Solution outlines for Rimmer midterm 1

1. The area of an equilateral triangle with side s is $A = \frac{\sqrt{3}}{4}s^2$, so the volume of the slice at x perpendicular to the x-axis with thickness dx is

$$\frac{\sqrt{3}}{4}s^2\,dx = \frac{\sqrt{3}}{4}(2\sqrt{x})^2\,dx = \sqrt{3}\,x\,dx.$$

The total volume is thus

$$V = \int_0^{\sqrt{2}} \sqrt{3}\,x\,dx = \frac{\sqrt{3}}{2}\,x^2\Big|_0^{\sqrt{2}} = \sqrt{3}$$

(A)

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2. Rotate the vertical section at x around the line $x = 4$ to get a shell with radius $4-x$, height $4x-2x^2$ and thickness dx. Since the volume of a shell is $2\pi rht$, the total volume will be

$$\begin{aligned} V &= 2\pi \int_0^2 (4-x)(4x-2x^2)\,dx = 2\pi \int_0^2 16x - 12x^2 + 2x^3\,dx \\ &= 2\pi\left(8x^2 - 4x^3 + \frac{x^4}{2}\right)\Big|_0^2 = \pi(64-64+16) = 16\pi \end{aligned}$$

(G)

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3. Rotate the vertical section at x around the x-axis to get a disk with radius $\sec^2 x$ and thickness dx. The volume of a disk is $\pi r^2 t$, so the total volume is

$$\begin{aligned} V &= \pi \int_0^{\pi/4} \sec^4 x\,dx = \pi \int_0^{\pi/4} (1+\tan^2 x)\sec^2 x\,dx = \pi \int_0^1 1+u^2\,du \\ &= \pi\left(u + \frac{u^3}{3}\right)\Big|_0^1 = \frac{4\pi}{3} \end{aligned}$$

(using the substitution $u = \tan x$)
(D)

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4. Not one of those. We need $ds = \sqrt{1+(y')^2}\,dx$:

$$y' = \frac{\sqrt{5}}{15}\cdot\frac{3}{2}(5x^2+2)^{1/2}\cdot 10x = \sqrt{5}\,x(5x^2+2)^{1/2}$$

so

$$(y')^2 = 5x^2(5x^2+2) = 25x^4+10x^2.$$

And so $1+(y')^2 = 25x^4+10x^2+1 = (5x^2+1)^2$, therefore $ds = (5x^2+1)\,dx$ and

$$L = \int_0^3 5x^2+1\,dx = \frac{5x^3}{3}+x\Big|_0^3 = 48$$

(E)

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5. Integrate by parts with $u = \ln x$ and $dv = 3x^{-2}\,dx$ to get

$$\begin{aligned}\int_{\sqrt{e}}^{e} 3x^{-2}\ln x\,dx &= -\frac{3\ln x}{x} - \int \frac{3}{x^2}\,dx = -\frac{3\ln x}{x} - \frac{3}{x}\Big|_{\sqrt{e}}^{e}\\ &= -\frac{3}{e}+\frac{3}{2\sqrt{e}}-\frac{3}{e}+\frac{3}{\sqrt{e}} = -\frac{6}{e}+\frac{9}{2\sqrt{e}}\end{aligned}$$

(H)

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6. Surface area is the integral of $2\pi r\,ds$. Around the y-axis, $r = x$. Since $y' = x^{-2/3}$, we have $1+(y')^2 = x^{-4/3}+1$ and so

$$\begin{aligned}SA &= 2\pi\int x\,dx = 2\pi\int_0^{3^{3/4}} x\sqrt{1+x^{-4/3}}\,dx = 2\pi\int_0^{3^{3/4}} x\cdot x^{-2/3}\sqrt{x^{4/3}+1}\,dx\\ &= 2\pi\int_0^{3^{3/4}} x^{1/3}\sqrt{x^{4/3}+1}\,dx = 2\pi\cdot\frac{3}{4}\int_0^3\sqrt{u+1}\,du = 2\pi\cdot\frac{3}{4}\cdot\frac{2}{3}(u+1)^{3/2}\Big|_0^3\\ &= \pi(4^{3/2}-1^{3/2}) = 7\pi\end{aligned}$$

(using the substitution $u = x^{4/3}$) (B)

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7. Since the density is constant, we can ignore it for the purpose of finding center of mass. There is a clever way to do this problem using symmetry, but we'll do it a more standard way.

We calculate $y_2 - y_1 = 4 + 2x - 2x^2 = -2(x+1)(x-2)$ so the curves intersect at $x = -1$ and $x = 2$ and y_2 is on top.

First we need to find the area:

$$A = \int_{-1}^{2} 4 + 2x - 2x^2 \, dx = \left(4x - x^2 - \frac{2x^3}{3}\right)\Big|_{-1}^{2} = 9.$$

Therefore

$$\overline{x} = \frac{1}{9}\int_{-1}^{2} x(4 - 2x - 2x^2) \, dx = \frac{1}{9}\left(2x^2 + \frac{2x^3}{3} - \frac{x^4}{2}\right)\Big|_{-1}^{2} = \frac{1}{2}$$

(G)