2 - 2x + 2)^2\,dx = \pi\int_1^2 x^4 - 4x^3 + 8x^2 - 8x + 4\,dx \\
&= \pi\left(\frac{x^5}{5} - x^4 + \frac{8x^3}{3} - 4x^2 + 4x\right)\bigg|_1^2 \\
&= \pi\left(\frac{31}{5} - 15 + \frac{56}{3} - 12 + 4\right) = \frac{28}{15}\pi
\end{aligned}$$

So the total volume is

$$\pi\left(\frac{28}{15} + \frac{1}{4}\right) = \frac{127}{60}\pi.$$

(A)

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4. Rotate vertical section around line $x = 2$ and get a shell with radius $2 - x$, height $2x - 2x^2$ and thickness dx. So the volume is

$$\begin{aligned} V &= 2\pi \int_0^1 (2-x)(2x-2x^2)\,dx = 2\pi \int_0^1 4x - 6x^2 + 2x^3\,dx \\ &= 2\pi \left(2x^2 - 2x^3 + \frac{x^4}{2} \right) \Big|_0^1 = \pi \end{aligned}$$

(C)

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5. Not one of those, alas. We need $ds = \sqrt{1 + (y')^2}\,dx$:

$$y' = \frac{2\sqrt{3}}{9} \cdot \frac{3}{2}(3x^2+1)^{1/2} \cdot 6x = 2\sqrt{3}\,x(3x^2+1)^{1/2}$$

so

$$(y')^2 = 12x^2(3x^2+1) = 36x^4 + 12x^2.$$

And so $1 + (y')^2 = 36x^4 + 12x^2 + 1 = (6x^2+1)^2$, therefore $ds = (6x^2+1)\,dx$ and

$$L = \int_{-1}^2 6x^2 + 1\,dx = 2x^3 + x\Big|_{-1}^2 = 21$$

(E)

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6. Surface area around x axis is $\displaystyle\int 2\pi y\,ds = \int 2\pi y \sqrt{1+(y')^2}\,dx$. We have

$$y = \sqrt{4x+4} = 2\sqrt{1+x} \qquad \text{so} \qquad y' \frac{1}{\sqrt{1+x}} \qquad \text{and} \qquad (y')^2 = \frac{1}{1+x}.$$

Therefore

$$1 + (y')^2 = 1 + \frac{1}{1+x} = \frac{x+2}{x+1}$$

and

$$SA = \int_0^8 2\pi\, 2\sqrt{1+x}\sqrt{\frac{x+2}{x+1}}\,dx = 4\pi \int_0^8 \sqrt{x+2}\,dx = \frac{8\pi}{3}(x+2)^{3/2}\Big|_0^8 = \frac{8\pi}{3}(10\sqrt{10} - 2\sqrt{2})$$

(A)

7. From 0 to $\pi/4$ the cosine curve is on top, and from $\pi/4$ to $\pi/2$ the sine curve is on top. But there is an obvious bilateral symmetry, so the total area is

$$A = 2\int_0^{\pi/4} \cos x - \sin x \, dx = 2(\sin x + \cos x)\Big|_0^{\pi/4} = 2(\sqrt{2} - 1).$$

(E)

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8. Since the density is constant, we can ignore it for the purposes of finding the center of mass. It seems easiest to do the centers of masses of the semicircle and the square separately and then treat them as points with different masses (according to their areas). The square is easy: $\overline{y}_1 = -1$ and $A_1 = 4$.

For the semicircle, we have $A_2 = \pi/2$, so

$$\overline{y}_2 = \frac{2}{\pi}\int_{-1}^{2} \frac{1}{2}(1 - x^2)\, dx = \frac{1}{\pi}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{1}{\pi}\left(2 - \frac{2}{3}\right) = \frac{4}{3\pi}.$$

So the center of mass of the whole thing has y-coordinate

$$\overline{y} = \frac{A_1\overline{y}_1 + A_2\overline{y}_2}{A_1 + A_2} = \frac{4(-1) + \frac{\pi}{2}\cdot\frac{4}{3\pi}}{4 + \frac{\pi}{2}} = \frac{-4 + \frac{2}{3}}{4 + \frac{\pi}{2}} = \frac{-20}{3(\pi + 8)}$$

(A)