Integrals

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Integrals

Start with dx — this means "a little bit of x" or "*a little change in* x"

If we add up a whole bunch of little changes in x, we get the "**total change of** x".

A **tautology** question

If you add up all of the *changes in* x as x changes from $x = 2$ to $x = 7$, what do you get?

A. 0 B. 2 C. 5 D. 7 E. It cannot be determined.

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We write this in integral notation as $\int_2^7 dx = 5$

Fundamental theorem

If $y = f(x)$, then we know that $dy = f'(x)\,dx$.
To add up all the "*little changes in y*" as x changes from 2 to 7, we should write

$$\int_2^7 f'(x)\,dx \qquad \text{or} \qquad \int_2^7 \frac{df}{dx}\,dx$$

. . . and the answer should be the **total change in** y **as** x **changes from 2 to 7**, in other words

$$\int_2^7 \frac{df}{dx}\,dx = f(7) - f(2).$$

This is the content of the fundamental theorem of calulus!

Fundamental theorem

The **Fundamental Theorem of Calculus** gives the connection between derivatives and integrals. It says that you can calculate

$$\int_a^b g(x)\,dx$$

precisely if you can find a function whose derivative is $g(x)$.

And the result is the difference between the value of the "anti-derivative" of g evaluated at $x = b$ minus the same function evaluated at $x = a$.

Antiderivatives

Basic antiderivative formulas

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{unless } n = -1, \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \sin x\,dx = -\cos x + C$$

$$\int e^x\,dx = e^x + C \quad \text{More generally,} \quad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C \qquad \int \frac{1}{1+x^2}\,dx = \arctan x + C$$

A quick example

Find the value of $\int_0^1 (1+x)^2\,dx$

A. 7/3

B. 0

C. 1

D. 5/3

E. 2

F. 1/3

G. 4/3

H. 2/3

Fundamental Theorem workout

Let

$$f(x) = \int_x^{x^2} t^2 \, dt$$

Find the value of $f'(1)$ — the *derivative of* f *at* $x = 1$.

A. 3

B. 8

C. 4

D. 0

E. 5

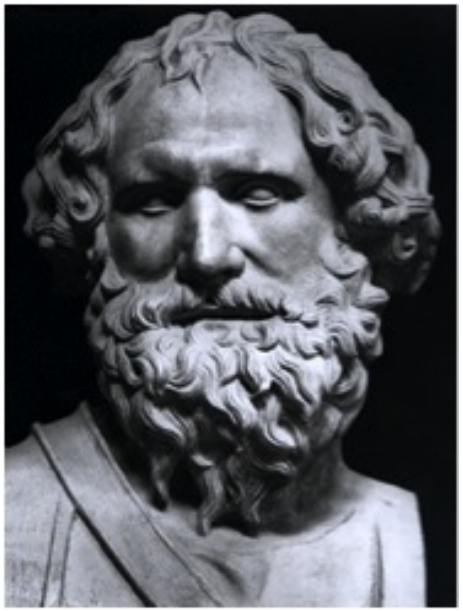
F. 2

G. 6

H. 1

Integrals and Areas

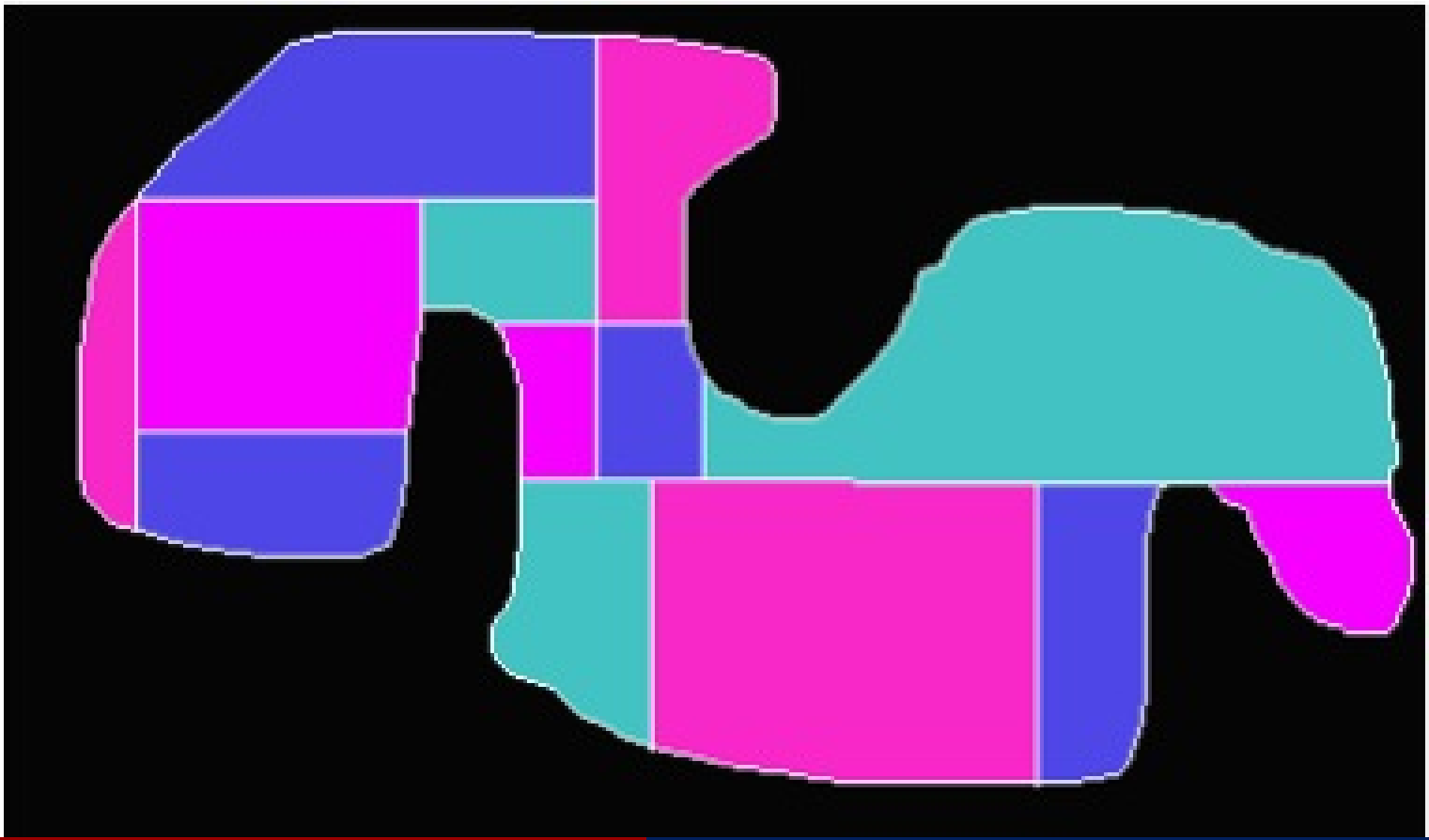
A problem that was around long before the invention of calculus is to find the area of a general plane region (with curved sides).



And a method of solution that goes all the way back to Archimedes is to divide the region up into lots of little regions, so that you can find the areas of almost all of the little regions and so that the total area of the ones you can measure is very small.

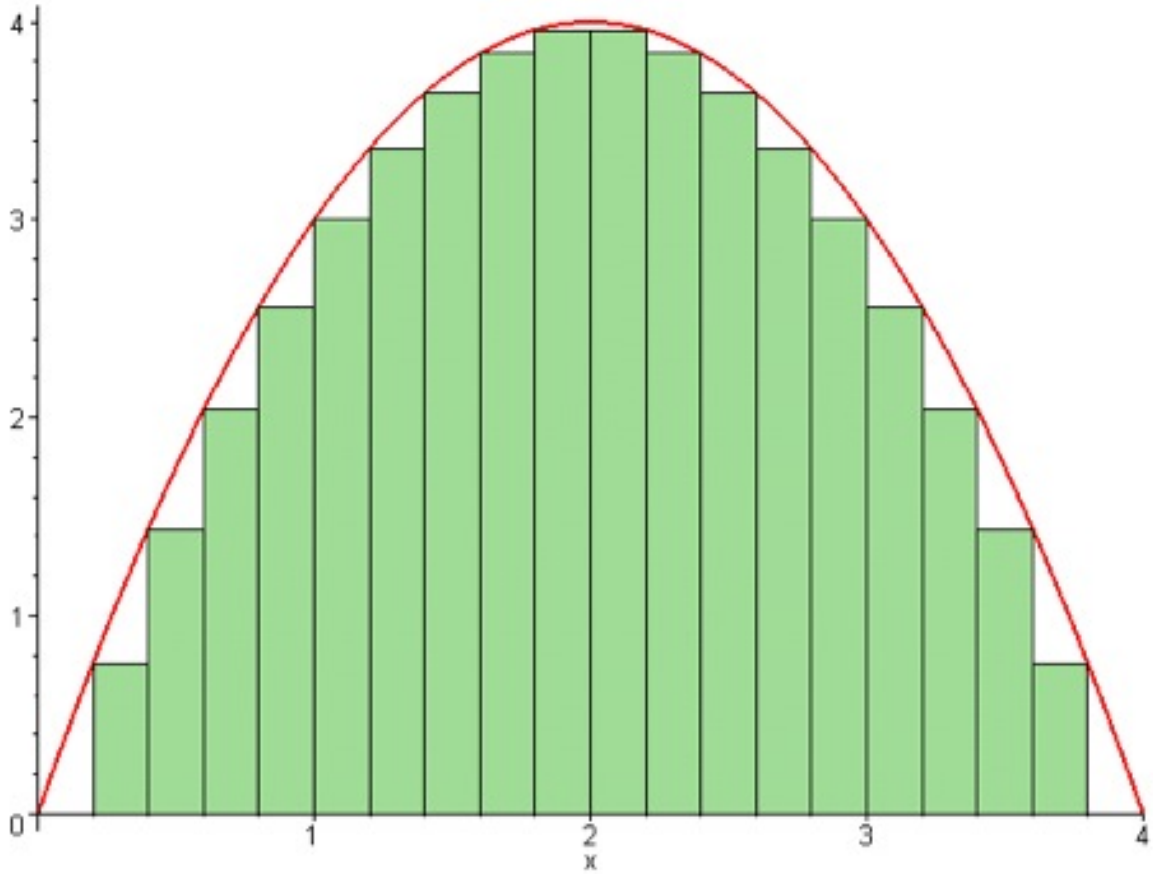
Integrals and Areas

By Newton's time, people realized that it would be sufficient to handle regions that had three straight sides and one curved side (or two or one straight sides — the important thing is that all the sides but one are straight). Essentially all regions can be divided up into this kind of pieces.

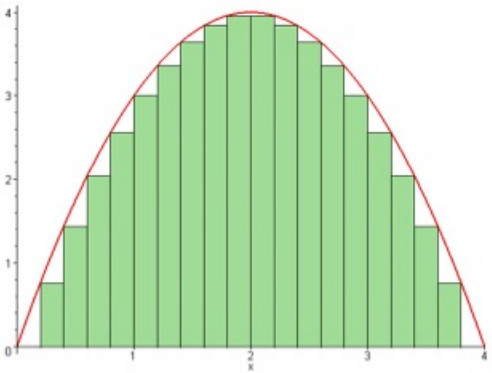


Integrals and Areas

These all-but-one-side-straight regions look like areas under the graphs of functions. And there is a standard strategy for calculating (at least approximately) such areas. For instance, to calculate the area between the graph of $y = 4x - x^2$ and the x-axis, we draw it and subdivide it as follows:



Calculating the area



Since the green pieces are all rectangles, their areas are easy to calculate. The blue parts under the curve are relatively small, so if we add up the areas of the rectangles, we won't be far from the area under the curve. For the record, the total areas of all the green rectangles shown here is $\frac{246}{25}$,

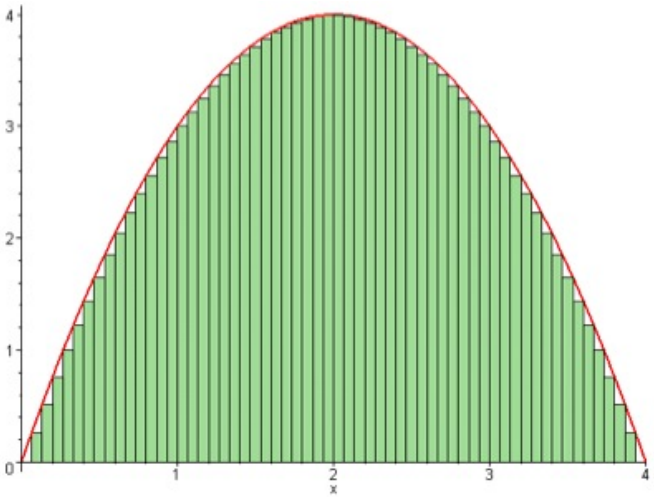
whereas the actual area under the curve is

$$\int_0^4 4x - x^2\,dx = \frac{32}{3}.$$

Also for the record, $\frac{246}{25} = 9.84$ while $\frac{32}{3}$ is about 10.6667.

Improving the estimate

We can improve the approximation by dividing the region into more rectangles:



Now there are 60 boxes instead of 20, and their total area is $\frac{7018}{675}$, which is about 10.397. Getting better. We can in fact take the limit of the green area as the number of rectangles goes to infinity, which will give the same value as the integral.

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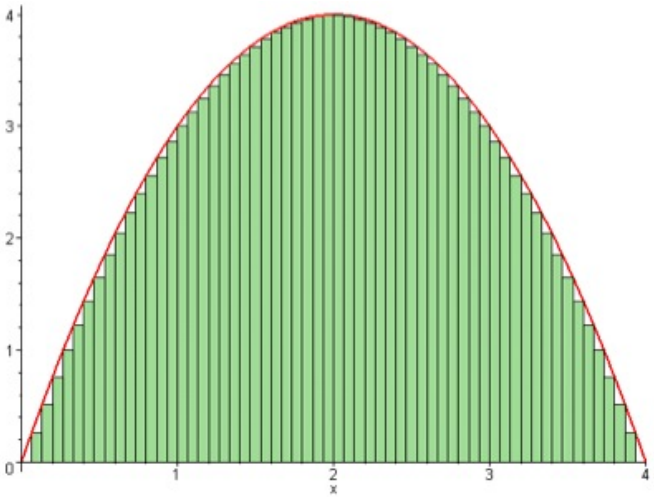

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This was Newton's and Leibniz's great discovery — derivatives and integrals are related and they are related to the area problem.

Area between two curves

A standard kind of problem is to find the area above one curve and below another (or to the left of one curve and to the right of another). This is easy using integrals.

Note that the "**area between the curve and the axis**" is a special case of this problem where one of the curves simply has the equation $y = 0$ (or perhaps $x = 0$).

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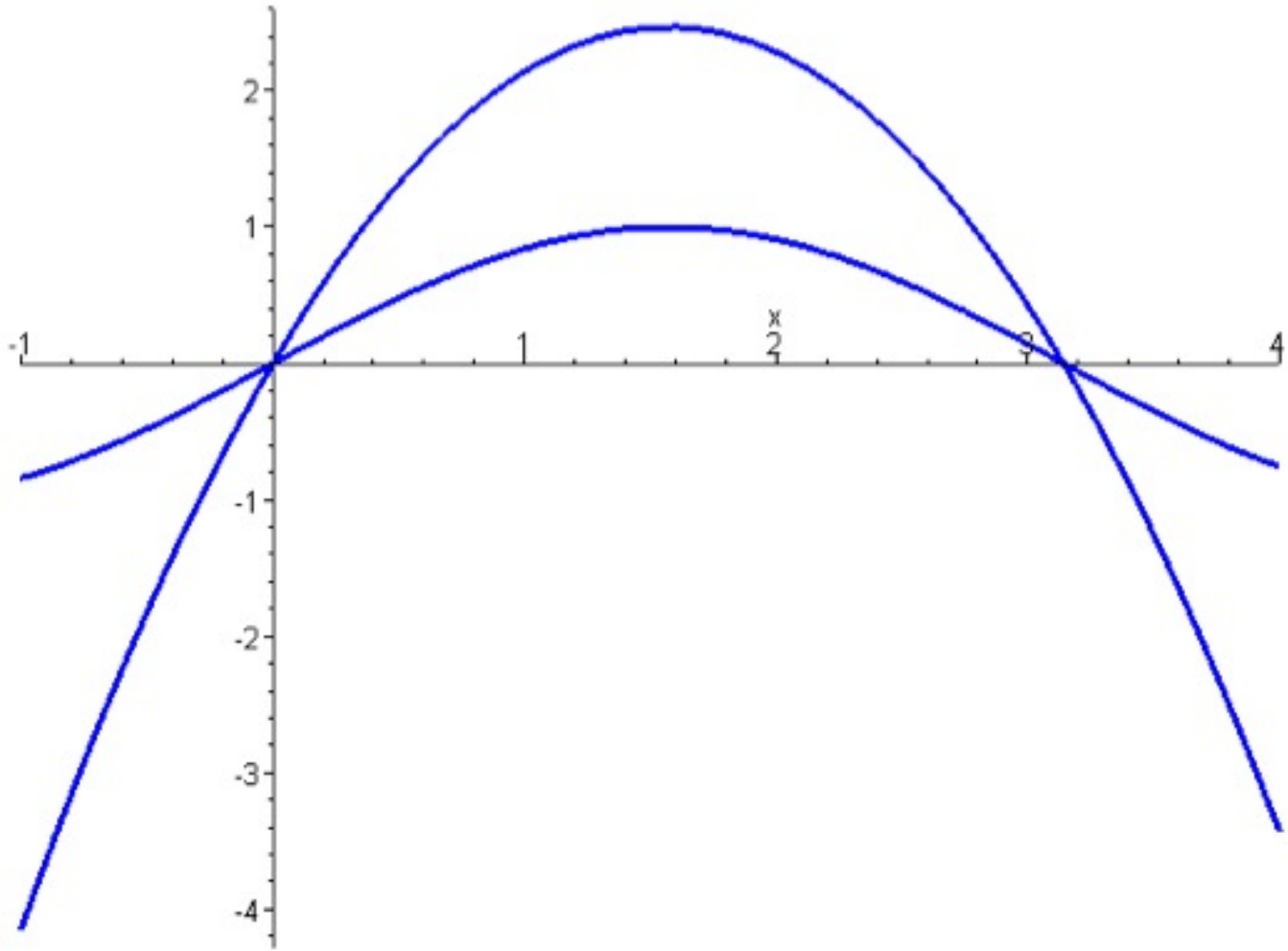
To solve an area problem:

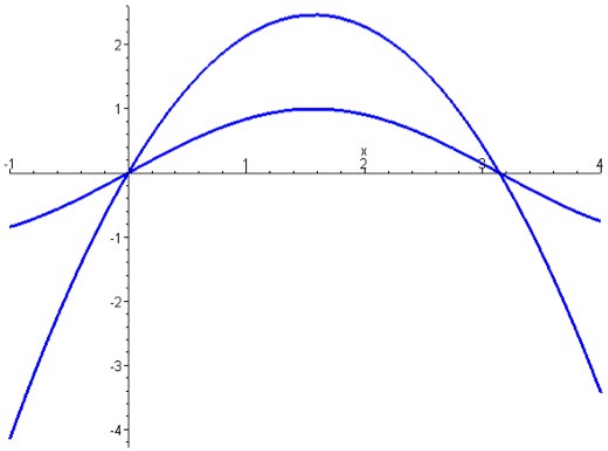
1. Graph the equations if possible
2. Find the points of intersection of the curves to determine the limits of integration, if none are given
3. Integrate the top curve's function minus the bottom curve's (or the right curve's minus the left curve's)

Example

Find the area between the graphs of $y = \sin x$ and $y = x(\pi - x)$.



Example



It's easy to see that the curves intersect on the x-axis, and the values of x at the intersection points are $x = 0$ and $x = \pi$.

The parabola is on top, so we integrate:

$$\int_0^\pi x(\pi - x) - \sin x \, dx = \frac{\pi^3}{6} - 2$$

And this many "square units" is the area between the two curves.

An area question

Find the area of the region bounded by the curves $y = 4x^2$ and $y = x^2 + 3$.

A. 1/2

B. 1

C. 3/2

D. 2

E. 5/2

F. 3

G. 7/2

H. 4

Position, velocity and acceleration

Since velocity is the derivative of position and acceleration is the derivative of velocity,
Velocity is the integral of acceleration and position is the integral of velocity.

(Of course, you must know the *starting values of position and/or velocity* in order to determine the constant[s] of integration.)

Position, velocity and acceleration

Example

An object moves in a force field so that its acceleration at time t is $a(t) = t^2 - t + 12$ (meters per second squared). Assuming that the object is moving at a speed of 5 meters per second at time $t = 0$, determine how far it travels in the first 10 seconds.

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First we determine the velocity, by integrating the acceleration. Because $v(0) = 5$, we can write the velocity as $v(t) = 5+$ a *definite* integral as follows:

$$v(t) = 5 + \int_0^t a(\tau)\, d\tau = 5 + \int_0^t \tau^2 - \tau + 12\, d\tau = 5 + \frac{t^3}{3} - \frac{t^2}{2} + 12t.$$

Position, velocity and acceleration

Example

An object moves in a force field so that its acceleration at time t is $a(t) = t^2 - t + 12$ (meters per second squared). Assuming that the object is moving at a speed of 5 meters per second at time $t = 0$, determine how far it travels in the first 10 seconds.

The distance the object moves in the first 10 seconds is the total change in position. In other words it is the integral of dx as t goes from 0 to 10. But $dx = v(t)\,dt$ so we can write:

$$(\text{distance traveled between } t = 0 \text{ and } t = 10) = \int_0^{10} v(t)\,dt$$

$$= \int_0^{10} 5 + \frac{t^3}{3} - \frac{t^2}{2} + 12t\,dt = \frac{3950}{3} = 1316.666\ldots \text{ meters}$$

Methods of integration

Before we get too involved with applications of the integral, we have to make sure we're good at calculating antiderivatives. There are four basic tricks that you have to learn (and hundreds of *ad hoc* ones that only work in special situations):

1. Integration by substitution (chain rule in reverse)
2. Trigonometric substitutions (using trig identities to your advantage)
3. Partial fractions (an algebraic trick that is good for more than doing integrals)
4. Integration by parts (the product rule in reverse)

We'll do #1 this week, and the others later. **LOTS** of practice is needed to master these!

Integration by substitution

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The idea is to remember the chain rule: If G is a function of u and u is a function of x, then the derivative of G with respect to x is:

$$\frac{dG}{dx} = G'(u)u'(x).$$

For instance, e^{x^2} could be thought of as e^u where $u = x^2$.
To differentiate e^{x^2} then, we use that the derivative of e^u is e^u:

$$\frac{d}{dx}e^{x^2} = \frac{d}{du}(e^u)\,\frac{d}{dx}(x^2) = e^u(2x) = 2xe^{x^2}.$$

Now we'll turn this around

A substitution integral

In the integral

$$\int x^3 \cos(x^4)\, dx$$

it looks as though x^4 should be considered as u,
in which case $du = d(x^4) = 4x^3\, dx$, or in other words $x^3\, dx = \frac{1}{4} du$.
And so:

$$\int x^3 \cos(x^4)\, dx = \int \cos u \, \frac{du}{4} = \frac{1}{4} \sin u + C = \frac{1}{4} \sin(x^4) + C.$$

Integration by substitution tips

In general, to carry out integration by substitution, you

1. Separate the integrand into factors.
2. Decide which factor is the most complicated.
3. Ask whether the *other* factors are (perhaps a constant times) the derivative of some compositional part of the complicated one

This provides the clue as to what to set equal to u.

Another substitution example

Calculate $\int \frac{x}{(2x^2+5)^3}\,dx$

Clearly the complicated factor is $\frac{1}{(2x^2+5)^3}$ in this case.

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The rest, $x\,dx$ is a constant times the differential of x^2 — but it's a good idea to try and make u substitute for a much of the complicated factor as possible.

And if you think about it, $x\,dx$ is also a constant times the differential of $2x^2+5$! So we let $u = 2x^2+5$ and then $du = 4x\,dx$, in other words $x\,dx = \frac{1}{4}du$. Therefore:

$$\int \frac{x}{(2x^2+5)^3}\,dx = \frac{1}{4}\int \frac{1}{u^3}\,du = -\frac{1}{8u^2} + C = -\frac{1}{8(2x^2+5)^2} + C.$$

Your turn!

$$\int_0^{\sqrt{\pi/2}} x \cos x^2 \, dx =$$

A. 0 B. $1/2$ C. 1 D. $\pi/2$ E. $\sqrt{\pi}$

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A. 0 B. $1/2$ C. 1 D. $\pi/2$ E. $\sqrt{\pi}$

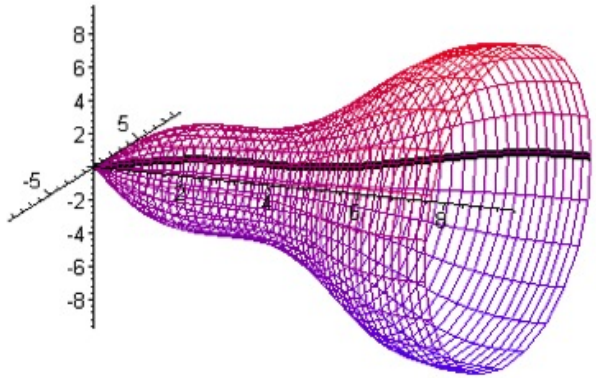
$$\int_0^{\pi/4} \sec^2 x \sin(\tan x) \, dx =$$

A. $\pi/2$ B. $1 - \pi/4$ C. $\sin 1$ D. $1 - \cos 1$

E. $\pi/2 - \sin 1$ F. $\pi/4 + \cos 1$ G. $1 + 3\pi/4$ H. $1 + \tan 1$

Surfaces of revolution

A surface of revolution is formed when a curve is revolved around a line into 3-dimensional space. The line is usually the x or y axis in the plane, but other lines sometimes occur. The curve sweeps out a surface with 360-degree rotational symmetry.

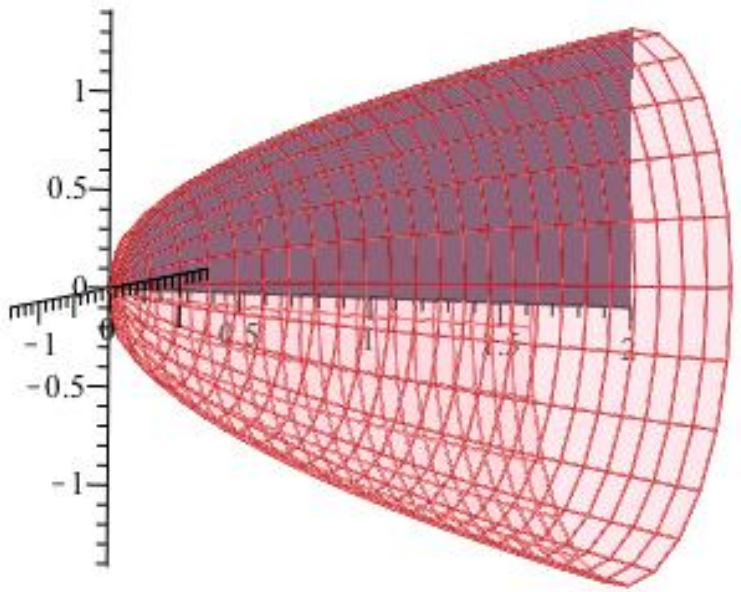


Interesting problems that can be solved by integration are to find the volume enclosed inside such a surface, or to find its surface area.

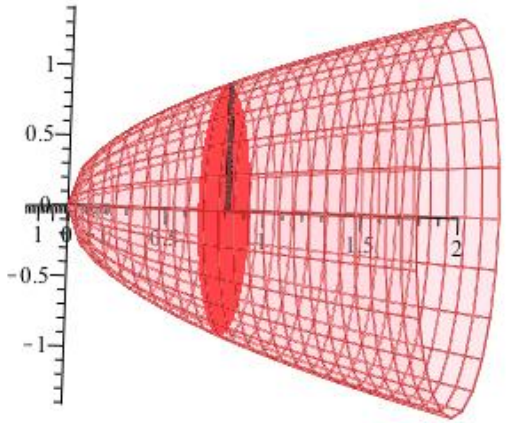
Volumes

You might already be familiar with finding volumes of revolution. Once a surface is formed by rotating a curve around the x-axis, you can sweep out the volume it encloses with disks perpendicular to the x-axis.

Here is the surface formed by rotating the part of the curve $y = \sqrt{x}$ for $0 \leq x \leq 2$ around the x-axis. We will calculate the volume of the solid obtained by rotating the region in the xy-plane between the curve and the x-axis around the x-axis, as shown here.

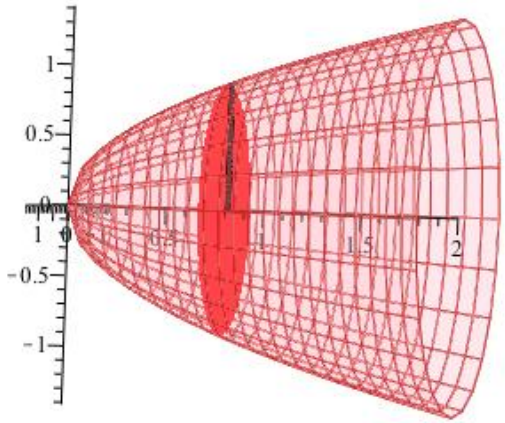


Calculate the volume



To calculate the volume enclosed inside the surface, we need to add up the volumes of all the disks for each value of x between 0 and 2. One such disk is shown here, with its radius.

Calculate the volume



To calculate the volume enclosed inside the surface, we need to add up the volumes of all the disks for each value of x between 0 and 2. One such disk is shown here, with its radius.

The disks are (approximately) cylinders turned sideways, and the disk centered at $(x, 0)$ has radius $\sqrt{x}$ and width (or height) dx. So the volume of the disk is $\pi r^2 h = \pi(\sqrt{x})^2\, dx = \pi x\, dx$.

To find the total volume of the solid we have to integrate this quantity for x from 0 to 2:

$$V = \int_0^2 \pi x\, dx = 2\pi \text{ cubic units.}$$

A formula for volume

In general, if the region between the x-axis and the part of the graph of $y = f(x)$ between $x = a$ and $x = b$ is revolved around the x-axis, the volume inside the resulting solid of revolution is calculated as:

$$V = \int_a^b \pi \left(f(x)\right)^2 \, dx$$

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The same sort of formula applied if we rotate the region between the y-axis and a curve around the y-axis (just change all the x's to y's): if the curve is given as $x = g(y)$ for $c \leq y \leq d$, then the volume is

$$V = \int_c^d \pi \left(g(y)\right)^2 \, dy$$

A different kind of volume problem

A different kind of problem is to rotate the region between a curve and the x-axis around the y-axis. For instance, let's look at the same region as before (between $y = 0$ and $y = \sqrt{x}$ for x between 0 and 2), but rotated around the y-axis instead.

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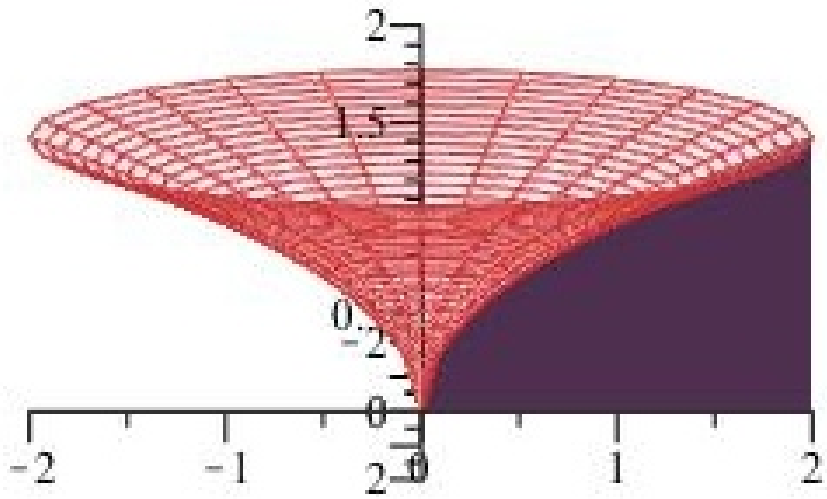
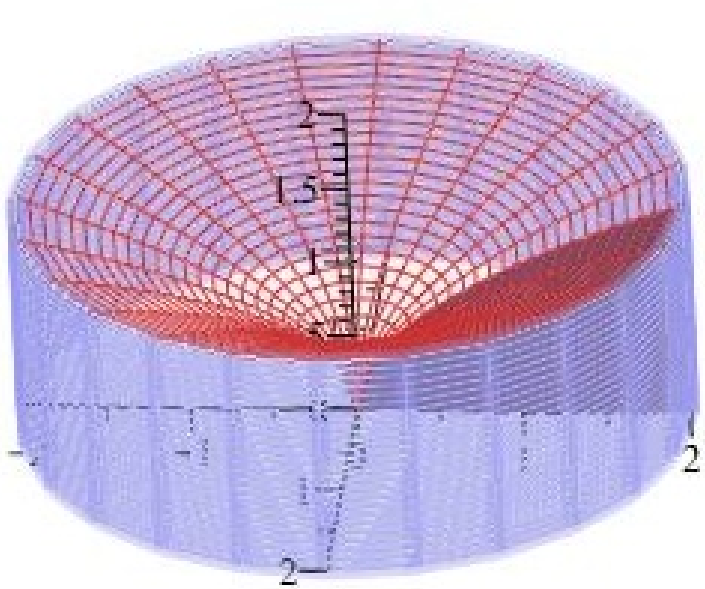
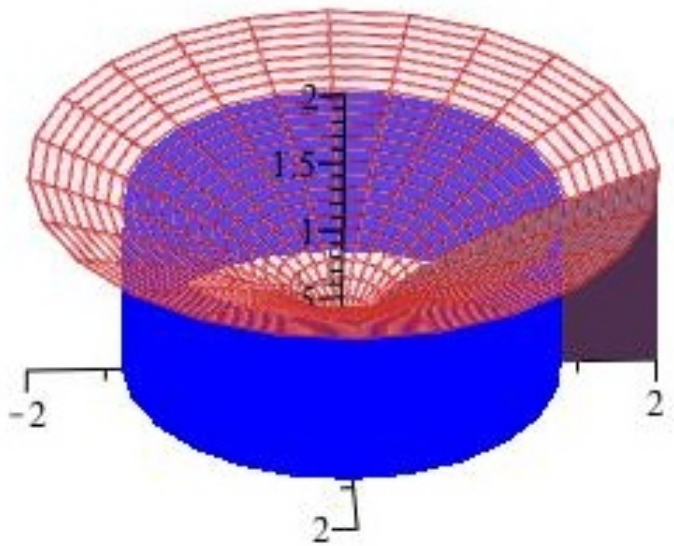
First, here's the surface swept out by the parabolic curve, together with the region in the xy-plane:



Volume by shells

Here is the solid swept out by the region — the outer (gray) surface is swept out by the vertical line $x = 2$.





If we rotate a thin rectangle of width dx and height $\sqrt{x}$ around the y-axis, we get a *cylindrical shell*. One such shell is shown in the figure on the left.

Volume by shells

Think of the shell as the label of a can (just the label, not the contents of the can!). If you cut the label along a vertical line, then it can be laid out as a rectangular sheet of paper, with length $2\pi x$, with height $\sqrt{x}$ and with thickness dx. The volume of the sheet of paper (and so the volume [approximately] of the cylindrical shell) that goes through the point $(x, 0)$ is thus

$$dV = 2\pi x\sqrt{x}\, dx.$$

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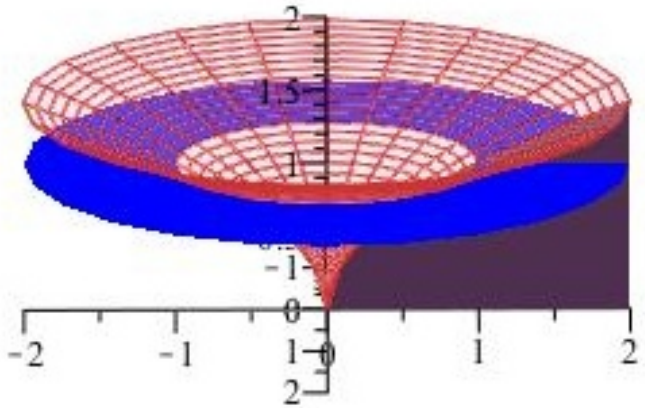
$$dV = 2\pi x\sqrt{x}\, dx.$$

So we can calculate the volume of the entire solid by adding up the volumes of all the shells for x starting from 0 and ending at 2:

$$V = \int_0^2 2\pi x^{3/2} dx = \frac{16\pi\sqrt{2}}{5} \text{ cubic units.}$$

Another approach

We could also calculate this last volume by rewriting the equation of the curve as $x = y^2$, and rotating thin horizontal rectangles of height dy that stretch horizontally from the curve $x = y^2$ to the vertical line $x = 2$. Such a horizontal rectangle produces a disk with a hole in the middle (a "washer"), as shown here:



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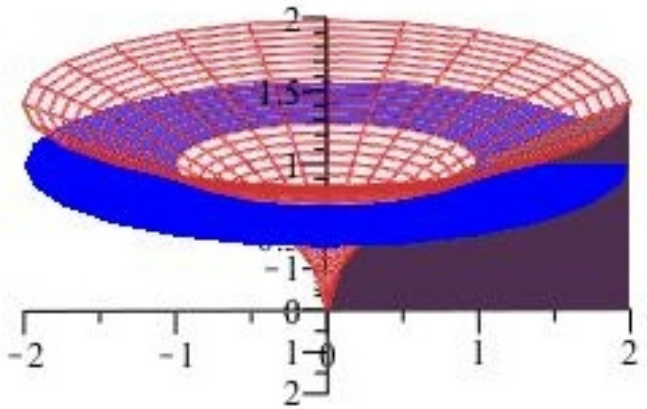

To calculate the volume of a washer, simply calculate the volume of the disk without the hole (for each y, it has radius 2), and subtract the volume of the hole (which is a disk of radius y^2 and height dy).

Volume by washers

The volume of a single washer is thus

$$dV = 4\pi\, dy - \pi(y^2)^2\, dy = (4\pi - \pi y^4)\, dy$$

and so the volume of the entire solid is

$$V = \int_0^{\sqrt{2}} 4\pi - \pi y^4\, dy = \frac{16\pi\sqrt{2}}{5} \text{ cubic units}$$

which agrees with what we got before.

Volumes of revolution: disks/washers or shells?

When calculating volumes of revolution, an important step is the decision of which approach to take. This decision is often framed as a choice between the disk (or washer) method and the shell method, but this is generally the wrong way to frame the question.

Rather, it is more important to ask whether choosing vertical (tall and thin, with width dx) or horizontal (short and wide, with height dy) sections of the region being rotated will result in simpler calculations of relevant lengths and integrals. A few examples will help illustrate this.

Example 1

Find the volume obtained by revolving the region between the parabola $x = 4y - y^2$ and the line $y = x$ around the x-axis.

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We begin by drawing the region described by the curves. For some values of x the (green) vertical sections sometimes have their lower endpoint on the parabola and their upper endpoint on the line, and for others, both endpoints are on the parabola. This means we would have to split the interval of integration into two pieces.

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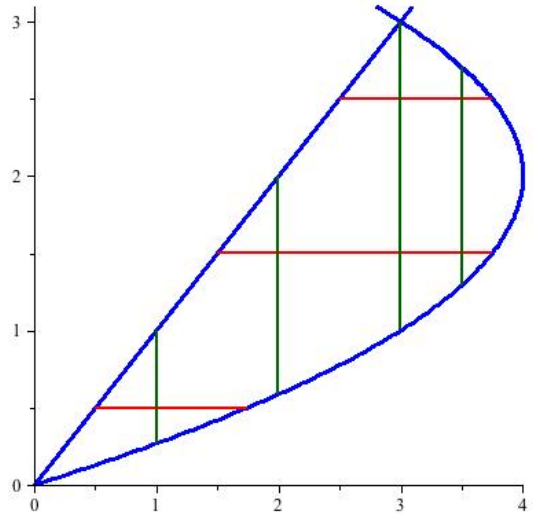
We begin by drawing the region described by the curves. For some values of x the (green) vertical sections sometimes have their lower endpoint on the parabola and their upper endpoint on the line, and for others, both endpoints are on the parabola. This means we would have to split the interval of integration into two pieces.

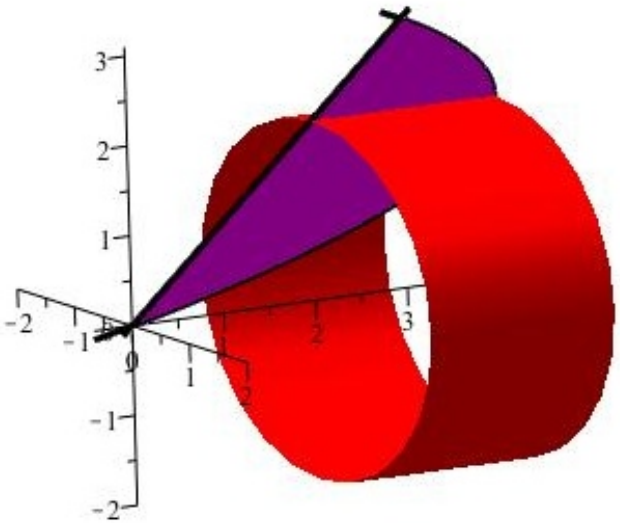
On the other hand, the (red) horizontal sections all have their left endpoints on the line and their right endpoints on the parabola.

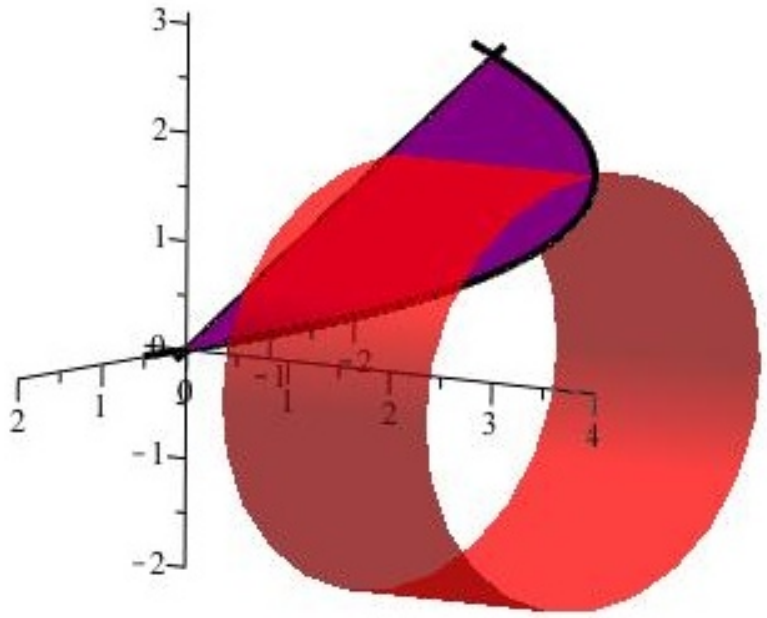


Example 1

Find the volume obtained by revolving the region between the parabola $x = 4y - y^2$ and the line $y = x$ around the x-axis.

So we would rather use horizontal sections. If we revolve a horizontal segment around the x-axis, we'll get a cylindrical shell, as shown here (from two different angles):





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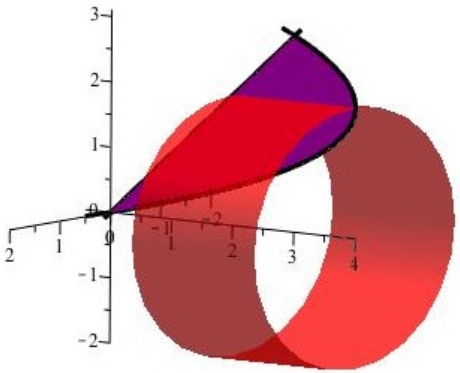


The thickness of the shell is dy, so we need to express everything in terms of y. The radius of the shell points from the x-axis to the shell, and so it is y itself. The height of the shell is the x-coordinate on the parabola minus the x-coordinate on the line, expressed in terms of y, so it is $(4y - y^2) - y = 3y - y^2$.

The volume of one shell is thus

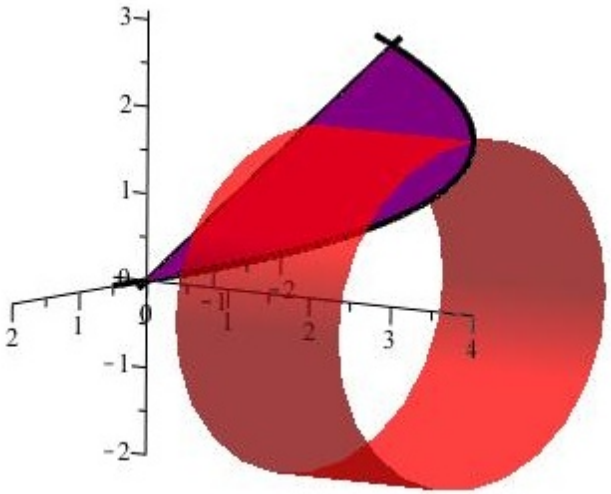
$$dV = 2\pi y(3y - y^2)\,dy = \pi(6y^2 - 2y^3)\,dy.$$

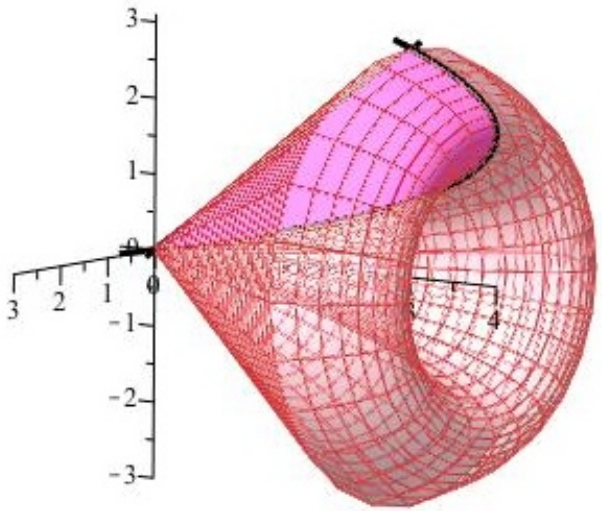
Example 1

Find the volume obtained by revolving the region between the parabola $x = 4y - y^2$ and the line $y = x$ around the x-axis.

Therefore the total volume of the solid is the sum of the volumes of the shells as y goes from 0 to 3, or:

$$V = \int_0^3 \pi(6y^2 - 2y^3)\,dy = \left(2\pi y^3 - \frac{\pi y^4}{2}\right)\Bigg|_0^3 = \frac{27\pi}{2} \text{ cubic units.}$$

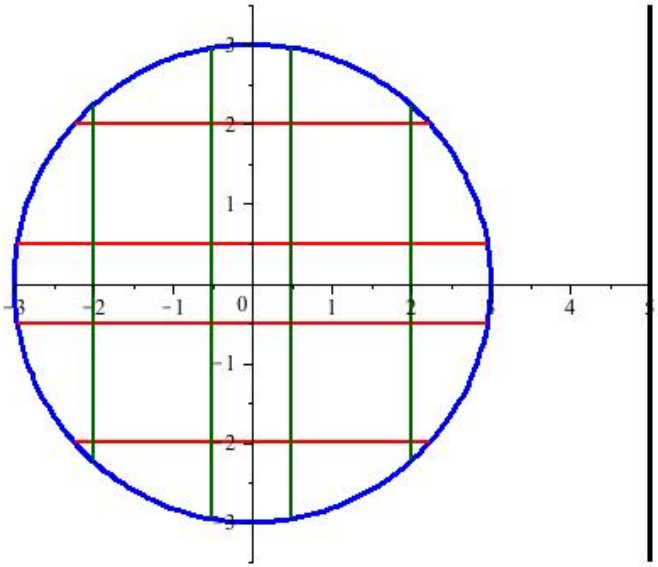




Example 2

Find the volume of the torus obtained by revolving the interior of the circle $x^2 + y^2 = 9$ around the line $x = 5$.

Since the region is a round disk, it is indifferent to whether we use horizontal or vertical sections, but perhaps this time the setup of the integral that results will be easier one way than the other. You can be the judge!



Here is the disk and the axis of rotation, with horizontal (red) sections, that go from the left half of the circle to the right half, as well as vertical (green) sections that go from the bottom half of the circle to the top half.

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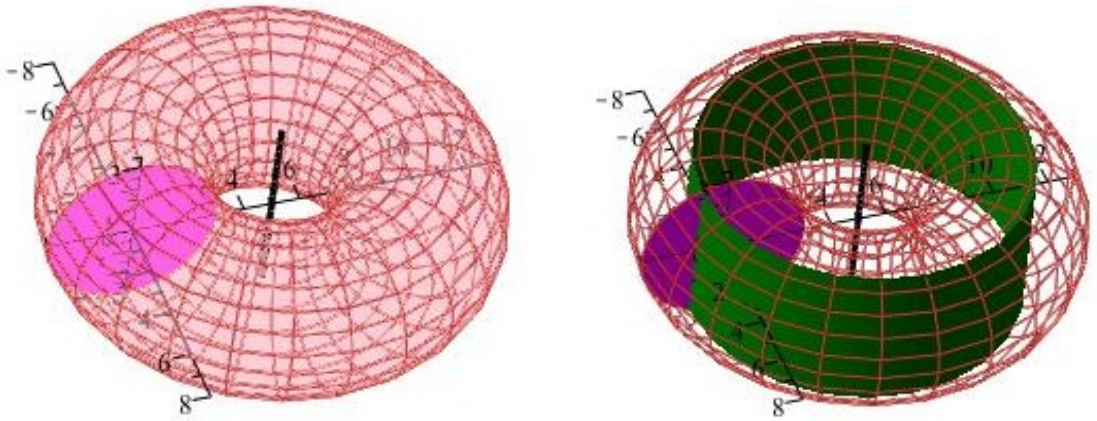

The green vertical sections are tall and thin, with width dx. The one at x goes from $y = -\sqrt{9 - x^2}$ to $y = +\sqrt{9 - x^2}$. When we rotate one of the green sections around the axis $x = 5$, we get a cylindrical shell of radius $5 - x$, height $2\sqrt{9 - x^2}$ and thickness dx. So the volume of a single shell is $dV = 2\pi(5 - x)\left(2\sqrt{9 - x^2}\right)\, dx$

We will integrate this for x going from -3 to 3 to get the volume of the torus.

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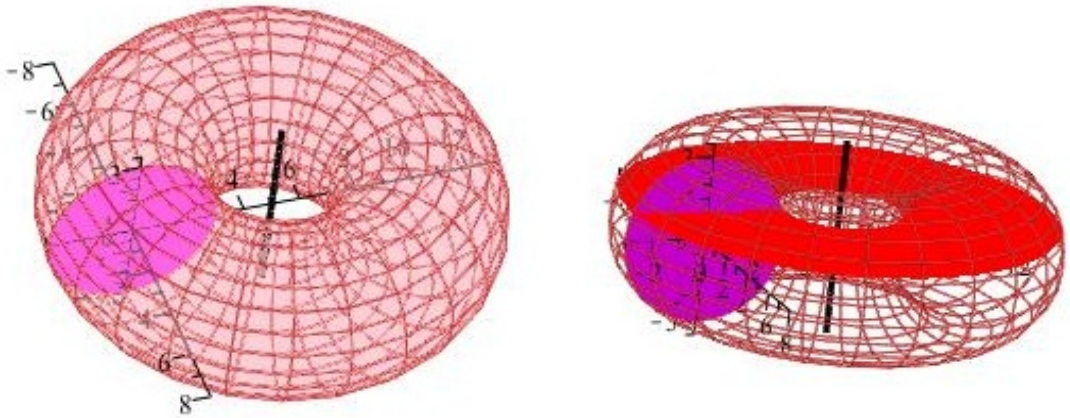
Find the volume of the torus obtained by revolving the interior of the circle $x^2 + y^2 = 9$ around the line $x = 5$.



The red horizontal sections are wide and short, with height dy. The one at height y goes from $x = -\sqrt{9 - y^2}$ to $x = +\sqrt{9 - y^2}$. When we rotate one of the red sections around the axis $x = 5$, we get a washer of outer radius $5 + \sqrt{9 - y^2}$, inner radius $5 - \sqrt{9 - y^2}$ and thickness dy. So the volume of a single washer is $dV = \pi \left((5 + \sqrt{9 - y^2})^2 - (5 - \sqrt{9 - y^2})^2 \right) dy$ We will integrate this for y going from -3 to 3 to get the volume of the torus.

Example 2

Find the volume of the torus obtained by revolving the interior of the circle $x^2 + y^2 = 9$ around the line $x = 5$.

Now we have two ways to calculate the volume. The shell way gives the integral:

$$V = \int_{-3}^{3} 2\pi(5 - x)\left(2\sqrt{9 - x^2}\right)\, dx$$

and the washer way gives:

$$\begin{aligned} V &= \int_{-3}^{3} \pi\left((5 + \sqrt{9 - y^2})^2 - (5 - \sqrt{9 - y^2})^2\right)\, dy \\ &= \int_{-3}^{3} 20\pi\sqrt{9 - y^2}\, dy. \end{aligned}$$

These integrals look hard, but we can evaluate them easily with some thought.

Example 2

Find the volume of the torus obtained by revolving the interior of the circle $x^2 + y^2 = 9$ around the line $x = 5$.

We will learn how to compute the antiderivative of $\sqrt{9 - y^2}$ later, but we can use the fact that the integral is the area between a curve and the axis to evaluate:

$$V = \int_{-3}^{3} 20\pi\sqrt{9 - y^2}\, dy = 20\pi \times (\text{area of semicircle of radius } 3)$$
$$= 20\pi\left(\frac{1}{2}\pi(3^2)\right) = 90\pi^2 \text{ cubic units}$$

We leave it to you to find a clever way to evaluate the other expression for the volume of the torus.

Volumes with known cross-sections

Another family of volume problems involves volumes of three-dimensional objects whose cross-sections in some direction all have the same shape.

For example:

Calculate the volume of the solid S if the base of S is the triangular region with vertices $(0, 0)$, $(2, 0)$ and $(0, 1)$ and cross-sections perpendicular to the x-axis are semicircles.

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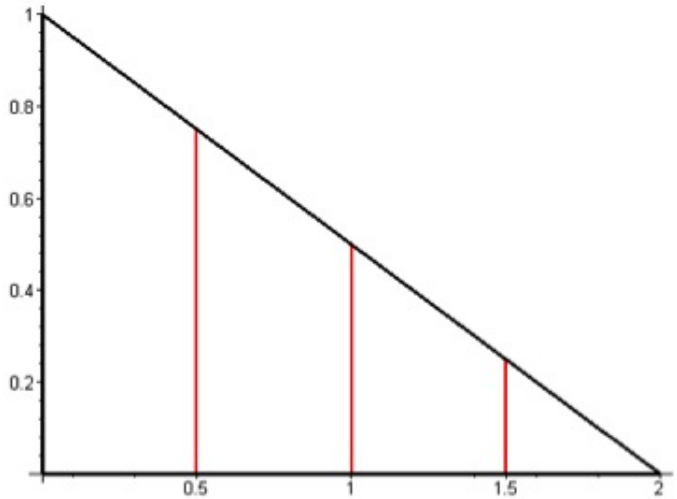
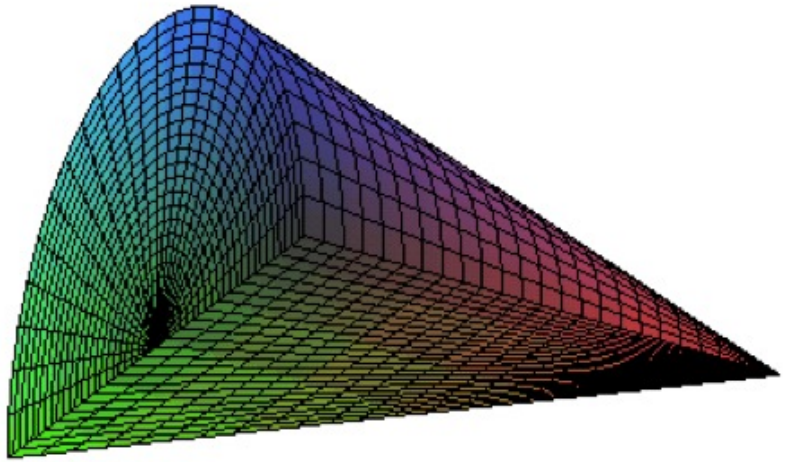
First we have to visualize the solid. Here is the base triangle, with a few vertical lines (perpendicular to the x-axis) drawn on it. These will be the diameters of the semicircles in the solid.

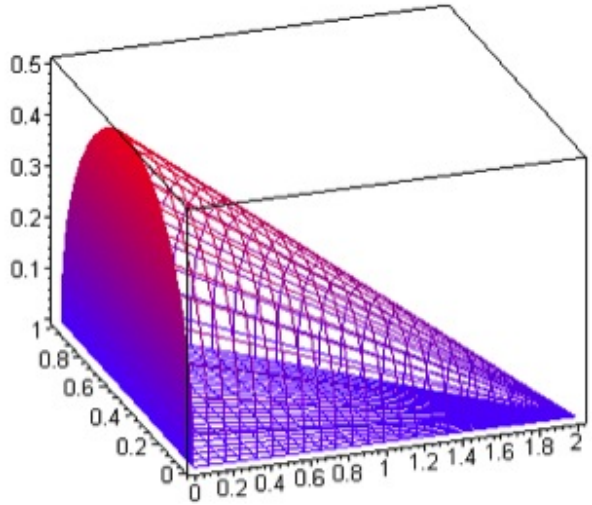


Volumes with known cross-sections

Here is a three-dimensional plot of the solid that has this triangle as the base and the semi-circular cross sections:



From that point of view you can see some of the base as well as the cross-section. We'll sweep out the volume with slices perpendicular to the x-axis, each of which looks like half a disk.



The volume of that object

Since the line connecting the two points $(0, 1)$ and $(2, 0)$ has equation $y = 1 - \frac{1}{2}x$, the centers of the half-disks are at the points $(x\,,\ \frac{1}{2} - \frac{1}{4}x)$ and their radii are likewise $\frac{1}{2} - \frac{1}{4}x$.

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So the little bit of volume at x is half the volume of a cylinder of radius $\frac{1}{2} - \frac{1}{4}x$ and height dx, in other words

$$dV = \frac{1}{2}\pi\left(\frac{1}{2} - \frac{1}{4}x\right)^2 dx$$

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Therefore the volume of the entire solid is

$$V = \frac{\pi}{2}\int_0^2 \left(\frac{1}{2} - \frac{1}{4}x\right)^2 dx = \frac{\pi}{12} \text{ cubic units.}$$

As a check. . .

Finally, we could also have calculated the volume by noticing that the solid S is half of a (skewed) cone of height 2 with base radius $\frac{1}{2}$.

Using the formula $V = \frac{1}{3}\pi r^2 h$ for a cone, we arrive at the same answer,

$$V = \frac{1}{2} \cdot \frac{1}{3}\pi \left(\frac{1}{2}\right)^2 (2) = \frac{\pi}{12} \text{ cubic units}$$

Arc length

The length of a curve in the plane is generally difficult to compute.
To do it, you must add up the little "bits of arc", ds.
A good approximation to ds is given by the Pythagorean theorem:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy.$$

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So we have that the length of the part of the curve $y = f(x)$ (or $x = g(y)$) from the point (a, b) to the point (c, d) is given by:

$$L = \int_a^c \sqrt{1 + (f'(x))^2}\, dx = \int_b^d \sqrt{(g'(y))^2 + 1}\, dy.$$

We can use one of these to find the length of any graph — provided we can do the integral that results!

Example

It's not easy to make up an arc length problem that yields a reasonable integral. The way textbooks (and professors making up exams) often do it is to find two functions whose derivatives multiply together to give $-\frac{1}{4}$ and then have the problem be to calculate the length of part of the curve $y = f(x)$, where $f(x)$ is the sum of those two functions. To illustrate:

Find the arclength of the part of the curve $y = \frac{x^3}{3} + \frac{1}{4x}$ for $1 \leq x \leq 2$.

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Find the arclength of the part of the curve $y = \frac{x^3}{3} + \frac{1}{4x}$ for $1 \leq x \leq 2$.

We'll solve the problem the "standard" way, then make some observations. We have $y' = x^2 - \dfrac{1}{4x^2}$, so $(y')^2 = x^4 - \dfrac{1}{2} + \dfrac{1}{16x^4}$.

Therefore $1 + (y')^2 = x^4 + \dfrac{1}{2} + \dfrac{1}{16x^4} = \left(x^2 + \dfrac{1}{4x^2}\right)^2$, and so

$$L = \int_1^2 x^2 + \frac{1}{4x^2}\,dx = \frac{x^3}{3} - \frac{1}{4x}\bigg|_1^2 = \frac{7}{3} + \frac{1}{8} = \frac{59}{24}$$

Some observations

In the preceding example, the derivatives of the two terms in $f(x)$ multiplied together to give $-\frac{1}{4}$, so that the middle term of $(f')^2$ turned out to be $-\frac{1}{2}$.

That meant that when we added 1, the $-\frac{1}{2}$ turned into $+\frac{1}{2}$, so that $1 + (f')^2$ was a perfect square.

And in fact, the integral of $\sqrt{1 + (f')^2}$ turned out to be the original function with the $+$ sign changed to a $-$. So we actually could have gone directly from the question to the final evaluation of the integral at the endpoints without any of the intermediate calculation.

Be alert for this trick in the homework problems (and on review problems from old exams) — it can be a real timesaver!

A harder example

As we will see, the simpler the function, the harder the arclength integral often turns out to be!

Find the arclength of the parabola $y = x^2$ for x between -1 and 1

Since $y' = 2x$, the element of arclength is $ds = \sqrt{1 + 4x^2}\, dx$ and the length we wish to calculate is

$$L = \int_{-1}^{1} \sqrt{1 + 4x^2}\, dx.$$

We'll do this integral later, using a trig substitution. But, appealing to Maple we get that

$$L = \int_{-1}^{1} \sqrt{1 + 4x^2}\, dx = \sqrt{5} - \frac{1}{2} \ln(\sqrt{5} - 2).$$

Area of a surface of revolution

The area of a surface of revolution can be calculated using a strategy similar to the disk method for volume: Slice the surface up into thin bands using slices perpendicular to the axis of revolution. Then get a good differential approximation for area of each band, and then add them up to get the area of the entire surface.

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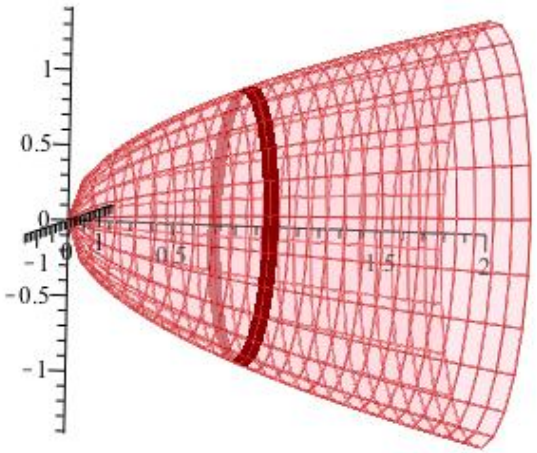
Here is the paraboloid obtained by rotating $y = \sqrt{x}$ for $0 \leq x \leq 2$ that we used before, together with one of the circular bands that sweep out its surface area:



To calculate the surface area...

...we need to determine the area of the bands. The one centered at $(x, 0)$ has radius $y = \sqrt{x}$ and width equal to $ds = \sqrt{dx^2 + dy^2}$. The radius of the band is its circumference times its width, or

$$d\sigma = 2\pi r\, ds.$$

In general, we can choose whether to integrate with respect to x or y — sometimes one way is clearly easier, but we'll do this one both ways so you can see how it's done.

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In general, we can choose whether to integrate with respect to x or y — sometimes one way is clearly easier, but we'll do this one both ways so you can see how it's done.

To integrate with respect to x, write $ds = \sqrt{1 + (y')^2}\, dx$. Since $y' = \dfrac{1}{2\sqrt{x}}$, we have $ds = \sqrt{1 + \dfrac{1}{4x}}\, dx$. And the radius is $\sqrt{x}$, so

$$\sigma = \int_0^2 d\sigma = \int_0^2 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx = \int_0^2 \pi\sqrt{4x+1}\, dx$$

$$= \frac{\pi}{4}\frac{2}{3}(4x+1)^{3/2}\Big|_0^2 = \frac{\pi}{6}(27 - 1) = \frac{13\pi}{3} \text{ square units}$$

The other way around

We can also calculate the surface area by integrating with respect to y. To do this, we express the curve as $x = y^2$ for $0 \leq y \leq \sqrt{2}$. Then the radius of the bands can be expressed simply as y, and

$$ds = \sqrt{(x')^2 + 1}\, dy = \sqrt{4y^2 + 1}\, dy.$$

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$$ds = \sqrt{(x')^2 + 1}\, dy = \sqrt{4y^2 + 1}\, dy.$$

So we have

$$\sigma = \int_0^{\sqrt{2}} d\sigma = \int_0^{\sqrt{2}} 2\pi r\, ds = \int_0^{\sqrt{2}} 2\pi y \sqrt{4y^2 + 1}\, dy$$

Make the substitution $u = 4y^2 + 1$ (so $du = 8y\, dy$) and obtain:

$$\sigma = \int_1^3 \frac{\pi}{4}\sqrt{u}\, du = \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^3 = \frac{\pi}{6}(27 - 1) = \frac{13\pi}{3} \text{ square units}$$

just as before.