# Improper integrals

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These are a special kind of limit. An *improper integral* is one where either

1. the interval of integration is infinite, or
2. the interval of integration includes a singularity of the function being integrated

or both.

# Improper integrals

**Examples of the first kind are**

$$\int_0^\infty e^{-x}\,dx \qquad \text{and} \qquad \int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$$

**Examples of the second kind are**

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \qquad \text{and} \qquad \int_{\pi/4}^{2\pi/3} \tan x\,dx$$

The second of these is subtle because the singularity of the integrand occurs in the interior of the interval of integration rather than at one of the endpoints.

# Dealing with improper integrals

No matter which kind of improper integral (or combination of improper integrals) we are faced with, the method of dealing with them is the same:

$\int_0^{\infty} e^{-x}\,dx$ means the same thing as $\lim_{b\to\infty} \int_0^b e^{-x}\,dx$

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$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ means the same thing as

$$\lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

# Dealing with improper integrals

No matter which kind of improper integral (or combination of improper integrals) we are faced with, the method of dealing with them is the same:

$\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ means the same thing as $\displaystyle\lim_{a\to 0+}\int_a^1 \frac{1}{\sqrt{x}}\,dx$

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$\displaystyle\int_{\pi/4}^{2\pi/3} \tan x\,dx$ means the same thing as

$$\lim_{b\to\frac{\pi}{2}-}\int_{\pi/4}^{b} \tan x\,dx + \lim_{a\to\frac{\pi}{2}+}\int_{a}^{2\pi/3} \tan x\,dx$$

The limit is basically a way of "sneaking up on the infinity" in the problem.

# OK, let's calculate some improper integrals

What is the value of the limit $\lim_{b\to\infty} \int_0^b e^{-x}\,dx$ and hence of the improper integral $\int_0^\infty e^{-x}\,dx$ ?

A. 0 B. 1 C. $e$ D. $e^{-b}$ E. $\infty$

What is the value of the limit $\lim_{a\to 0+} \int_a^1 \frac{1}{\sqrt{x}}\,dx$ and hence of the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$?

A. 0 B. 1 C. 2 D. $\sqrt{2}$ E. $\infty$

# Convergent and divergent improper integrals

It is possible that the limit used to define an improper integral fails to exist or is infinite. In this case, the improper integral is said to *diverge* (or *be divergent*).

If the limit does exist and is finite, then the improper integral *converges*.

For example, the two integrals you just did both converge.

And

$$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0+} \ln 1 - \ln a = \infty$$

is an example of a divergent improper integral.

# The other examples

Recall that $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$.

If it is convergent, what is the value of $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ ?

A. 0 B. $\frac{\pi}{4}$ C. $\frac{\pi}{2}$ D. $\pi$ E. divergent

If it is convergent, what is the value of $\displaystyle\int_{\pi/4}^{2\pi/3} \tan x\,dx$ ?

A. 0 B. $\frac{1}{2}\ln\frac{3}{2}$ C. $\frac{1}{2}\ln 6$ D. 1 E. divergent

# Using estimation to show convergence or divergence

Sometimes it is possible to show that an improper integral converges or diverges without actually evaluating it:

Since $\dfrac{1}{x^4+x+7} < \dfrac{1}{x^4}$ for all positive values of $x$, we have that

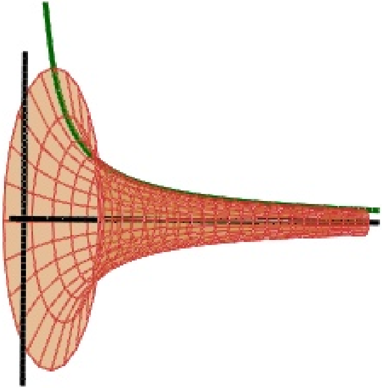
$$\int_1^b \frac{1}{x^4+x+7}\,dx < \int_1^b \frac{1}{x^4}\,dx = \frac{1}{3} - \frac{1}{3b^3}$$

for all $b > 1$. So the limit of the first integral as $b \to \infty$ must be finite because it increases as $b$ does but it is bounded above (by $\frac{1}{3}$).

This shows that the improper integral $\displaystyle\int_1^\infty \frac{1}{x^4+x+7}\,dx$ is convergent.

# A puzzling example

Consider the surface obtained by rotating the graph of $y = \dfrac{1}{x}$ for $x > 1$ around the $x$-axis.



We can calculate the volume of the surface via disks as

$$V = \int_1^\infty \frac{\pi}{x^2}\,dx$$

and

$$V = \int_1^\infty \frac{\pi}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\pi}{x^2}\,dx = \lim_{b\to\infty} \left(-\frac{\pi}{b} + \frac{\pi}{1}\right) = \pi \text{ cubic units}$$

# But what about the surface area?

The surface area is equal to

$$\sigma = \int_1^\infty 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx = \int_1^\infty \frac{2\pi}{x}\sqrt{1+\frac{1}{x^4}}\,dx.$$

This integral is difficult (impossible) to evaluate directly, but it's easy to see that the integrand can be bounded from below by something simpler:

$$\frac{2\pi}{x}\sqrt{1+\frac{1}{x^4}} \geq \frac{2\pi}{x}$$

for all $x \geq 1$.

But the integral $\displaystyle\int_1^\infty \frac{2\pi}{x}\,dx$ is divergent, so the surface has *infinite* surface area.

This surfaces is sometimes called "Gabriel's horn" — it is a surface that can be "filled with water" but not "painted".