Solution outlines for Rimmer midterm 3

1. We'll make careful use of Top Ten limit number 9:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a.$$

as follows:

$$\lim_{n\to\infty}\left(\frac{n}{n-2}\right)^{n/2} = \lim_{n\to\infty}\left[\left(1+\frac{2}{n-2}\right)^{n-2}\right]^{1/2}\left(\frac{n}{n-2}\right) = \lim_{n\to\infty}(e^2)^{1/2}\cdot 1 = e$$

(E)

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2. Since $3^{n+1}4^{-n} = 3(\frac{3}{4})^n$, the series is geometric with $r = \frac{3}{4} < 1$ and thus convergent. Its first term is $a = \frac{9}{4}$ so the sum is

$$\frac{\frac{9}{4}}{1-\frac{3}{4}} = 9$$

(A)

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3. I. The numerator is $2n^2+3n$ and the denominator is $3n^4+\cdots$. So the series converges by limit comparison with $\sum \frac{1}{n^2}$

II. Since the limit as $n\to\infty$ of $n^{1/n}$ is 1, the series diverges by the nth term test.

III. Use the integral test (and the substitution $u = \ln x$):

$$\int_2^\infty \frac{1}{x(\ln x)^{5/3}} = \int_{\ln 2}^\infty \frac{1}{u^{5/3}}\,du = -\frac{3}{2u^{2/3}}\bigg|_{\ln 2}^{\infty} < \infty.$$

Since the integral converges, so does the series.

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4. I. Since e and π are positive, the series converges by the alternating series test, since $n^{e/\pi}$ increases to infinity. But $e < \pi$ so the series of absolute values is a p-series with $p < 1$, hence divergent. Therefore the original series converges conditionally.

II. Since $\ln n < n^{1/4}$ for n large, we have

$$\sum\frac{\ln n}{n^{7/4}} < \sum\frac{n^{1/4}}{n^{7/4}} = \sum\frac{1}{n^{3/2}}$$

and the latter is a convergent p-series. So the original series converges absolutely.

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5. Use the ratio test:

$$\lim_{n\to\infty} \frac{|x-1|n(\ln n)^2}{(n+1)(\ln(n+1))^2} = |x-1|$$

We need $|x-1| < 1$ for the ratio test to guarantee convergence, i.e, $-1 < x-1 < 1$ or $0 < x < 2$.

Next,we have to check the endpoints. Using the integral test, since

$$\int_*^\infty \frac{dx}{x(\ln x)^2} = \int_*^\infty \frac{1}{u^2}\,dx = -\frac{1}{u}\bigg|_*^\infty < \infty$$

converges, then the series at both endpoints converge absolutely. The interval of convergence is thus $[0,2]$.

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6. The easiest way is to use that we know

$$\ln(1+Z) = Z - \frac{Z^2}{2} + \frac{Z^3}{3} - \cdots$$

Since $x = 1 + (x-1)$, we can substitute $Z = x-1$ in the above to get

$$\ln x = \ln 1 + (x-1) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots$$

Without the $\cdots$ this is $T_3(x)$. So

$$T_3(2) = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{3}$$

(B)

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7. Since we're going to integrate, which increases powers by 1, we only need the series of the integrand up to x^3. Next, since the arctan series begins with x, we just need the terms of e^{2x} up to x^2. So we have

$$e^{2x}\arctan x = \left(1 + 2x + \frac{(2x)^2}{2} + \cdots\right)\left(x - \frac{x^3}{3} + \cdots\right) = x + 2x^2 - \frac{x^3}{3} + 2x^3 + \cdots$$

To get the x^4 term of the integral, we just have to integrate the x^3 term of this:

$$\int \frac{5}{3}x^3\,dx = \frac{5}{12}x^4$$

(we can ignore the "$+C$" for this purpose), so the coefficient is $\frac{5}{12}$.

(H)