More on Integrals and their Applications

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Integration by parts

This is another way to try and integrate products.
In fact, it is the opposite of the product rule for derivatives:

Product rule for derivatives

$$d(uv) = u\,dv + v\,du$$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Use this when you have a product under the integral sign, and it appears that integrating one factor and differentiating the other will make the resulting integral easier.

Example

$$\int x^2 e^x \, dx$$

The is no other rule for this, so we try *parts*.

If we differentiate x^2. we'll get $2x$, which seems simpler.
And integrating e^x doesn't change anything.
So let $u = x^2$ and $dv = e^x \, dx$. Then $du = 2x \, dx$ and $v = e^x$.
The integration by parts formula then gives us:

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx.$$

Example (continued)

$$\int x^2 e^x \, dx = x^2 e^x - \int 2xe^x \, dx.$$

Continuing, we can use parts again on the latter integral (with $u = 2x$ and $dv = e^x \, dx$) to get:

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2xe^x + \int 2e^x \, dx \\ &= x^2 e^x - 2xe^x + 2e^x + C \end{aligned}$$

Now you try one:

$$\int x \sin 2x \, dx \qquad \ldots \text{contrast this with} \qquad \int x \sin x^2 \, dx$$

Tips and tricks

There is a method called the "tic-tac-toe method", or *tabular integration* for repeated integrations by parts.

You must use different strategies for choosing u and dv to integrate $x \ln x$ or $x \arctan x$. Some people find the mnemonic "LIPET" useful – for

Logarithmic, Inverse trig, Polynomials, Exponential, Trigonometric

When you have a product of two of these kinds of functions, the leftmost one should be u, and the rightmost dv.

There is also an algebraic trick for the integral of an exponential times sine or cosine.

We'll explore several examples of this in class.

A couple of problems for you to do:

Find the value of $\int_0^{\pi} x \cos x \, dx$

A. π	B. 2π	C. 2	D. 0
E. -2	F. 1	G. $1/2$	H. $\pi/2$

Evaluate $\int_0^{1} x \ln x \, dx$

A. $-1/4$	B. $-1/2$	C. 0	D. $1/4$	E. $1/2$

Trigonometric substitution integrals

These are just substitutions like the ordinary *u*-substitution method, but some aspects are a little tricky and sometimes surprising. There are two kinds:

The first kind: products of powers of sine and cosine

To integrate products of powers of sine and cosine (such as $\sin^3 x \cos^6 x$), you need the identity:

$$\sin^2 x + \cos^2 x = 1 \qquad \text{(everybody knows that one!)}$$

and the double-angle formulas:

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \text{and} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

For integrals of products of powers of sine and cosine. . .

The trick is as follows:

- If both the power of sine and the power of cosine are **even**, then use the double angle formulas to divide both in half. Keep doing this until at least one of the powers is odd.
- Once the power of at least one of sine or cosine is **odd**, then let $u =$ the **other** function, and use the Pythagorean identity to convert all but one power of the odd-powered function into the u function, and the last power will be the du in a regular substitution.

A couple of examples will help. . .

First example

Evaluate $\int \sin^5 2x \cos^2 2x \, dx$

Since the power of sine is odd, we make the substitution $u = \cos 2x$. Then $du = -2 \sin 2x \, dx$, which uses up one power of $\sin 2x$, and we rewrite the other four as

$$\sin^4 2x = (1 - \cos^2 2x)^2 = (1 - u^2)^2.$$

So the whole integral becomes

$$\int \sin^5 2x \cos^2 2x \, dx = \int (1 - \cos^2 2x)^2 \cos^2 2x \sin 2x \, dx$$
$$= \frac{1}{2} \int -(1 - u^2)^2 u^2 \, du = \frac{1}{2} \int -(u^2 - 2u^4 + u^6) \, du$$
$$= -\frac{1}{2} \left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} \right) + C = -\frac{\cos^3 2x}{6} + \frac{\cos^5 2x}{5} - \frac{\cos^7 2x}{14} + C$$

Here's a nasty one:

Evaluate $\int \sin^4 x \cos^4 x \, dx$

Both powers are even, so we use the double-angle formula trick:

$$\begin{aligned}\sin^4 x \cos^4 x &= (\sin^2 x)^2 (\cos^2 x)^2 \\ &= \left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right)^2 \\ &= \frac{1}{16}\left(1 - 2\cos^2 2x + \cos^4 2x\right)\end{aligned}$$

In the last two terms, the powers are still even so we use the identities again to get what's on the next slide:

Trig identity city!

$$
\begin{aligned}
\frac{1}{16}\left(1 - 2\cos^2 2x \right. & \left. + \cos^4 2x\right) \\
&= \frac{1}{16}\left(1 - (1+\cos 4x) + \left(\frac{1+\cos 4x}{2}\right)^2\right) \\
&= \frac{1}{64}\left(1 - 2\cos 4x + \cos^2 4x\right) \\
&= \frac{1}{64}\left(1 - 2\cos 4x + \frac{1+\cos 8x}{2}\right) \\
&= \frac{1}{128}(3 - 4\cos 4x + \cos 8x)
\end{aligned}
$$

...and (finally!) this is something we can integrate!

At last...

$$
\begin{aligned}
\int \sin^4 x \cos^4 x\, dx &= \frac{1}{128}\int 3 - 4\cos 4x + \cos 8x\, dx \\
&= \frac{1}{128}\left(3x - \sin 4x + \frac{1}{8}\sin 8x\right) + C \\
&= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C
\end{aligned}
$$

Wow. I told you it was a nasty one!

A couple of problems for you to do:

Here's one with an odd power: $\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx$

A. $2/15$	B. $4/15$	C. $2/5$	D. $8/15$
E. $2/3$	F. $4/5$	G. $14/15$	H. 1

Try this one: $\int_0^{\pi} \cos^4 x \, dx$

A. 2	B. π	C. $\pi - \frac{1}{2}$	D. $\sqrt{2}\,\pi$	E. $3\pi/8$

Secants and tangents

A similar trick can be used to integrate products of powers of secants and tangents, with a slight twist:

To integrate $\int \tan^a x \sec^b x \, dx$:

1. If the power of secant is **even**, then let $u = \tan x$, so that $du = \sec^2 x \, dx$, and you can covert the other powers of secant (if any) into tangents using the identity $\sec^2 x = 1 + \tan^2 x$. (If the power of secant is zero, then it usually helps to [perhaps repeatedly] use the identity $\tan^2 x = \sec^2 x - 1$ to create some).
2. If the power of secant is **odd**, then integration by parts is called for — we'll do an example of that next.

Example: $\int \sec^3 x\, dx$

I'd rather integrate $\sec^2 x$ then $\sec x$, so break up the integrand into a product by setting $u = \sec x$ and $dv = \sec^2 x\, dx$ so that $du = \sec x \tan x\, dx$ and $v = \tan x$. Then

$$\begin{aligned}
\int \sec^3 x\, dx &= \sec x \tan x - \int \tan x \sec x \tan x\, dx \\
&= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx \\
&= \sec x \tan x + \int \sec x\, dx - \int sec^3 x\, dx \\
&= \sec x \tan x + \ln(\sec x + \tan x) - \int sec^3 x\, dx
\end{aligned}$$

so we had to integrate $\sec x$ anyhow — but now we can do the "add the integral back to the other side" trick.

Example: $\int \sec^3 x \, dx$

So far we have:

$$\int \sec^3 x \, dx = \sec x \tan x + \ln(\sec x + \tan x) - \int sec^3 x \, dx$$

and if we add the integral of $\sec^3 x \, dx$ to both sides and divide by 2 we can conclude that

$$\int \sec^3 x \, dx = \frac{1}{2} (\sec x \tan x + \ln(\sec x + \tan x)) + C.$$

There are also integrals like $\int \cos 2x \sin 3x \, dx$ and in class we'll show how to use the addition formulas for sine and cosine to handle these.

Trig subsitutions II

The second and more important kind of trig substitution happens when there is a sum or difference of squares in the integrand — usually one of the squares is a constant and the other involves the variable.

If some other substitution doesn't suggest itself first, then try the Pythagorean trigonometric identity that has the same pattern of signs as the one in the problem:

$$\begin{array}{lcl} \text{constant}^2 - \text{expression}^2 & \leftrightarrow & \cos^2 = 1 - \sin^2 \\ \text{constant}^2 + \text{expression}^2 & \leftrightarrow & \sec^2 = 1 + \tan^2 \\ \text{expression}^2 - \text{constant}^2 & \leftrightarrow & \tan^2 = \sec^2 - 1 \end{array}$$

Example: $\int \frac{x^2}{\sqrt{5-x^2}}\,dx$

If you think about it, the substitution $u = 5 - x^2$ won't work, because of the extra factor of x in the numerator.

But $5 - x^2$ is vaguely reminiscent of $1 - \sin^2$, so let

$$x^2 = 5\sin^2\theta.$$

Then

$$x = \sqrt{5}\,\sin\theta \qquad dx = \sqrt{5}\,\cos\theta$$

$$\text{and} \quad 5 - x^2 = 5 - 5\sin^2\theta = 5\cos^2\theta.$$

We make all the substitutions and get:

$$\int \frac{x^2}{\sqrt{5-x^2}}\,dx = \int \frac{5\sin^2\theta}{\sqrt{5}\,\cos\theta}\,\sqrt{5}\cos\theta\,d\theta = \int 5\sin^2\theta\,d\theta$$

So far, $\displaystyle\int \frac{x^2}{\sqrt{5-x^2}}\,dx = \int 5\sin^2\theta\,d\theta$ where $x = \sqrt{5}\sin\theta$

The latter is a trig integral of the first kind, and we can use the double-angle formula to get

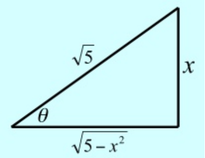
$$\int 5\sin^2\theta\,d\theta = \frac{5}{2}\int 1-\cos 2\theta\,d\theta = \frac{5}{2}\theta - \frac{5}{4}\sin 2\theta + C.$$

Now we have to get back to x's.

Since $\sin\theta = \dfrac{x}{\sqrt{5}}$, we have $\theta = \arcsin\left(\dfrac{x}{\sqrt{5}}\right)$.

Also, $\sin 2\theta = 2\sin\theta\cos\theta$ and

from the triangle we see that $\cos\theta = \dfrac{\sqrt{5-x^2}}{\sqrt{5}}$.



The conclusion is

$$\begin{aligned}
\int \frac{x^2}{\sqrt{5-x^2}}\,dx &= \frac{5}{2}\theta - \frac{5}{4}\sin 2\theta + C \\
&= \frac{5}{2}\theta - \frac{5}{2}\sin\theta\cos\theta + C \\
&= \frac{5}{2}\arcsin\left(\frac{x}{\sqrt{5}}\right) - \frac{5}{2}\,\frac{x}{\sqrt{5}}\,\frac{\sqrt{5-x^2}}{\sqrt{5}} + C \\
&= \frac{5}{2}\arcsin\left(\frac{x}{\sqrt{5}}\right) - \frac{1}{2}x\sqrt{5-x^2} + C
\end{aligned}$$

Quite a bit of work for one integral!

Examples for you...

Here are two for you two work on.

Notice the subtle difference in the integrand that changes entirely the method used:

$$\int x\sqrt{4-x^2}\,dx$$

$$\int x^2\sqrt{4-x^2}\,dx$$

Partial Fractions

Last but not least is integration by *partial fractions*. This method is based on an algebraic trick. It works to integrate a rational function (quotient of polynomials) when the degree of the denominator is **greater** than the degree of the numerator (otherwise "when in doubt, divide it out") and you can factor the denominator completely.

In full generality, partial fractions works when the denominator has repeated and/or quadratic factors, but we will begin with the case of distinct linear factors (i.e., the denominator factors into linear factors and they're all different). This part also uses the (easy) fact that

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + C.$$

Basic partial fractions

The basic idea of partial fractions is to take a rational function of the form

$$\frac{p(x)}{(x-a)(x-b)(x-c)\cdots}$$

(where there can be more or fewer factors in the denominator, but the degree of $p(x)$ *must* be less than the number of factors in the denominator) and rewrite it as a sum:

$$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \cdots$$

for some constants A, B, C (etc..).

For instance,

You can rewrite $\dfrac{2x+1}{x^2+4x+3}$ first by factoring the denominator:

$$\frac{2x+1}{(x+1)(x+3)}$$

and then in partial fractions as

$$\frac{-\frac{1}{2}}{x+1} + \frac{\frac{5}{2}}{x+3}.$$

This leads naturally to two questions:

1. Why do partial fractions?
2. How to do partial fractions?

Two reasons for why:

The first reason is that partial fractions help us to do integrals.

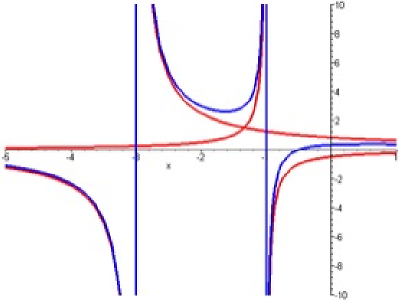
From the previous example, we see that we don't know how to evaluate $\int \frac{2x+1}{x^2+4x+3}\,dx$ straightaway, but...

$$\begin{aligned}\int \frac{2x+1}{x^2+4x+3}\,dx &= \int \frac{\frac{5}{2}}{x+3}\,dx - \int \frac{\frac{1}{2}}{x+1}\,dx \\ &= \frac{5}{2}\ln(x+3) - \frac{1}{2}\ln(x+1) + C\end{aligned}$$

The second reason for "why":

Partial fractions help us understand the behavior of a rational function near its "most interesting" points.

For the same example, we graph the function $\dfrac{2x+1}{x^2+4x+3}$ in blue, and the partial fractions $\dfrac{\frac{5}{2}}{x+3}$ and $\dfrac{-\frac{1}{2}}{x+1}$ in red:



One or the other of the red curves mimics the behavior of the blue one at each singularity.

OK, now for the "how"

First, we give the official version, then a short-cut.

Official version: It is a general fact that the original function can be decomposed into a sum with *one term for each factor in the denominator*. So (in the example above), write

$$\frac{2x+1}{x^2+4x+3} = \frac{A}{x+3} + \frac{B}{x+1}$$

where the constants A and B are to be determined. To determine A and B, pick two values of x other than $x = -3$ or $x = -1$, substitute them into the equation, and then solve the resulting two equations for the two unknowns A and B.

For instance, we can put $x = 0$ and get $\frac{1}{3} = \frac{1}{3}A + B$ and for $x = 1$, get $\frac{3}{8} = \frac{1}{4}A + \frac{1}{2}B$.

Solve to get $A = \frac{5}{2}$ and $B = -\frac{1}{2}$ as we did before.

"Short cut"

1. Write the fraction with denominator in factored form, and leave blanks in the numerators of the partial fractions:

$$\frac{2x+1}{(x+3)(x+1)} = \frac{\_}{x+3} + \frac{\_}{x+1}$$

2. To get the numerator that goes over $x+3$, put your hand over the $x+3$ factor in the fraction on the left and set $x=-3$ in the rest. You should end up with $5/2$.
3. To get the numerator that goes over $x+1$, cover the $x+1$ and set $x=-1$ (and you get $-1/2$).

It's that simple

Another example: $\int \frac{x^2+1}{x^2-x}\,dx$

First, the numerator and denominator have the same degree. So we have to divide it out before we can do partial fractions:

$$\frac{x^2-1}{x^2-x} = 1 + \frac{x+1}{x^2-x} = 1 + \frac{x+1}{x(x-1)}$$

and we can use partial fractions on the second term:

$$1 + \frac{x+1}{x(x-1)} = 1 + \frac{A}{x} + \frac{B}{x-1}$$

Use either the official or short-cut method to get $A = -1$ and $B = 2$, and so:

$$\int \frac{x^2+1}{x^2-x}\,dx = \int 1 - \frac{1}{x} + \frac{2}{x+1}\,dx = x - \ln x + 2\ln(x+1) + C.$$

A couple of examples for you to try:

Find $\displaystyle\int_2^3 \frac{dx}{x(x-1)}$

A. $3/2$	B. $4/3$	C. $\ln 2$	D. $\ln 3$
E. $\ln(3/2)$	F. $\ln(4/3)$	G. $\ln(2/3)$	H. $\frac{3}{2}\ln 2$

Calculate $\displaystyle\int_3^4 \frac{4x-6}{x^2-3x+2}\,dx$

A. $\ln(4/3)$	B. $2+\arctan 3$	C. $2\ln 3$
D. $\ln(12/5)$	E. $\pi/3-\arctan(1/4)$	

Nasty partial fraction examples

$$\int \frac{3x^2 + 17x + 30}{x^3 + 7x^2 + 19x + 13}\, dx$$

$$\int \frac{4x^2 + 2x + 16}{x^4 + 8x^2 + 16}\, dx$$

$$\int \frac{9x^3 - 3x^2 + 2x - 16}{(x-2)(x-1)(x^2 + 2x + 2)}\, dx$$

Can we do that arclength integral?

Recall the problem about the arclength of the parabola:

Find the arclength of the parabola $y = x^2$ for x between -1 and 1

Since $y' = 2x$, the element of arclength is $ds = \sqrt{1+4x^2}\,dx$ and the length we wish to calculate is

$$L = \int_{-1}^{1} \sqrt{1+4x^2}\,dx.$$

This is a trig substitution integral. With the identity $\tan^2 + 1 = \sec^2$ in mind we let $4x^2 = \tan^2\theta$, in other words $x = \frac{1}{2}\tan\theta$ and so $dx = \frac{1}{2}\sec^2\theta\,d\theta$.
These substitutions transform the integral into

$$\int \sqrt{1+4x^2}\,dt = \int \sqrt{1+\tan^2\theta}\,\frac{1}{2}\sec^2\theta\,d\theta = \frac{1}{2}\int \sec^3\theta\,d\theta.$$

That arclength integral

We've integrated $\sec^3\theta$ before, when we did integration by parts, and got that

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln(\sec\theta + \tan\theta) + C.$$

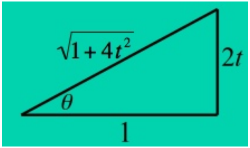
So for our arclength integral we have:

$$\begin{aligned} L = \int_{-1}^{1} \sqrt{1+4x^2}\, dx &= \int_{x=-1}^{x=1} \frac{1}{2}\sec^3\theta\, d\theta \\ &= \frac{1}{4}\sec\theta\tan\theta + \frac{1}{4}\ln(\sec\theta + \tan\theta)\Bigg|_{x=-1}^{x=1} \end{aligned}$$

where $x = \frac{1}{2}\tan\theta$, or $\tan\theta = 2x$.

So far $L = \frac{1}{4}\sec\theta\tan\theta + \frac{1}{4}\ln(\sec\theta + \tan\theta)\Big|_{x=-1}^{x=1}$

Becuase $x = \frac{1}{2}\tan\theta$, we can use the triangle to see that $\tan\theta = 2x$ and $\sec\theta = \sqrt{1+4x^2}$. Therefore



$$L = \int_{-1}^{1} \sqrt{1+4x^2}\,dx = \frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\ln\left(2x + \sqrt{1+4x^2}\right)\Big|_{-1}^{1}$$

$$= \sqrt{5} + \frac{1}{4}\ln\left(\frac{\sqrt{5}+2}{\sqrt{5}-2}\right)$$

Even though it looks a little different, this is the same answer Maple got (to see this, use the fact that $(\sqrt{5}+2)(\sqrt{5}-2) = 1$).