

Solution outlines for Rimmer midterm 2

1. The form $\sqrt{1-x^2}$ leads us to the trig substitution $x = \sin \theta$. So

$$\begin{aligned}\int_0^1 15x^3 \sqrt{1-x^2} dx &= \int_0^{\pi/2} 15 \sin^3 \theta \cos^2 \theta d\theta = - \int_1^0 15(1-u^2)u^2 du \\ &= \int_0^1 15(u^2 - u^4) du = 15 \left(\frac{1}{3} - \frac{1}{5} \right) = 5 - 3 = 2\end{aligned}$$

(D)

2. First let $u = x^2$, then do partial fractions:

$$\begin{aligned}\int_3^4 \frac{16x}{x^4 - 16} dx &= \int_9^{16} \frac{8}{u^2 - 16} du = \int_9^{16} \frac{8}{(u-4)(u+4)} du \\ &= \int_9^{16} \frac{1}{u-4} - \frac{1}{u+4} du = \ln \frac{u-4}{u+4} \Big|_9^{16} = \ln \frac{12}{20} - \frac{5}{13} = \ln \frac{3 \cdot 13}{5 \cdot 5} = \ln \frac{39}{25}\end{aligned}$$

(G)

3. First let $u = \sqrt{x}$, and later let $u = 2 \tan \theta$ and get

$$\int_{12}^{\infty} \frac{1}{\sqrt{x}(x+4)} dx = \int_{\sqrt{12}}^{\infty} \frac{2}{u^2+4} du = \int_{\pi/3}^{\pi/2} \frac{4 \sec^2 \theta}{4 \sec^2 \theta} d\theta = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}.$$

(note that $\sqrt{12} = 2\sqrt{3}$ and $\tan \pi/3 = \sqrt{3}$).

(C)

4. If $f(x)$ is to be a probability density, then we need $C > 0$ and

$$\begin{aligned}1 &= \int_4^{\infty} \frac{8C}{(x-2)(x+2)} = \int_4^{\infty} \frac{2C}{x-2} - \frac{2C}{x+2} dx = 2C \ln \frac{x-2}{x+2} \Big|_4^{\infty} \\ &= -2C \ln \frac{1}{3} = 2C \ln \frac{1}{3} = 2C \ln 3 = C \ln 9.\end{aligned}$$

Therefore $C = \frac{1}{\ln 9}$.

(B)

5. Integrate by parts with $u = x$ and $dv = e^{-sx}$ (be careful to treat s as a constant during the integration!):

$$L\{x\} = \int_0^\infty x e^{-sx} dx = -\frac{1}{s} e^{-sx} - \frac{1}{s^2} e^{-sx} \Big|_0^\infty = \frac{1}{s^2}$$

since we know the limits of e^{-sx} and of $x e^{-sx}$ are zero as long as $s > 0$ (or more generally if s has a positive real part).

(A)

6. This is a linear differential equation with $P = \frac{e^x}{1+e^x}$ and $Q = \frac{\sin x}{1+e^x}$. Since the numerator of P is the derivative of its denominator, we have $\int P = \ln(1+e^x)$ and $e^{\int P} = 1+e^x$. Therefore

$$y = e^{-\int P} \int Q e^{\int P} = \frac{1}{1+e^x} \int \sin x dx = \frac{1}{1+e^x} (C - \cos x).$$

Since $y(0) = 1/2$, we can conclude that $C = 2$ and so

$$y = \frac{2 - \cos x}{1 + e^x}.$$

Therefore $y(\pi) = \frac{3}{1+e^\pi}$.

(C)

7. This is a separable equation which can rewrite as $\frac{dy}{y} = \frac{\ln x}{x} dx$. Integrate both sides to get

$$\ln y = \frac{1}{2} (\ln x)^2 + C.$$

Since $y(1) = 4$, we see that $C = \ln 4$, and so when $x = e$, we have

$$\ln y = \frac{1}{2} + \ln 4 \quad \text{or} \quad y = 4\sqrt{e}$$

(F)