Note: I did the wrong problem in section 10.9 at first (#16 instead of #10) So #10 appears at the end (but before 10.10 #58)

Sec 10.8 p630 #22

1

$f(x) = \frac{x^2}{x+1}$

Recall $\frac{1}{1-\square} = 1 + \square + \square^2 + \square^3 + \cdots$ Let $\square = -x$,

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$

$\frac{x^2}{1+x} = x^2(1 - x + x^2 - x^3 + \cdots)$

$= x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=2}^{\infty} (-1)^n x^n.$

Sec 10.8 p630 #24

$f(x) = 2x^3 + x^2 + 3x - 8$ $a=1$ $f(1) = -2$

$f'(x) = 6x^2 + 2x + 3$ $f'(1) = 11$

$f''(x) = 12x + 2$ $f''(1) = 14$

$f'''(x) = 12$ $f'''(1) = 12$

$f^{(n)}(x) = 0$ for $n \geq 4$ $f^{(n)}(1) = 0, n \geq 4$

So $f(x) = -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3.$

Sec 10.8 p630 #32

$f(x) = \sqrt{x+1} = (x+1)^{1/2}$, $a=0$

$f(x) = (x+1)^{1/2}$ $f(0) = 1$

$f'(x) = \frac{1}{2}(x+1)^{-1/2}$ $f'(0) = 1/2$

$f''(x) = \frac{-1}{2^2}(x+1)^{-3/2}$ $f''(0) = -1/2^2$

$f'''(x) = +\frac{1 \cdot 3}{2^3}(x+1)^{-5/2}$ $f'''(0) = \frac{1 \cdot 3}{2^3}$

$f''''(x) = -\frac{1 \cdot 3 \cdot 5}{2^4}(x+1)^{-7/2}$ $f''''(0) = \frac{-1 \cdot 3 \cdot 5}{2^4}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n+1} 3 \cdot 5 \cdots (2n-3)}{2^n}(x-1)^{\frac{-(2n-1)}{2}}$

$f^{(n)}(0) = \frac{(-1)^{n+1} 3 \cdot 5 \cdots (2n-3)}{2^n}$

($n=0,1$ do not fit this pattern)

$\Rightarrow \sqrt{x+1} = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdots (2n-3)}{2^n\, n!} x^n$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

Sec 10.8 p630 #8

$f(x) = \tan x$ $a = \pi/4$ $f(\frac{\pi}{4}) = 1$

$f'(x) = \sec^2(x)$ $f'(\frac{\pi}{4}) = 2$

$f''(x) = 2\sec^2(x)\tan x$ $f''(\frac{\pi}{4}) = 4$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$ $f'''(\frac{\pi}{4}) = 16$

degree 0: $P_0 = 1$

degree 1: $P_1 = 1 + 2(x - \frac{\pi}{4})$

degree 2: $P_2 = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2$

degree 3: $P_3 = 1 + 2(x - \frac{\pi}{4}) + 2(x - \frac{\pi}{4})^2 + \frac{8}{3}(x - \frac{\pi}{4})^3$

Sec 10.8 p630 #12

$f(x) = xe^x$; $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

So $xe^x = x(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$

$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$

Sec 10.9 p637 #8

arctan$(3x^4)$.

we know

$$\arctan \square = \square - \frac{\square^3}{3} + \frac{\square^5}{5} - \cdots$$

So

$$\arctan(3x^4) = 3x^4 - \frac{(3x^4)^3}{3} + \frac{(3x^4)^5}{5} - \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n 3^{2n+1} \frac{x^{8n+4}}{2n+1}$$

Sec 10.9 p637 #16

$x^2 \cos x^2$

$$\cos \square = 1 - \frac{\square^2}{2!} + \frac{\square^4}{4!} - \cdots$$

so

$$\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$$

and

$$x^2 \cos x^2 = x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!}$$

Sec 10.9 p637 #20

$x \ln(1+2x)$

$$\ln(1+\square) = \square - \frac{\square^2}{2} + \frac{\square^3}{3} - \frac{\square^4}{4} + \cdots$$

$$\ln(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots$$

$$x \ln(1+2x) = 2x^2 - \frac{2^2 x^3}{2} + \frac{2^3 x^4}{3} - \frac{2^4 x^5}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n x^{n+1}}{n}$$

Sec 10.9 p637 #30

$$\frac{\ln(1+x)}{1-x} = (1+x+x^2+x^3+x^4+\cdots)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$= x + (1-\tfrac{1}{2})x^2 + (1 - \tfrac{1}{2} + \tfrac{1}{3})x^3 + (1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4})x^4 + \cdots$$

$$= x + \tfrac{1}{2}x^2 + \tfrac{5}{6}x^3 + \tfrac{7}{12}x^4 + \cdots$$

Sec 10.9 p637 #36

$R_4 = \dfrac{M (\frac{1}{2})^5}{5!}$ where M is the max of e^x on $[0, \frac{1}{2}]$

We know $e < 3$ so $e^x < e^{1/2} < 3^{1/2} < 1.75$ on $[0, \frac{1}{2}]$

An estimate of the error is thus:

$$R_4 < \frac{1.75}{2^5 \cdot 5!} = \frac{7}{4 \cdot 32 \cdot 120} = \frac{7}{128 \cdot 120} \approx 0.00046$$

Sec 10.9 p637 #40

Error estimate is $R_1 < \dfrac{M (0.01)^2}{2!}$ where M

is the max of $\left|\dfrac{d^2}{dx^2}\sqrt{1+x}\right|$ on $[-0.01, 0.01]$

$$\frac{d}{dx}(1+x)^{1/2} = \tfrac{1}{2}(1+x)^{-1/2}, \quad \frac{d^2}{dx^2}(1+x)^{1/2} = -\tfrac{1}{4}(1+x)^{-3/2}$$

Max of this occurs when $x = -0.01$: $\frac{1}{4}(0.99)^{-3/2}$

So $R_1 < \dfrac{\frac{1}{4}(0.99)^{-3/2}(0.01)^2}{2}$ Now $(0.99)^{-3/2} < (0.81)^{-3/2} = (0.9)^{-3} = \dfrac{1}{0.729} < 1.5$

So $R_1 < \dfrac{1.5\,(0.01)^2}{8}$

$= 0.00001875$

Sec 10.9 p637 #50

(a) First, note the Taylor expansion of $\sin x$ around $a = \pi$ is

$$\sin x = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \cdots$$

So if P is near π we have

$\sin P = -(P-\pi) + \cdots$ and error estimate (alternating series) is

$$\left|\sin P - (-(P-\pi))\right| < \left|\frac{(P-\pi)^3}{3!}\right|$$

ie $$\left|(\sin P + P) - \pi\right| < \left|\frac{(P-\pi)^3}{6}\right|$$

If $|P-\pi| < 10^{-n}$, then

$$|P + \sin P - \pi| < \tfrac{1}{6}\, 10^{-3n}$$

ie 3 times as many correct decimals

b) Try with $n=2$:

$3.14 + \sin(3.14) =$

$3.14 + 0.001592653\ldots$

$= 3.141592653\ldots$

$\pi = 3.14159265\,4\ldots$!!

3

Sec 10.10 p645 #8

$(1+x^2)^{-1/3}$

Binomial

$$(1+\square)^{-1/3} = 1 - \tfrac{1}{3}\square + \frac{\frac{1}{3}\cdot\frac{4}{3}}{2!}\square^2 - \frac{\frac{1}{3}\cdot\frac{4}{3}\cdot\frac{7}{3}}{3!}\square^3 + \cdots$$

$$(1+x^2)^{-1/3} = 1 - \tfrac{1}{3}x^2 + \frac{1\cdot 4}{3^2\cdot 2!}x^4 - \frac{1\cdot 4\cdot 7}{3^3\cdot 3!}x^6 + \cdots$$

Sec 10.10 p645 #16

$$\int_0^{0.4} \frac{e^{-x}-1}{x}\,dx = \int_0^{0.4} \frac{1}{x}\left(-x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)dx$$

$$= \int_0^{0.4}\left(-1 + \frac{x}{2!} - \frac{x^2}{3!} + \cdots\right)dx = -x + \frac{x^2}{2\cdot 2!} - \frac{x^3}{3\cdot 3!} + \frac{x^4}{4\cdot 4!} + \cdots \Big|_0^{0.4}$$

Now $\frac{0.4^5}{5\cdot 5!} \approx 0.000017$ (too big), $\frac{0.4^6}{6\cdot 6!} \approx 9.5\times 10^{-7} < 10^{-5}$ (small enough)

So to the nearest 0.00001, the integral is

$$-0.4 + \frac{0.4^2}{2\cdot 2!} - \frac{0.4^3}{3\cdot 3!} + \frac{0.4^4}{4\cdot 4!} - \frac{0.4^5}{5\cdot 5!}$$

$$= -0.4 + 0.04 - 0.003555\ldots + 0.0002666\ldots - 0.0000170666\ldots$$

$$= -0.3633059555\ldots$$

Sec 10.10 p645, #20

$$\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1} 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \cdots \Big|_0^{0.1}$$

Now $\frac{(0.1)^7}{7\cdot 3!} = \frac{10^{-7}}{42} < 10^{-8}$

so to within 10^{-8}, integral is

$$0.1 - \frac{0.1^3}{3} + \frac{0.1^5}{10} = \begin{array}{r} 0.100010 \\ -0.000333 \\ \hline 0.099677 \end{array}$$

Sec 10.10 p645 #24

The next term in the approximation is $\int_0^1 \frac{t^4}{8!}dt = \frac{1}{5\cdot 8!}$ The error is less than this because the series is alternating

Sec 10.10 p645, #34

$$\lim_{y\to 0} \frac{\arctan y - \sin y}{y^3\cos y}$$

$$= \lim_{y\to 0} \frac{\left(\cancel{y} - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right) - \left(\cancel{y} - \frac{y^3}{6} + \frac{y^5}{120} - \cdots\right)}{y^3\left(1 - \frac{y^2}{2} + \cdots\right)}$$

$$= \lim_{y\to 0} \frac{y^3\left(-\frac{1}{6} + \cdots\right)}{y^3(1 - \cdots)} = -\frac{1}{6}.$$

Sec 10.10 p645 #42

$$\left(\tfrac{1}{4}\right)^3 + \left(\tfrac{1}{4}\right)^4 + \left(\tfrac{1}{4}\right)^5 + \left(\tfrac{1}{4}\right)^6 + \cdots$$

geometric — 1st term is $\frac{1}{64}$, $r = 1/4$

$$\text{Sum} = \frac{1/64}{1-1/4} = \frac{1/64}{3/4} = \frac{1}{48}$$

Sec 10.10 p645 #46

$$\frac{2}{3} - \frac{2^3}{3^3\cdot 3} + \frac{2^5}{3^5\cdot 5} - \frac{2^7}{3^7\cdot 7}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(2/3)^{2n+1}}{2n+1}$$

$$= \arctan 2/3$$

Sec 10.9 p637 #10

$$\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2}\ \frac{1}{1-\boxed{\frac{x}{2}}}$$

but $\frac{1}{1-\square} = 1 + \square + \square^2 + \cdots$

So

$$\frac{1}{2-x} = \frac{1}{2}\left[1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \cdots\right]$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$$

Sec 10.10 p645 # 58

(a) Let $f(x) = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \binom{m}{3}x^3 + \cdots$

Then $f'(x) = \sum_{k=1}^{\infty} k\binom{m}{k} x^{k-1} = \binom{m}{1} + 2\binom{m}{2}x + 3\binom{m}{3}x^2 + 4\binom{m}{4}x^3 + \cdots$

$= \sum_{k=1}^{\infty} m\binom{m-1}{k-1} x^{k-1} = m\binom{m-1}{0} + m\binom{m-1}{1}x + m\binom{m-1}{2}x^2 + m\binom{m-1}{3}x^3 + \cdots$

(For instance $4\binom{m}{4} = 4\dfrac{m(m-1)(m-2)(m-3)}{4\cdot 3\cdot 2\cdot 1} = \dfrac{m[(m-1)(m-2)(m-3)]}{3\cdot 2\cdot 1} = m\binom{m-1}{3}$).

So $(1+x)f'(x) = \sum_{k=1}^{\infty} m\binom{m-1}{k-1}(x^{k-1} + x^k) = m(1+x) + m\binom{m-1}{1}(x+x^2) + m\binom{m-1}{2}(x^2+x^3) + \cdots$

$= m\left[1 + \left(1 + \binom{m-1}{1}\right)x + \left(\binom{m-1}{1} + \binom{m-1}{2}\right)x^2 + \left(\binom{m-1}{2} + \binom{m-1}{3}\right)x^3 + \cdots\right]$

$\Rightarrow (1+x)f'(x) = m\left(1 + \sum_{k=1}^{\infty}\left(\binom{m-1}{k-1} + \binom{m-1}{k}\right)x^k\right)$

But $\binom{m-1}{k-1} + \binom{m-1}{k} = \binom{m}{k}$, because $\binom{m-1}{k-1} + \binom{m-1}{k} = \dfrac{(m-1)\cdots(m-k+1)}{(k-1)!} + \dfrac{(m-1)(m-2)\cdots(m-k)}{k!}$

$= \dfrac{(m-1)\cdots(m-k+1)}{(k-1)!}\left[1 + \dfrac{m-k}{k}\right]$

$= \dfrac{(m-1)\cdots(m-k+1)}{(k-1)!}\left[\dfrac{m}{k}\right] = \dfrac{m(m-1)\cdots(m-k+1)}{k!} = \binom{m}{k}$

$\Rightarrow (1+x)f'(x) = m\left(1 + \sum_{k=1}^{\infty}\binom{m}{k}x^k\right) = mf(x).$

$\Rightarrow f'(x) = \dfrac{mf(x)}{1+x}.$

(b) Let $g(x) = (1+x)^{-m}f(x)$. Then $g'(x) = -m(1+x)^{-m-1}f(x) + (1+x)^{-m}f'(x)$

$= -m(1+x)^{-m-1}f(x) + (1+x)^{-m}\dfrac{mf(x)}{1+x}$

$= m(1+x)^{-m-1}f(x)\left[-1+1\right] = 0.$

So $g(x)$ is a <u>constant</u>

(c) $g(0) = (1+0)^{-m}f(0) = 1 \Rightarrow g(x)$ is the constant 1.

$\Rightarrow (1+x)^{-m}f(x) = 1 \Rightarrow f(x) = (1+x)^m$ ✓