

**MATH 104 – Practice Problems for Final Exam - Selected (so far) Hints
and answers**

1(a) The two curves intersect at $(0,0)$ and at $(\frac{1}{2}, 3)$. And $3 \sin(\pi x) > 6x$ for $0 < x < 1/2$, so the area is given by the (elementary) integral

$$\int_0^{1/2} 3 \sin(\pi x) - 6x \, dx = \frac{3}{4} \frac{4 - \pi}{\pi}.$$

1(b) This is a trig substitution integral, let $x = 2 \tan \theta$. You'll need a triangle. The result is $\ln((3 + \sqrt{13})/2)$.

1(c) This is easy – integrate $\sqrt{x} - x$ from 0 to 1.

2(a) To do this one, compute $\pi \int_0^{1/2} (3 \sin(\pi x))^2 - (6x)^2 \, dx$, which is $3\pi/4$.

2(b) To do this one, compute $2\pi \int_0^{1/2} x(3 \sin(\pi x) - 6x) \, dx$, which is $6/\pi - \pi/2$.

2(c) To do this one, compute $\pi \int_0^3 \frac{1}{x^2 + 4} \, dx$, which is $\frac{\pi}{2} \arctan(\frac{3}{2})$.

2(d) To do this one, compute $2\pi \int_0^3 \frac{x}{\sqrt{x^2 + 4}} \, dx$, which is $2\pi(\sqrt{13} - 2)$.

2(e) Washers: integrate $\pi(x - x^4)$ from 0 to 1.

2(f) Shells: integrate $2\pi x(\sqrt{x} - x^2)$ from 0 to 1. Why is the answer the same as the previous problem?

2(g) Shells: integrate $2\pi(1 - x)(\sqrt{x} - x^2)$ from 0 to 1.

2(h) Washers: integrate $\pi((\sqrt{x} + 1)^2 - (x^2 + 1)^2)$ from 0 to 1.

3(a) Answer: $(\frac{\sqrt{5}}{2} + \frac{1}{8} \ln(9 + 4\sqrt{5}))$.

3(b) Answer: $\frac{58\sqrt{58}}{27} - \frac{13\sqrt{13}}{27}$.

3(c) Answer: infinite (the graph goes to negative infinity at $x = 0$).

3(d) Answer: $\ln(2 + \sqrt{3}) - \frac{1}{2} \ln(2) + \ln(2 - \sqrt{2})$

3(e) Answer: $\frac{1}{2} \ln(2 + \sqrt{2}) - \frac{1}{2} \ln(2 - \sqrt{2}) + \sqrt{1 + e^2} - \sqrt{2} + \frac{1}{2} \ln(\sqrt{1 + e^2} - 1) - \frac{1}{2} \ln(\sqrt{1 + e^2} + 1)$.

4(a) Answer: $\frac{\pi}{6}(5\sqrt{5} - 1)$.

4(b) Answer: $\frac{\pi}{64}(36\sqrt{5} - \ln(9 + 4\sqrt{5}))$.

4(e) Answer: $\pi(e\sqrt{1+e^2} - \sqrt{2} + \ln(e + \sqrt{1+e^2}) - \ln(1 + \sqrt{2}))$.

5(a) This is integration by parts, let $u = \ln(2x)$ and $dv = x^4 dx$. The answer is $\frac{1}{5}x^5 \ln(2x) - \frac{1}{25}x^5 + C$.

5(b) Parts again, this time $u = x^2$ and $dv = \cos(3x) dx$. You'll have to do parts twice, or else use tabular integration. The result is $\frac{1}{3}x^2 \sin(3x) + \frac{2}{9}x \cos(3x) - \frac{2}{27} \sin(3x) + C$.

5(c) Partial fractions – the denominator factors as $(x+3)(x+5)$ and the result is $\frac{3}{2} \ln(x+5) - \frac{1}{2} \ln(x+3) + C$.

5(d) Trig substitution – since $4x^2$ should equal $\sin^2 \theta$, let $x = \frac{1}{2} \sin \theta$. The resulting integral has $\cos^2 \theta$, and so you'll have to use the trig identity for that. The answer is $\frac{1}{2}x\sqrt{1-4x^2} + \frac{1}{4} \arcsin(2x) + C$. (You could probably also do this by parts, letting $u = \sqrt{1-4x^2}$ and $dv = dx$.)

5(e) Start with a desperation substitution $u = \sqrt{x}$, or $x = u^2$ with $dx = 2u du$, so the integral becomes

$$\int \frac{2u}{u-1} du.$$

Divide it out, then it's easy. The result is $2\sqrt{x} + 2 \ln(\sqrt{x} - 1) + C$.

5(f) First let $u = \ln x$, then you have to integrate $\sin^2 u$ using the trig identity. The result is $\frac{1}{2} \ln x - \frac{1}{4} \sin(2 \ln x) + C$.

5(g) Lots of substitutions – $u = \ln x$ changes it to the integral of $\sec^u / \sqrt{1 - \tan u}$. Then let $v = \tan u$ to get the integral of $1/\sqrt{1-u}$. Then it's not too bad. The result is $-2\sqrt{1 - \tan(\ln x)} + C$.

6(a) Use the identity $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$, multiply out, and use the identity again.

6(b) Substitute $x = 4 \tan \theta$, the answer is $\sqrt{x^2 + 16} + 2 \ln(\sqrt{x^2 + 16} - 4) - 2 \ln(\sqrt{x^2 + 16} + 4) + C$, I think

6(c) Substitute $e^t = 2 \tan \theta$, the answer is $\frac{1}{2} \arctan(\frac{1}{2}e^t) + C$.

6(d) Start with $e^x = \sin \theta$.

6(e) First let $u = \sqrt{x}$, then integrate by parts. Get $2 \cos \sqrt{x} + 2\sqrt{x} \sin \sqrt{x} + C$.

7(a) Answer: 1

7(b) Answer: $\frac{2}{3} \ln 2$

7(c) Answer: $\pi/2$.

8(a) Answer: $k = -3$, $E = \frac{1}{3}$, median = $\frac{1}{3} \ln 2$, $P(x > 1) = e^{-3}$.

8(b) Answer: $k = 1/\pi$, E does not exist, median = 0, $P(x > 1) = \frac{1}{4}$.

8(c) Answer: $k = 1/(6 \ln 2 - 2)$, $E = (2 \ln 2 + 1)/(6 \ln 2 - 2)$, median ≈ 1.15124 , $P(x > 1) = (8 \ln 2 - 3 \ln 3 - 1)/(6 \ln 2 - 2)$.

9(a) The general solution of the equation is $y = 2 + Ce^{-x}$. For $y(0) = 1$, you need $C = -1$, so the solution of the problem is $y = 2 - e^{-x}$.

9(b) Answer: $y = 1 - e^{-(x^3+3x)/3}$.

9(c) Answer: $y = 4e^{x^2/2} - 1$.

10(a) Answer: $y = 2^x$

10(b) Answer: $y = 2/(2 - x)$

10(c) Answer: $y = (3 - 2\sqrt{2})x^2 + (2\sqrt{2} - 2)x + 1$

10(d) Answer: $y = 2(\frac{3}{2})^x - 1$.

11(a) Answer: $\frac{3}{2}$

11(b) Answer: e^{-1}

11(c) Answer: 5

12(a) Converges by the ratio test (the ratio is $1/3$).

12(b) Diverges by limit comparison to the harmonic series.

12(c) Converges by (limit) comparison to the sum of $1/n^2$.

12(d) Diverges – the denominator approaches 1 (since e^{-n} approaches zero), but the numerator doesn't approach anything (let alone zero), so it fails the test for divergence (n th term test).

12(e) Converges by the ratio test – the ratio is 0.

13(a), (c) and (e) – since these series converged without $(-1)^n$, their corresponding alternating versions converge absolutely.

13(b) converges conditionally, since these terms approach zero and we already know that the series of absolute values diverges.

13(d) Still diverges, since the terms don't approach zero.

14(a) Converges by the integral test (same integral as in 7(a) – after a substitution $u = \ln x$ you're integrating $1/u^2$, which converges).

14(b) Since $\ln(n!) = \ln 1 + \ln 2 + \cdots + \ln n < 1 + 2 + \cdots + n < n^2$, this series converges by (limit) comparison with the sum of $1/n^2$.

14(c) For large n , since $1/n \approx 0$, we have $\tan(1/n) \approx 1/n$. So this series can be limit-compared to the sum of $1/(n \ln n)$, which diverges by the integral test.

14(d) The ratio test works to show this one converges (the limit is zero).

15(a), (b) and (d) converge absolutely as in problem 13.

15(c) Converges conditionally, since the terms decrease and approach zero (alternating series test), but the series of absolute values diverged.

16(a) Ratio test gives convergence for $-3 < x < -1$, and at $x = -3$ get the (conditionally convergent) alternating harmonic series, and at $x = -1$ get the (divergent) harmonic series, so interval of convergence is $[-3, 1)$.

16(b) Ratio test gives convergence for $2 < x < 4$, and the series converges (absolutely) at both endpoints by integral test or comparison with $\sum 1/n^2$. so interval of convergence is $[2, 4]$.

16(c) Ratio test gives convergence for $1 - e < x < 1 + e$, and at $x = 1 + e$ get the alternating harmonic series, at $x = 1 - e$ get the plain harmonic series, so the interval of convergence is $(1 - e, 1 + e]$.

16(d) Ratio test gives convergence for $-1 < x < 1$, and at the endpoints, the terms don't approach zero, so the interval of convergence is $(-1, 1)$.

17(a) The series in 16(a) is related to the geometric series. Since the integral of $(x+2)^{n-1}$ is $(x+2)^n/n$, the whole series is the integral of $\sum (x+2)^{n-1}$, which is geometric with first term 1 and ratio $(x+2)$. So the entire series represents the integral of $1/(1 - (x+2)) = 1/(-1 - x)$, which is $-\ln(-1 - x)$.

17(c) Answer: $-\ln\left(1 + \frac{x-1}{e}\right)$.

17(d) Answer: $\frac{x^2 + x}{(1-x)^3}$.

18(a) Substitute $-x^2$ into the series for e^x and get

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

18(b) As in part (a), the series for $\cos t^3$ is

$$\cos t^3 = \sum_{n=0}^{\infty} \frac{(-1)^n t^{6n}}{(2n)!}.$$

Then integrate this from 0 to x and get

$$\int_0^x \cos t^3 dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!(6n+1)}.$$

18(c) Answer: $\sqrt{x} = 1 + \frac{x-1}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdots (2n-3)}{2^n n!} (x-1)^n$

18(d) Answer: $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}.$

19(a) Since

$$\cos \sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!},$$

the integral is

$$\sum_{n=0}^{\infty} \frac{(-1)^n (0.2)^{n+1}}{(2n)!(n+1)} = 0.2 - \frac{(0.2)^2}{2! \cdot 2} + \frac{(0.2)^3}{4! \cdot 3} - \dots$$

Since this is an alternating series, and the last term shown here is less than 0.001, we can stop after the first two terms, to get

$$\int_0^{0.2} \cos \sqrt{x} \, dx \approx 0.2 - \frac{0.04}{4} = 0.190$$

to within 0.001.

20. The Maclaurin series for $x^3 \cos x^2$ is $x^3 - \frac{x^7}{2!} + \frac{x^{11}}{4!} - \frac{x^{15}}{6!} - \dots$. Since there's no x^{13} term, $f^{(13)}(0) = 0$.