Rimmer MT 4 

#1 I. $\displaystyle\sum_{n=1}^{\infty} \frac{n(n+3)}{(n+1)(n+5)(n+11)}$ limit comparison with $\sum \frac{1}{n}$ (harmonic, divergent)

$$\lim_{n\to\infty} \frac{n(n+3)}{(n+1)(n+5)(n+11)} \cdot \frac{n}{1} = 1 \Rightarrow \text{series diverges.}$$

II. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ ROOT TEST(!!)

$$\lim_{n\to\infty} \left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$ (Famous limit #6).

$\frac{1}{e} < 1$ so series converges

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#2. I. $\displaystyle\sum_{n=2}^{\infty} \frac{(2n)!}{3^{n^2}}$ RATIO TEST.

$$\lim_{n\to\infty} \frac{(2(n+1))!}{3^{(n+1)^2}} \cdot \frac{3^{n^2}}{(2n)!} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{3^{2n+1}} = 0 < 1$$

so series converges

II. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{4/3}}$ INTEGRAL TEST

$$\int_2^{\infty} \frac{1}{x(\ln x)^{4/3}}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^{4/3}}\,du = -\frac{3}{u^{1/3}}\Bigg|_{\ln 2}^{\infty}$$ converges

$u = \ln x$

$du = \frac{1}{x}\,dx$

so series converges

#3. I. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^{e/\pi}}$ terms $\frac{1}{n^{e/\pi}}$ are decreasing to zero since $e/\pi > 0$, so series converges by alternating series test.

And $e<\pi$ so $\sum \frac{1}{n^{e/\pi}}$ diverges as a p-series with $p = \frac{e}{\pi} < 1$

So original series converges conditionally

II. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{(\ln 2)^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{\ln 2}\left(-\frac{3}{\ln 2}\right)^n$ is a divergent geometric series since $|r| = \left|\frac{-3}{\ln 2}\right| > 1$.

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#4. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)(3x-1)^n}{3^n n^3}$

RATIO TEST: $\displaystyle\lim_{n\to\infty}\left|\frac{(n+4)(3x-1)^{n+1}}{3^{n+1}(n+1)^3}\cdot\frac{3^n n^3}{(n+3)(3x-1)^n}\right| = \lim_{n\to\infty}\left(\frac{n+4}{n+3}\right)\left(\frac{n}{n+1}\right)^3\frac{|3x-1|}{3}$

$= \dfrac{|3x-1|}{3} < 1$ means $-1 < \dfrac{3x-1}{3} < 1$

$-3 < 3x-1 < 3$

$-2 < 3x < 4$

$-\frac{2}{3} < x < \frac{4}{3}$

Check endpoints:

$x = \frac{4}{3}$: $\displaystyle\sum_{n=1}^{\infty}\frac{(n+3)3^n}{3^n n^3} = \sum_{n=1}^{\infty}\frac{(n+3)}{n^3}$

converges by limit comp. with $\sum\frac{1}{n^2}$.

$x = -\frac{2}{3}$ $\displaystyle\sum_{n=1}^{\infty}\frac{(n+3)(-3)^n}{3^n n^3} = \sum_{n=1}^{\infty}\frac{(-1)^n(n+3)}{n^3}$

(absolutely) convergent alternating series

$\Rightarrow$ interval of convergence is $-\frac{2}{3} \le x \le \frac{4}{3}$ B.

#5. $@\ x=4$

$$f(x)=\sqrt{x^2+9} \qquad 5$$

$$f'(x)=\frac{x}{\sqrt{x^2+9}} \qquad \frac{4}{5}$$

$$f''(x)=\frac{\sqrt{x^2+9}-\frac{x^2}{\sqrt{x^2+9}}}{x^2+9} \qquad \frac{5-\frac{16}{5}}{25}=\frac{9}{125}$$

So $$P_2(x)=5+\frac{4}{5}(x-4)+\frac{9}{125}\frac{(x-4)^2}{2!}$$

$$=5+\frac{4}{5}(x-4)+\frac{9}{250}(x-4)^2.$$

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#6. $$f(x)=\frac{xe^x}{\ln(1+x)}=\frac{x\left(1+x+\frac{x^2}{2}+\cdots\right)}{x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots}=\frac{x\left(1+x+\frac{x^2}{2}+\cdots\right)}{x\left(1-\frac{x}{2}+\frac{x^2}{3}-\cdots\right)}$$

(Strictly speaking, $f$ doesn't have a Maclaurin series because it has a (removable) singularity at $x=0$. Define $f(0)=1$ to fix this)

$$=\frac{1+x+\frac{x^2}{2}+\cdots}{1-\frac{x}{2}+\frac{x^2}{3}-\cdots}$$

<u>LONG</u> DIVISION:

$$\begin{array}{r|l} & 1+\frac{3}{2}x+\frac{11}{12}x^2+\cdots \\ \hline 1-\frac{1}{2}x+\frac{1}{3}x^2-\cdots & 1+x+\frac{1}{2}x^2+\cdots \\ & 1-\frac{1}{2}x+\frac{1}{3}x^2+\cdots \\ \hline & \frac{3}{2}x+\frac{1}{6}x^2+\cdots \\ & \frac{3}{2}x-\frac{3}{4}x^2+\cdots \\ \hline & \frac{11}{12}x^2+\cdots \end{array}$$

So $f(x)=1+\frac{3}{2}x+\frac{11}{12}x^2+\cdots$

#7. $L = \sum_{n=1}^{\infty} \frac{n(-1)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} n\left(-\frac{1}{2}\right)^{n-1}$

comes from putting $x = -\frac{1}{2}$ in $\sum_{n=1}^{\infty} n x^{n-1} = \left(\sum_{n=0}^{\infty} x^n\right)' = \frac{d}{dx}\left(\frac{1}{1-x}\right)$

$$= \frac{1}{(1-x)^2}$$

So $L = \frac{1}{(1-(-\frac{1}{2}))^2} = \frac{1}{(\frac{3}{2})^2} = \frac{4}{9}.$

$M = 1 - \ln 9 + \frac{(\ln 9)^2}{2!} - \frac{(\ln 9)^3}{3!} + \cdots$

comes from putting $x = -\ln 9$ in $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

So $M = e^{-\ln 9} = \frac{1}{9}$

$\Rightarrow \frac{L}{M} = \frac{4/9}{1/9} = 4$ (D)