Sec 10.5 p609 # 4

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$$

RATIO TEST:

$$\lim_{n\to\infty} \frac{2^{n+2}}{(n+1)3^{n}} \cdot \frac{n3^{n-1}}{2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{2}{3}\frac{n}{n+1} = \frac{2}{3} < 1$$

So series converges by Ratio test.

Sec 10.7 p624 # 56

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

a) $$\frac{d}{dx}e^x = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= e^x \ !!$$

b) $$\int e^x dx = \int 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots dx$$

$$= C + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= (C-1) + e^x.$$

c) $$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$e^x \cdot e^{-x} = 1 + (1-1)x + (\tfrac{1}{2} - 1 + \tfrac{1}{2})x^2$$
$$+ (\tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{6})x^3$$
$$+ (\tfrac{1}{24} - \tfrac{1}{6} + \tfrac{1}{4} - \tfrac{1}{6} + \tfrac{1}{24})x^4$$
$$+ (\tfrac{1}{120} - \tfrac{1}{24} + \tfrac{1}{12} - \tfrac{1}{12} + \tfrac{1}{24} - \tfrac{1}{120})x^5$$
$$+ \cdots$$

$$= 1 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5 + \cdots$$

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Sec 10.5 p609 #8

$$\sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$$

RATIO TEST:

$$\lim_{n\to\infty} \frac{(n+1)5^{n+1}}{(2n+5)\ln(n+2)} \frac{(2n+3)\ln(n+1)}{n5^n}$$

$$= \lim_{n\to\infty} \frac{n+1}{n}\cdot\frac{2n+3}{2n+5}\cdot\frac{\ln(n+1)}{\ln(n+2)}\cdot\frac{5^{n+1}}{5^n}$$

$$= 1\cdot 1\cdot 1\cdot 5 > 1$$

$\Rightarrow$ series diverges by R.T.

Sec 10.5 p609 #14

$$\sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right)$$

ROOT TEST:

$$\lim_{n\to\infty}\left(\sin^n\frac{1}{\sqrt{n}}\right)^{1/n} = \lim_{n\to\infty}\sin\left(\frac{1}{\sqrt{n}}\right)$$

$$= \sin 0 = 0 < 1$$

$\Rightarrow$ Series converges by root test

Sec 10.5 p609 #20

$$\sum_{n=1}^{\infty}\frac{n!}{10^n}$$ RATIO TEST:

$$\lim_{n\to\infty}\frac{(n+1)!}{10^{n+1}}\cdot\frac{10^n}{n!} = \lim_{n\to\infty}\frac{(n+1)!}{n!}\frac{10^n}{10^{n+1}}$$

$$= \lim_{n\to\infty}(n+1)10 = \infty > 1$$

$\Rightarrow$ Series diverges by ratio test

Sec 10.5 p609 #26

$$\sum_{n=1}^{\infty}\left(1-\frac{1}{3n}\right)^n$$ n^{th} term test

$$\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^n = e^{-1/3} \neq 0$$

(Use "Famous limit #5: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$)

So series diverges by nth term test

Sec 10.5 p609 #30

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n^2}\right)^n$$ Root test

$$\lim_{n\to\infty}\left(\left(\frac{1}{n}-\frac{1}{n^2}\right)^n\right)^{1/n} = \lim_{n\to\infty}\frac{1}{n}-\frac{1}{n^2} = 0-0 = 0 <$$

So series converges by root test

Sec 10.5 p609 #38

$$\sum_{n=1}^{\infty}\frac{n!}{(-n)^n}$$ RATIO TEST:

$$\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \lim_{n\to\infty}\frac{(n+1)\,n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n} =$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$ (Famous limit #5 again)

So series converges absolutely by ratio test

Sec 10.5 p609 #48

$a_1 = 3 \qquad a_n = \frac{n}{n+1}a_n$

$n=1$: $a_2 = \frac{1}{2}a_1 = \frac{3}{2}$

$n=2$: $a_3 = \frac{2}{3}a_2 = \frac{3}{3}$

$n=3$: $a_4 = \frac{3}{4}a_3 = \frac{3}{4}$

$n=4$: $a_5 = \frac{4}{5}a_4 = \frac{3}{5}$

in general: $a_n = \frac{3}{n}$

So series is $\sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}\frac{3}{n}$ which is the divergent harmonic series.

Sec 10.5 p609 #56

$$\sum_{n=1}^{\infty}\frac{(3n)!}{n!(n+1)!(n+2)!}$$ RATIO TEST.

$$\lim_{n\to\infty}\frac{(3(n+1))!}{(n+1)!(n+2)!(n+3)!}\cdot\frac{n!(n+1)!(n+2)!}{(3n)!}$$

$$= \lim_{n\to\infty}\frac{(3n+3)(3n+2)(3n+1)}{(n+3)(n+2)(n+1)} = 3\cdot 3\cdot 3 = 27 >$$

So series diverges by RATIO test.

Sec 10.5 p609 # 58

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^n}{n^{n^2}} = \sum_{n=1}^{\infty}\left(\frac{-n!}{n^n}\right)^n$$

Root test: ← n^{th} root of $|n^{th}$ term$|$

$$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1}{n \quad n \quad n \quad n\ n\ n}$$

$$= \boxed{\lim_{n\to\infty} \underbrace{\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n}\cdots\frac{3}{n}\cdot\frac{2}{n}}_{\text{each of these factors is } \le 1}\cdot\frac{1}{n} \le \lim_{n\to\infty}\frac{1}{n} = 0}$$

But $a_n \geq 0$ so $\lim_{n\to\infty} a_n^{1/n} \geq 0 \Rightarrow$

$\lim_{n\to\infty} a_n^{1/n} = 0$ and root test implies convergence
(<1) (absolutely)

Sec 10.6 p615 #10

$$\sum_{n=2}^{\infty} (-1)^{n+1}\frac{1}{\ln n}$$ Since $\frac{1}{\ln n} > 0$ for $n > 1$,

$\ln n$ increases (so $\frac{1}{\ln n}$ decreases) and

$\lim_{n\to\infty} \frac{1}{\ln n} = 0$, Series converges by

A.S. test

Sec 10.6 p615 #20

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n!}{2^n}$$ RATIO TEST:

$$\lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}}\cdot\frac{2^n}{n!} = \lim_{n\to\infty}\frac{n+1}{2} = \infty > 1$$

Since ratio > 1, terms don't go to zero

so fails nth term test and series diverges

Sec 10.6 p615 #30

$$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n-\ln n}$$ Since $\lim_{n\to\infty}\frac{\ln n}{n} = 0$,

we have $\lim_{n\to\infty}\frac{\ln n}{n-\ln n} = 0$

so series converges by A.S. test.

But $\frac{\ln n}{n-\ln n} > \frac{1}{n}$ and $\sum \frac{1}{n}$ diverges, so

$\sum \frac{\ln n}{n - \ln n}$ diverges by comparison. $\Rightarrow \sum (-1)^n \frac{\ln n}{n-\ln n}$

converges conditionally

Sec 10.6 p615 # 36

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}.$$ Terms $\to 0$ so

series converges by A.S. test. But

$\sum \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{\cos n\pi}{n}$ converges conditionally.

Sec 10.6 p615 #42

$$\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n^2+n} - n\right)$$

$$\lim_{n\to\infty} \frac{(\sqrt{n^2+n}-n)(\sqrt{n^2+n}+n)}{\sqrt{n^2+n}+n} = \lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}+n}$$

$$= 1/2 \neq 0$$

So series fails n^{th} term test

and diverges

Sec 10.6 p615 # 50

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{10^n} = \frac{1}{10} - \frac{1}{100} + \frac{1}{1000} - \frac{1}{1000} + \cdots$$

by A.S., error is less than 1^{st} omitted

term, ie Error $< \frac{1}{100000} = 0.00001$

Sec 10.6 p615 # 54

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n}{n^2+1}$$ (Conditionally convergent)

For error < 0.001, need the first

omitted term to be less than 0.001.

So solve $\frac{n}{n^2+1} < 0.001$ ie $0 < n^2 - 1000n + 1$

Note $n = 999$ doesn't work but

$n = 1000$ does.

So the 1000^{th} term should be the

first omitted term and you need

999 terms to get error < 0.001

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Sec 10.7 p624 #6

$\sum_{n=0}^{\infty} (2x)^n$ geometric with

$r=2x$, so series converges (absolutely) for $-1<2x<1$, ie $-\frac{1}{2}<x<\frac{1}{2}$ and diverges otherwise

(interval of conv: $(-\frac{1}{2},\frac{1}{2})$, radius $=\frac{1}{2}$)

Sec 10.7 p624 #10

$\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ RATIO TEST:

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{(x-1)^n}\right| = \lim_{n\to\infty}|x-1|\frac{\sqrt{n}}{\sqrt{n+1}}$$

$= |x-1| < 1$ means $0<x<2$

At $x=0$: $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges conditionally (A.S. test, p-series $p=1/2$)

At $x=2$: $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges

So interval of convergence is $[0,2)$

radius of convergence is 1.

Sec 10.7 p624 #14

$\sum_{n=1}^{\infty}\frac{(x-1)^n}{n^3 3^n}$: RATIO TEST:

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(n+1)^3 3^{n+1}}\cdot\frac{n^3 3^n}{(x-1)^n}\right| = \lim_{n\to\infty}\frac{|x-1|}{3}\frac{n^3}{(n+1)^3}$$

$= \frac{|x-1|}{3} < 1$ means $-2<x<4$

At $x=-2$: $\sum\frac{(-3)^n}{n^3 3^n} = \sum\frac{(-1)^n}{n^3}$

converges absolutely by A.S. test, p-series $p=3$

At $x=4$: $\sum\frac{3^n}{n^3 3^n} = \sum\frac{1}{n^3}$ converges (abs.)

So interval of convergence is $[-2,4]$.

Radius of convergence is 3

Sec 10.7 p624 #28

$\sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^n$ RATIO TEST:

$$\lim_{n\to\infty}\left|\frac{(-2)^{n+1}(n+2)(x-1)^{n+1}}{(-2)^n(n+1)(x-1)^n}\right| = \lim_{n\to\infty} 2|x-1|\frac{n+2}{n+1}$$

$= 2|x-1| < 1$ means $\frac{1}{2}<x<\frac{3}{2}$

At $x=\frac{1}{2}$: $\sum_{n=0}^{\infty}(n+1)$ diverges (nth term test)

At $x=\frac{3}{2}$: $\sum_{n=0}^{\infty}(-1)^n(x-1)^n$ diverges

So interval of convergence is $(\frac{1}{2},\frac{3}{2})$

Radius of convergence is $1/2$

Sec 10.7 p624 #34

$\sum_{n=1}^{\infty}\frac{3\cdot5\cdot7\cdots(2n+1)}{n^2 2^n}x^{n+1}$ RATIO TEST:

$$\lim_{n\to\infty}\left|\frac{3\cdot5\cdot7\cdots(2n+1)(2n+3)\,x^{n+2}}{(n+1)^2\,2^{n+1}}\cdot\frac{n^2\cdot2^n}{3\cdot5\cdot7\cdots(2n+1)\,x^{n+1}}\right|$$

$$= \lim_{n\to\infty}\frac{(2n+3)|x|}{2}\frac{n^2}{(n+1)^2} = \infty$$

So the series converges only for $x=0$

Radius of convergence is 0.

Sec 10.7 p624 #38

$\sum_{n=1}^{\infty}\left(\frac{2\cdot4\cdot6\cdots(2n)}{2\cdot5\cdot8\cdots(3n-1)}\right)^2 x^n$ RATIO TEST

$$\lim_{n\to\infty}\left|\left(\frac{2\cdot4\cdot6\cdots(2n)(2n+2)}{2\cdot5\cdot8\cdots(3n-1)(3n+2)}\right)^2\frac{x^{n+1}}{x^n}\left(\frac{2\cdot5\cdot8\cdots(3n-1)}{2\cdot4\cdot6\cdots(2n)}\right)^2\right|$$

$$= \lim_{n\to\infty}\frac{(2n+2)^2}{(3n+2)^2}|x| = \frac{4}{9}|x|$$

so series converges (at least) for $-\frac{9}{4}<x<\frac{9}{4}$

and radius of convergence is $\frac{9}{4}$.

Sec 10.7 p624 #46

$\sum_{n=0}^{\infty}(\ln x)^n$: geometric series, $a=1$, $r=\ln x$

converges to $\frac{1}{1-\ln x}$ provided $-1<\ln x<1$,

i.e. for $\frac{1}{e}<x<e$.