7.4 p452 # 4

Faster than x^2:

e: $x^3 - x^2$

g. $(1.1)^x$

Slower than x^2.

c. $x^2 e^{-x}$

d $\log_{10}(x^2)$

f. $\left(\frac{1}{10}\right)^x$

Same rate:

a. $x^2 + \sqrt{x}$

b. $10x^2$

h. $x^2 + 100x$

7.4 p452 #8

As $x \to \infty$.

$$\underset{c}{(\ln 2)^x} < \underset{b}{x^2} < \underset{a}{2^x} < \underset{d}{e^x}$$

7.4 p452 #10 As $x \to \infty$

a. $\frac{1}{x+3} = \Theta\left(\frac{1}{x}\right)$ True.

b. $\frac{1}{x} + \frac{1}{x^2} = \Theta\left(\frac{1}{x}\right)$ True

c. $\frac{1}{x} - \frac{1}{x^2} = o\left(\frac{1}{x}\right)$ False

d. $2 + \cos x = \Theta(2)$ True

e. $e^x + x = \Theta(e^x)$ True

f. $x \ln x = o(x^2)$ true

g. $\ln(\ln x) = \Theta(\ln x)$ True

(in fact $\ln(\ln x) = o(\ln x)$)

h. $\ln x = o(\ln(x^2+1))$ False

(but $\ln x = \Theta(\ln(x^2+1))$)

10.1 p581 #18

$$\frac{-3}{2}, -\frac{1}{6}, \frac{1}{12}, \frac{3}{20}, \frac{5}{30}, \ldots$$

$$\frac{-3}{1\cdot 2}, \frac{-1}{2\cdot 3}, \frac{1}{3\cdot 4}, \frac{3}{4\cdot 5}, \frac{5}{5\cdot 6} \ldots$$

$$a_n = \frac{2n-5}{n(n+1)}$$

10.1 p581 #36

$$a_n = (-1)^n\left(1-\frac{1}{n}\right).$$

$\lim_{n\to\infty} 1-\frac{1}{n} = 1$, but the even-numbered terms approach $+1$ while the odd-numbered terms approach -1. $\Rightarrow$ the sequence diverges

10.1 p581 #48

$$a_n = \frac{3^n}{n^3} \qquad \lim_{n\to\infty} \frac{3^n}{n^3} \overset{L'H}{=} \lim_{n\to\infty} \frac{3^n \ln 3}{3n^2}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{3^n (\ln 3)^2}{6n}$$

$$\overset{L'H}{=} \lim_{n\to\infty} \frac{3^n (\ln 3)^3}{6} = \infty$$

$\Rightarrow$ the sequence diverges

10.1 p581 #54

$$a_n = \left(1-\frac{1}{n}\right)^n.$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{n(\ln(1-\frac{1}{n}))} = \lim_{n\to\infty} e^{\frac{\ln(1-\frac{1}{n})}{1/n}}$$

$$\overset{L'H}{=} \lim_{n\to\infty} e^{\frac{1/n^2}{1-1/n} / -\frac{1}{n^2}} = \lim_{n\to\infty} e^{-1/(1-\frac{1}{n})} = e^{-1}$$

The sequence converges to $1/e$

10.1 p581 #72

$$a_n = \left(1-1/n^2\right)^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{n\ln(1-\frac{1}{n^2})}$$

$$= \lim_{n\to\infty} e^{\frac{\ln(1-\frac{1}{n^2})}{1/n}}$$

$$\overset{L'H}{=} \lim_{n\to\infty} e^{\frac{-2/n^3}{1-\frac{1}{n^2}} / -1/n^2}$$

$$= \lim_{n\to\infty} e^{-\frac{2}{n(1-\frac{1}{n^2})}} = e^0 = 1$$

$\Rightarrow$ The sequence converges to 1.

10.1 p581 #84

$$a_n = \sqrt[n]{n^2+n} = (n^2+n)^{1/n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\frac{\ln(n^2+n)}{n}}$$

$$\overset{L'H}{=} \lim_{n\to\infty} e^{\frac{2n+1}{n^2+n}/1} = e^0 = 1.$$

The sequence converges to 1

10.1 p581 #88

$$a_n = \frac{1}{\sqrt{n^2-1} - \sqrt{n^2+n}}$$

$$= \frac{\sqrt{n^2+1} + \sqrt{n^2+n}}{(\sqrt{n^2-1} - \sqrt{n^2+n})(\sqrt{n^2-1} + \sqrt{n^2+n})}$$

$$= \frac{\sqrt{n^2+1} + \sqrt{n^2+n}}{n^2-1-(n^2+n)}$$

$$= \frac{-(\sqrt{n^2+1} + \sqrt{n^2+n})}{n+1}$$

$$\lim_{n\to\infty} a_n \overset{L'H}{=} \lim_{n\to\infty} -\left(\frac{n}{\sqrt{n^2+1}} + \frac{n+1/2}{\sqrt{n^2+n}}\right)$$

$$= 2.$$

The sequence converges to 2.

10.1 p581 #94

$a_1 = 0, \; a_{n+1} = \sqrt{8 + 2a_n}$

Let $\lim\limits_{n\to\infty} a_n = L$ then

$$L = \sqrt{8+2L} \Rightarrow L^2 = 8 + 2L$$

$$L^2 - 2L - 8 = 0, \; (L-4)(L+2) = 0$$

so either $L = 4$ or $L = -2$. But all the terms are positive so L can't be -2. Thus $\lim\limits_{n\to\infty} a_n = 4$.

10.1 p581 #98

$a_{n+1} = \sqrt{1 + a_n}$ If $\lim\limits_{n\to\infty} a_n = L$

then $L = \sqrt{1+L} \Rightarrow L^2 = 1 + L$

$$\Rightarrow L^2 - L - 1 = 0 \Rightarrow L = \frac{1 \pm \sqrt{1+4}}{2}$$

$L = \frac{1 \pm \sqrt{5}}{2}$, but L can't be negative,

so $L = \frac{1 + \sqrt{5}}{2}$.

10.2 p591 #12

$$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n}\right) = \sum_{n=0}^{\infty} \frac{5}{2^n} - \sum_{n=0}^{\infty} \frac{1}{3^n}$$

geometric $a = 5$, $r = \frac{1}{2}$ geometric $a = 1$, $r = 1/3$

$$\text{Sum} = \frac{5}{1-\frac{1}{2}} - \frac{1}{1 - 1/3} = 10 - \frac{3}{2} = \frac{17}{2}.$$

10.2 p591 #24

$$1.\overline{414} = 1 + \sum_{n=1}^{\infty} \frac{414}{1000^n}$$

$\to$ geometric $a = \frac{414}{1000}$, $r = \frac{1}{1000}$

$$= 1 + \frac{\frac{414}{1000}}{1 - \frac{1}{1000}} = 1 + \frac{414}{999}$$

$$= \frac{1413}{999}$$

10.2 p591 #30

$\sum\limits_{n=1}^{\infty} \frac{n}{n^2+3}$ $\lim\limits_{n\to\infty} \frac{n}{n^2+3} = 0$, so

n^{th} term test is inconclusive.

(Series diverges, though, by the integral test e.g.)

10.2 p591 #34

$$\sum_{n=0}^{\infty} \cos \pi n = \sum_{n=0}^{\infty} (-1)^n$$

$\lim\limits_{n\to\infty} (-1)^n$ does not exist, so this series diverges by the n^{th} term test

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10.2 p591 #38

$$\sum_{n=1}^{\infty} (\tan n - \tan(n-1))$$

$= (\tan 1 - \tan 0) + (\tan 2 - \tan 1) + \cdots$

The n^{th} partial sum is $\tan n$.

But $\tan n$ does not have a limit as $n \to \infty$, so the series diverges.

10.2 p591 #54

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}$$ geometric with $a = 1$, $r = -1/5$

Since $|r| < 1$, series converges to $\frac{1}{1-(-1/5)} = \frac{5}{6}$.

Sec 10.2 p591 #60

$\sum_{n=1}^{\infty} (1-\frac{1}{n})^n$

From #54 of sec 10.1, we know $\lim_{n\to\infty}(1-\frac{1}{n})^n = \frac{1}{e} \neq 0$

so series diverges by the n^{th} term test

Sec 10.2 p591 #76

$\sum_{n=0}^{\infty} (-\frac{1}{2})^n (x-3)^n$

Geometric with $a=1$ and $r = -\frac{1}{2}(x-3)$.

Series will converge if

$|r| = |-\frac{1}{2}(x-3)| = \frac{1}{2}|x-3| < 1$

ie $|x-3|<2$ or $1 < x < 5$

Sum: $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}(x-3)} = \frac{2}{5-x}$

Sec 10.2 p591 #86

Since $\sum a_n$ converges we know that $a_n \to 0$ as $n\to\infty$. But then $\frac{1}{a_n} \to \infty$ as $n\to\infty$ so the series $\sum \frac{1}{a_n}$ diverges by the n^{th} term test

Sec 10.2 p591 #96

a) Each time every side of the figure is replaced by four sides, each of which has $\frac{1}{3}$ the length of the original side. So $L_1 = 3$, $L_2 = \frac{4}{3}\cdot 3$, $L_3 = (\frac{4}{3})^2 \cdot 3$

etc. $L_n = (\frac{4}{3})^{n-1}\cdot 3 \to \infty$ as $n\to\infty$.

Note: the number of sides of C_n is $3\cdot 4^{n-1}$.

Sec 10.3 p598 #4 $\sum_{n=1}^{\infty} \frac{1}{n+4}$

terms are positive and decreasing,

$\int_1^{\infty} \frac{1}{x+4}dx = \ln(x+4)\Big|_1^{\infty} = \infty$ so series diverges

Sec 10.3 p598 #6

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ terms are positive and decreasing

$\int_2^{\infty} \frac{1}{x(\ln x)^2}dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}du = -\frac{1}{u}\Big|_{\ln 2}^{\infty} = \ln 2$

$u = \ln x$

$du = 1/x \, dx$

$\Rightarrow$ series converges

Sec 10.3 p598 #16

$\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = -\underbrace{\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}}$ this series has positive decreasing terms

$\int_1^{\infty} \frac{2}{x^{3/2}}dx = -\frac{4}{x^{1/2}}\Big|_1^{\infty} = 4 \Rightarrow$ series converges

Sec 10.3 p598 #22

$\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$ $\lim_{n\to\infty} \frac{5^n}{4^n+3} \overset{L'H}{=} \lim_{n\to\infty} \frac{5^n \ln 5}{4^n \ln 4} = \infty$

Series diverges by n^{th} term test.

Sec 10.3 p598 #30 $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$ Since $\ln 3 > 1$, this is a geometric series with $|r|<1$, so it converges (to $\frac{1/\ln 3}{1-1/\ln 3} = \frac{1}{\ln 3 - 1}$).

b) Each time, a triangle of area $\frac{A_1}{9^n}$ is added on each side, so the first time $3\cdot\frac{A_1}{9}$, then $3\cdot 4\cdot\frac{A_1}{9^2}$, then $3\cdot 4^2\cdot\frac{A_1}{9^3}$ etc. So the total area is $A_1(1 + \frac{3}{9} + \frac{3\cdot 4}{9^2} + \frac{3\cdot 4^2}{9^3} + \cdots)$

$= A_1(1 + \sum_{n=0}^{\infty} \frac{1}{3}(\frac{4}{9})^n)$

$= A_1(1 + \frac{1/3}{1-4/9}) = A_1(1+\frac{3}{5}) = \frac{8}{5}A_1$

Sec 10.3 p598 # 34

$\sum_{n=1}^{\infty} n \tan\frac{1}{n}$

$\lim_{n\to\infty} n\tan\frac{1}{n} = \lim_{n\to\infty} \frac{\tan\frac{1}{n}}{1/n}$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{-1/n^2 \sec^2 1/n}{-1/n^2}$

$= \lim_{n\to\infty} \sec^2 1/n = \sec 0 = 1 \neq 0$

So series diverges by nth term test

Sec 10.3 p598 # 38

$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ terms $\to 0$, decrease

so $\int_1^{\infty} \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln(x^2+1)\Big|_1^{\infty} = \infty$

$\Rightarrow$ series diverges

Sec 10.3 p598 #60

a) the sum is bounded as follows:

$$\sum_{K=1}^{10}\frac{1}{K^4} + \int_{11}^{\infty}\frac{1}{x^4}dx < \sum_{K=1}^{\infty}\frac{1}{K^4} < \sum_{K=1}^{10}\frac{1}{K^4} + \int_{10}^{\infty}\frac{1}{x^4}dx$$

$$\sum_{K=1}^{10}\frac{1}{K^4} + \frac{1}{3\cdot 11^3} < \sum_{K=1}^{\infty}\frac{1}{K^4} < \sum_{K=1}^{10}\frac{1}{K^4} + \frac{1}{3\cdot 10^3}$$

$$\begin{array}{c}1.082036583\\ +0.000250438\end{array} < \sum_{K=1}^{\infty}\frac{1}{K^4} < \begin{array}{c}1.082036583\\ +0.0003333\ldots\end{array}$$

$$1.082287 < \sum_{K=1}^{\infty}\frac{1}{K^4} < 1.082370$$

b) the difference between the two bounds is about 0.000083, so their average, namely 1.082328468 is off by less than half of this, or 0.0000415

Sec 10.4 p603 #4

$\sum_{n=}^{\infty} \frac{n+2}{2n^2-n}$. Well, $\frac{n+2}{n^2-n} > \frac{n}{n^2} = \frac{1}{n}$,

and $\sum \frac{1}{n}$ is the (divergent) harmonic series.

So $\sum \frac{n+2}{n^2-n}$ diverges by the comparison test.

Sec 10.4 p603 #6

$\sum_{n=1}^{\infty} \frac{1}{n3^n}$. Well, $\frac{1}{n3^n} < \frac{1}{3^n}$, and

$\sum \frac{1}{3^n}$ is a convergent geometric series ($r = \frac{1}{3} < 1$).

So $\sum \frac{1}{n3^n}$ converges by the comparison test

Sec 10.4 p603 #12

$\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$ (Regular comparison: $\frac{2^n}{3+4^n} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n$, so this converges by comparison to Meg Ryan series)

Limit compare with $\sum \frac{2^n}{4^n} = \sum \frac{1}{2^n}$.

$$\lim_{n\to\infty} \frac{\frac{2^n}{3+4^n}}{2^n/4^n} = \lim_{n\to\infty} \frac{4^n}{3+4^n} = 1$$

So two series do the same thing (namely converge since $\sum \frac{1}{2^n}$ converges)

~~Sec 10.4~~ p603 #16

$\sum \ln\left(1+\frac{1}{n^2}\right)$

Limit comp. to $\sum \frac{1}{n^2}$ (convergent p-series $p = 2 > 1$)

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{-2}{n^3}\,\frac{1}{1+\frac{1}{n^2}}}{-2/n^3}$$

$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$ So $\sum \ln\left(1+\frac{1}{n^2}\right)$

converges by LCT with $\sum \frac{1}{n^2}$.

Sec 10.4 p603 # 22

$\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$ n^{th} term is like $\frac{n}{n^2\sqrt{n}} = \frac{1}{n^{3/2}}$

Limit comparison with $\sum \frac{1}{n^{3/2}}$ (convergent p-series $p=3/2>1$)

$$\lim_{n\to\infty} \frac{\frac{n+1}{n^2\sqrt{n}}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

so our series converges by L.C.T.

Sec 10.4 p603 # 28

$\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ Since $\ln n < \sqrt{n}$,

compare to $\sum \frac{1}{n^2}$ (convergent p-series $p=2$)

$$\frac{(\ln n)^2}{n^3} < \frac{(\sqrt{n})^2}{n^3} = \frac{1}{n^2}.$$

So $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges by C.T.

Sec 10.4 p603 # 34

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ like $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

So compare to $\sum \frac{1}{n^{3/2}}$ (convergent p-series, $p=3/2$)

$$\frac{\sqrt{n}}{n^2+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

So $\sum \frac{\sqrt{n}}{n^2+1}$ converges by C.T.

Sec 10.4 p603 # 40

$\sum \frac{2^n+3^n}{3^n+4^n}$ compare to $\sum \frac{2^n+3^n}{4^n}$ ← sum of two convergent geo. series $r=\frac{2}{4}, \frac{3}{4}$

$$\frac{2^n+3^n}{3^n+4^n} < \frac{2^n+3^n}{4^n} \text{ so}$$

$\sum \frac{2^n+3^n}{3^n+4^n}$ converges by C.T.

Sec 10.4 p603 #46

$\sum_{n=1}^{\infty} \tan\frac{1}{n}$ For small x, $\tan x \simeq x$

so limit compare to $\sum \frac{1}{n}$ (divergent harmonic series)

$$\lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{-\frac{1}{n^2}\sec^2\frac{1}{n}}{-\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \sec^2\frac{1}{n} = 1$$

So $\sum \tan\frac{1}{n}$ diverges by LCT with $\sum \frac{1}{n}$

Sec 10.4 p603 # 52

$$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{2-1/n}}$$

Limit-compare to $\sum \frac{1}{n^2}$ (convergent p-series $p=2$)

$$\lim_{n\to\infty} \frac{\frac{1}{n^{2-1/n}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^{2-1/n}} = \lim_{n\to\infty} n^{1/n}$$

$$= \lim_{n\to\infty} e^{\frac{\ln n}{n}} \overset{L'H}{=} \lim_{n\to\infty} e^{\frac{1}{n}} = 1$$

So $\sum \frac{\sqrt[n]{n}}{n^2}$ converges by L.C.T.

Sec 10.4 p603 # 60

$a_n > 0$ and $\lim_{n\to\infty} n^2 a_n = 0$

So for large n we have $n^2 a_n < 1$

or $a_n < \frac{1}{n^2}$. So compare $\sum a_n$

with $\sum \frac{1}{n^2}$, which converges,

so $\sum a_n$ converges, too.