

Welcome to Math 104

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Welcome to the course

- Math 104 – Calculus I
- Topics: Quick review of Math 103 topics, methods and applications of integration, infinite series and applications
- Pace and workload:
 - Moves *very fast*
 - Demanding workload, but *help is available!*
 - YOU ARE ADULTS - how much do you need to practice each topic?
- Emphasis on applications - what is this stuff good for?
- Opportunities to interact with professor, TA, and other students

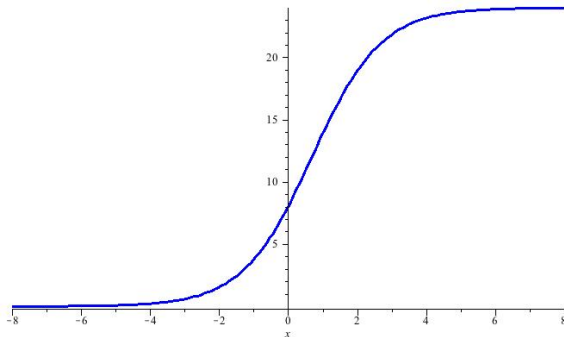
Outline for week 1

- Review of functions and graphs
- Review of limits
- Review of derivatives – idea of velocity, tangent and normal lines to curves
- Review of related rates and max/min problems

Functions and Graphs

The idea of a function and of the graph of a function should be very familiar.

$$y = f(x) = \frac{24}{1 + 2e^{-x}}$$



Questions for discussion

- Describe the graph of the function $f(x)$ (use calculus vocabulary as appropriate).
- The graph intersects the y -axis at one point. What is it (how do you find it)?
- How do you know there are no other points where the graph intersects the y -axis?
- Does the graph intersect the x -axis? How do you know?
- There is one especially interesting point on the graph. How do you find it?
- Where might this function come from?

Kinds of functions that should be familiar

- Linear, quadratic
- Polynomials, quotients of polynomials
- Powers and roots
- Exponential, logarithmic
- Trigonometric functions (sine, cosine, tangent, secant, cotangent, cosecant)
- Hyperbolic functions ($\sinh$, $\cosh$, $\tanh$, sech , coth , csch)

Quick Question

The **domain** of the function $f(x) = \frac{\sqrt{1-x}}{x^2 - 2x}$ is

- A. All x except $x = 0, x = 2$
- B. All $x \leq 1$ except $x = 0$.
- C. All $x \geq 1$ except $x = 2$.
- D. All $x \leq 1$.
- E. All $x \geq 1$.

Quick Question

Which of the following has a graph that is **symmetric with respect to the y-axis**?

A. $y = \frac{x-1}{x}$

B. $y = 2x^4 - 1$

C. $y = x^3 - 2x$

D. $y = x^5 - 2x^2$

E. $y = \frac{x}{x^3 + 3}$

Quick Question

The **period** of the function $f(x) = \sin\left(\frac{3\pi x}{5}\right)$ is

- A. 3
- B. $3/5$
- C. $10/3$
- D. $6/5$
- E. 5

Not-so-quick Question

If $\log_a(5^a) = \frac{a}{3}$, then $a =$

- A. 5
- B. 15
- C. 25
- D. 125
- E. None of these

Limits

- The concept of limit underlies all of calculus
- Derivatives, integrals and series are all different kinds of limits

First some notation and a few basic facts.

Let f be a function, and let a and L be fixed numbers.

Then $\lim_{x \rightarrow a} f(x) = L$ is read

“The limit of $f(x)$ as x approaches a is L ”

You probably have an intuitive idea of what this means.

And we can do examples:

For many functions. . .

. . . and many values of a , it is true that

$$\lim_{x \rightarrow a} f(x) = f(a)$$

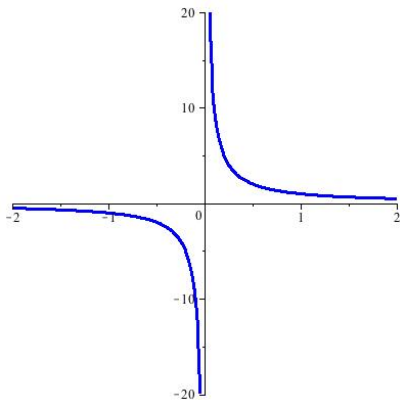
And it is usually apparent when this is not true.

“Interesting” things happen when $f(a)$ is not well-defined, or there is something “singular” about f at a .

Top ten famous limits

1. $\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$

$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$



2. $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$

$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$

Top ten limits

3. If $0 < x < 1$ then $\lim_{n \rightarrow \infty} x^n = 0$

If $x > 1$ then $\lim_{n \rightarrow \infty} x^n = \infty$

4. $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ and $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$.

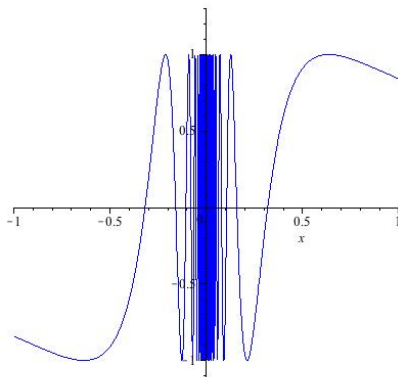
5. $\lim_{x \rightarrow -\infty} e^x = 0$ and $\lim_{x \rightarrow \infty} e^x = \infty$.

Top ten limits

6. For *any* value of n , $\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0$

and for any *positive* value of n , $\lim_{x \rightarrow \infty} \frac{\ln x}{x^n} = 0$.

7. $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ *does not exist*



Top ten limits

$$8. \lim_{x \rightarrow 0^+} x \ln x = 0.$$

$$9. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

10. If $f(x)$ is differentiable at $x = a$, then

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a).$$

Basic properties of limits

I. Arithmetic of limits:

If both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist, then

$$\lim_{x \rightarrow a} f(x) \pm g(x) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

$$\lim_{x \rightarrow a} f(x)g(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

and if $\lim_{x \rightarrow a} g(x) \neq 0$ then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

II. Two-sided and one-sided limits:

$\lim_{x \rightarrow a} f(x) = L$ if and only if *both* $\lim_{x \rightarrow a+} f(x) = L$ and $\lim_{x \rightarrow a-} f(x) = L$

III. Monotonicity:

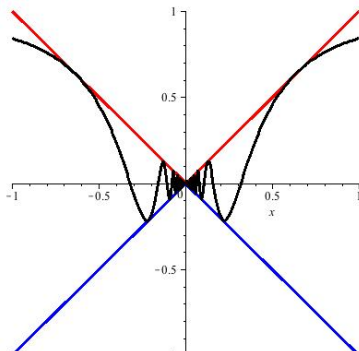
If $f(x) \leq g(x)$ for all x near a , then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$.

Basic properties of limits

IV. Squeeze theorem:

If $f(x) \leq g(x) \leq h(x)$ for all x near a , and if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x)$, then $\lim_{x \rightarrow a} g(x)$ exists and is equal to the common value of the other two limits.

Example: $-|x| \leq x \sin \frac{1}{x} \leq |x| \implies \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0.$



Let's work through a few:

1. $\lim_{x \rightarrow 2} \frac{x+5}{x+2}$

2. $\lim_{x \rightarrow -2} \frac{x+5}{x+2}$

3. $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x+2}$

Now you try this one:

$$\lim_{t \rightarrow 0} \frac{\sqrt{2-t} - \sqrt{2}}{t} =$$

A. 0

E. -1

B. ∞

F. $\sqrt{2}$

C. $-\frac{1}{2}$

G. -2

D. $\frac{1}{2\sqrt{2}}$

H. $-\frac{1}{2\sqrt{2}}$

A function is **continuous at $x = a$** if it is true that

$$\lim_{x \rightarrow a} f(x) = f(a).$$

The existence of both the limit and of $f(a)$ is implicit here.

Functions that are continuous at every point on an interval are called “*continuous on the interval*”

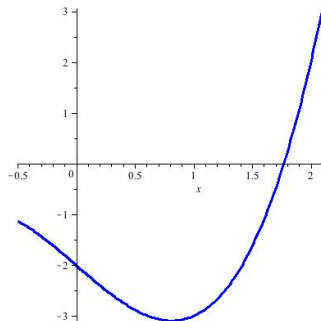
Intermediate value theorem

The most important property of continuous functions is the “common sense” **Intermediate Value Theorem**:

Suppose f is continuous on the interval $[a, b]$, and $f(a) = m$ and $f(b) = M$, with $m < M$. Then for any number p between m and M there is a solution in $[a, b]$ of the equation $f(x) = p$.

Application of the intermediate-value theorem

Consider the function $f(x) = x^3 - 2x - 2$ on the interval $[0, 2]$.



Since $f(0) = -2$ and $f(2) = +2$, there must be a root of $f(x) = 0$ in between $x = 0$ and $x = 2$.

A naive way to look for it is the “*bisection method*” — try the number halfway between the two closest places you know of where f has opposite signs.

$$f(x) = x^3 - 2x - 2$$

We know that $f(0) = -2$ and $f(2) = 2$, so there is a root in between. Choose the halfway point, $x = 1$.

Since $f(1) = -3 < 0$, we now know (and of course we could already see from the graph) that there is a root between $x = 1$ and $x = 2$. So try halfway between again:

$$f(1.5) = -1.625$$

So the root is between $x = 1.5$ and $x = 2$. Try $x = 1.75$:

$$f(1.75) = -0.140625$$

$$f(x) = x^3 - 2x - 2$$

We had $f(1.75) < 0$ and $f(2) > 0$. So the root is between $x = 1.75$ and $x = 2$. Try the average, $x = 1.875$:

$$f(1.875) = 0.841796875$$

f is positive here, so the root is between $x = 1.75$ and $x = 1.875$. Try their average, $x = 1.8125$:

$$f(1.8125) = 0.329345703$$

So the root is between $x = 1.75$ and $x = 1.8125$. One more:

$$f(1.78125) = 0.089141846$$

So now we know the root is between $x = 1.75$ and $x = 1.8125$.

You could write a computer program to continue this process to any desired accuracy.

Let's discuss it:

- 1 What, in a few words, is the derivative of a function?
- 2 What are some things you learn about the graph of a function from its derivative?
- 3 What are some applications of the derivative?
- 4 What is a differential? What does $dy = f'(x)dx$ mean?

Derivatives (continued)

Derivatives give a comparison between the rates of change of two variables:

When x changes by so much, then y changes by so much times the amount x changes by.

Derivatives are like “exchange rates”:

7/15/15: 1 Euro = 1.0950 Dollars

7/16/15: 1 Euro = 1.0875 Dollars

Definition of the derivative:

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Common derivative formulas:

$$\frac{d}{dx}(x^p) = px^{p-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg}{dx} + \frac{df}{dx}g(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

Let's do some examples ...

Derivative question #1

Find $f'(1)$ if $f(x) = \sqrt[5]{x} + \frac{1}{x^{9/5}}$.

A. $1/5$

E. $-1/5$

B. $2/5$

F. $4/5$

C. $-8/5$

G. $8/5$

D. $-2/5$

H. $-4/5$

Derivative question #2

Find the equation of the line tangent to $y = \frac{8}{\sqrt{4+3x}}$ at the point $(4, 2)$.

A. $6x + y = 26$

E. $5x + 21y = 62$

B. $4x + 2y = 20$

F. $4x + 15y = 46$

C. $3x - 4y = 4$

G. $3x + 16y = 44$

D. $7x + 18y = 64$

H. $2x - y = 6$

Derivative question #3

Calculate $\frac{d^2f}{dx^2}$ if $f(x) = \frac{e^x}{x}$.

A. $\frac{e^x(x+4)}{x^4}$

B. $\frac{e^x(x^2-1)}{x^4}$

C. $\frac{e^x(x^2+x)}{x^4}$

D. $\frac{e^x(x^2+3)}{x^4}$

E. $\frac{e^x(x-2)}{x^3}$

F. $\frac{e^x(x^2+5x)}{x^3}$

G. $\frac{e^x(x^2-2x+2)}{x^3}$

H. $\frac{e^x(x^3-4x^2+3)}{x^3}$

Derivative question #4

What is the largest interval on which the function $f(x) = \frac{x}{x^2 + 1}$ is **concave upward**?

A. $(0, 1)$

E. $(1, \sqrt{3})$

B. $(1, 2)$

F. $(\sqrt{3}, \infty)$

C. $(1, \infty)$

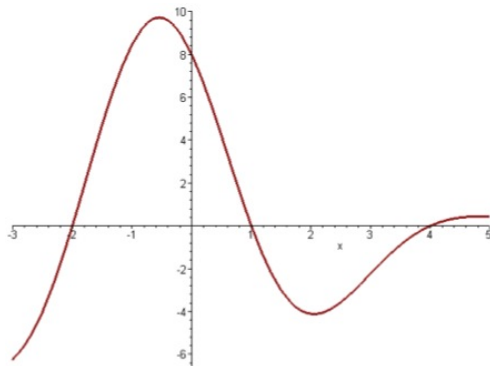
G. $(\sqrt{2}, \infty)$

D. $(0, \infty)$

H. $(\frac{1}{2}, \infty)$

Discussion

Here is the graph of a function. Draw a graph of its derivative.



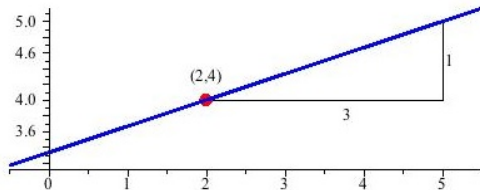
The meaning and uses of derivatives, in particular:

- ① The idea of linear approximation
 - ② How second derivatives are related to quadratic functions
 - ③ Together, these two ideas help solve max/min problems
- The derivative and second derivative provide us with a way of comparing other functions with (and approximating them by) linear and quadratic functions.
 - Before you can do that, though, you need to understand linear and quadratic functions

Linear functions

Linear functions of one variable in the plane are determined by one point and one number (the slope):

$$y = 4 + \frac{1}{3}(x - 2)$$



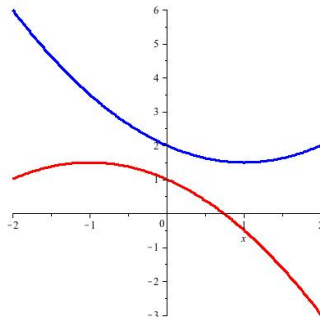
- Linear functions occur in calculus as differential approximations to more complicated functions (or as “first-order Taylor polynomials”)

$$f(x) \approx f(a) + f'(a)(x - a).$$

Quadratic functions

Quadratic functions have parabolas as their graphs:

$$y = \frac{x^2}{2} - x + 2 = \frac{1}{2}(x-1)^2 + \frac{3}{2} \quad y = -\frac{x^2}{2} - x + 1 = -\frac{1}{2}(x+1)^2 + \frac{3}{2}$$



Quadratic functions occur as “second-order Taylor polynomials”:

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Quadratic approximations

Near a *critical point* of f , in other words a point $x = a$ where $f'(a) = 0$, the quadratic approximation

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

becomes

$$f(x) \approx f(a) + \frac{f''(a)}{2}(x - a)^2$$

so that the point $(a, f(a))$ is a relative maximum or minimum according to the sign of $f''(a)$:

- If $f''(a) > 0$ then $(a, f(a))$ is a **relative minimum**, and if $f''(a) < 0$ then $(a, f(a))$ is a **relative maximum**.

Max and min

Recall that to find the maximum and minimum values of a function on any interval, we should look at three kinds of points:

- 1 The *critical points* of the function. These are the points where the derivative of the function is equal to zero.
- 2 The places where the derivative of the function fails to exist (sometimes these are called critical points as well)
- 3 The endpoints of the interval. If the interval is unbounded, this means paying attention to

$$\lim_{x \rightarrow \infty} f(x) \quad \text{and/or} \quad \lim_{x \rightarrow -\infty} f(x).$$

Position, velocity and acceleration

An application of first and second derivatives in physics: You know that if $y = f(t)$ represents the position of an object moving along a line, then $v = f'(t)$ is its velocity and $a = f''(t)$ is its acceleration.

Example: For falling objects (near the surface of the earth, and neglecting air resistance), $y = y_0 + v_0 t - 16t^2$ is the height (in feet) of the object at time t (seconds), where y_0 is the initial height (at time $t = 0$) of the object, and v_0 is its initial velocity.

Related rates

“Related rates” problems are applications of the chain rule. This is one of the big ideas that makes calculus important.

If you know how fast a quantity z changes when y changes (in other words, $\frac{dz}{dy}$) and you also know how fast y changes when x changes (in other words, $\frac{dy}{dx}$), then you know how fast z changes when x changes:

$$\frac{dz}{dx} = \frac{dz}{dy} \frac{dy}{dx}.$$

Implicit differentiation

It is also valuable to remember the idea of *implicit differentiation* — or differentiating with respect to a variable that isn't there —

The derivative of $f(y)$ **with respect to** x is $f'(y)\frac{dy}{dx}$.

The idea is that “**differentiating both sides of an equation with respect to x** ” (or any other variable) is a legal (and useful!) operation.

This is best illustrated via examples. . .

Related Rates Greatest Hits

1. A light is at the top of a 16-ft pole. A boy 5 ft tall walks away from the pole at a rate of 4 ft/sec. At what rate is the tip of his shadow moving when he is 18 ft from the pole? At what rate is the length of his shadow increasing?
2. A man on a dock is pulling in a boat by means of a rope attached to the bow of the boat 1 ft above the water level and passing through a simple pulley located on the dock 8 ft above water level. If he pulls in the rope at a rate of 2 ft/sec, how fast is the boat approaching the dock when the bow of the boat is 25 ft from a point on the water directly below the pulley?

Related Rates Greatest Hits

3. A weather balloon is rising vertically at a rate of 2 ft/sec . An observer is situated 100 yds from a point on the ground directly below the balloon. At what rate is the distance between the balloon and the observer changing when the altitude of the balloon is 500 ft ?
4. The ends of a water trough 8 ft long are equilateral triangles whose sides are 2 ft long. If water is being pumped into the trough at a rate of 5 cu ft/min , find the rate at which the water level is rising when the depth is 8 in .
5. Gas is escaping from a spherical balloon at a rate of 10 cu ft/hr . At what rate is the radius changing when the volume is 400 cu ft ?