Solution outlines for Gressman midterm 3

1. The graphs of $y = x$ and $y = x^2$ intersect at $x = 0$ and $x = 1$. The area of the region is thus

$$A = \int_0^1 x - x^2\, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

and so

$$\overline{y} = 6 \cdot \frac{1}{2} \int_0^1 x^2 - x^4 = 3\left(\frac{1}{3} - \frac{1}{5}\right) = 1 - \frac{3}{5} = \frac{2}{5}$$

(B)

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2. The ratio test gives

$$\lim \frac{2|x+3|(n+1)^2}{3(n+2)^2} = \frac{2}{3}|x+3|$$

For the ratio test to guarantee convergence, we need $\frac{2}{3}|x+3| < 1$, or $-\frac{3}{2} < x + 3 < \frac{3}{2}$ which gives $-\frac{9}{2} < x < -\frac{3}{2}$.

At the endpoints, the absolute value of the nth term is $\dfrac{1}{(n+1)^2}$, the nth term of a convergent p-series, so the series converges (absolutely) at both endpoints and the interval of convergence is $[-\frac{9}{2}, -\frac{3}{2}]$.
(B)

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3. If the series $\sum a_n$ converges then $a_n \to 0$, in which case $e^{a_n} \to 1$, and so the series $\sum e^{a_n}$ fails the nth term test and diverges.
(E)

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4. For $f(x)$ to be a pdf, we need $C > 0$ and

$$1 = \int_1^2 C\sqrt{x-1}\, dx = \frac{2}{3} C (x-1)^{3/2} \Big|_1^2 = \frac{2}{3} C$$

so $C = \frac{3}{2}$. The mean of f is:

$$\mu = \int_1^2 \frac{3}{2} x\sqrt{x-1}\, dx = \int_0^1 \frac{3}{2}(u+1)\sqrt{u}\, du = \frac{3}{2}\int_0^1 u^{3/2} + u^{1/2}\, du = \frac{3}{2}\left(\frac{2}{5} + \frac{2}{3}\right) = 1 + \frac{3}{5} = \frac{8}{5}.$$

(F)

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5. Rotate the vertical section at x with width dx to obtain a disk with radius $e^{-x/2}$. The volume is

$$V = \int \pi r^2 t = \int_0^{\ln 2} \pi e^{-x}\, dx = \left. -\pi e^{-x} \right|_0^{\ln 2} = \pi\left(-\frac{1}{2}+1\right) = \frac{\pi}{2}$$

(F)

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6. Use the substitution $u = \cos x$:

$$\int_0^{\pi} \sin^3 x \cos^4 x\, dx = \int_{-1}^{1} (1-u^2)u^4\, du = \left.\left(\frac{u^5}{5} - \frac{u^7}{7}\right)\right|_{-1}^{1} = \frac{4}{35}.$$

(D)

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7. This is a linear differential equation, in proper form it is

$$y' - \frac{2}{x}\, y = x^2$$

so $P = -\dfrac{2}{x}$ and $Q = x^2$, so $\int P = -2\ln x$ and $e^{\int P} = \dfrac{1}{x^2}$. Therefore

$$y = e^{-\int P} \int Q\, e^{\int P} = x^2 \int 1\, dx = x^2(x+C) = x^3 + Cx.$$

The condition $y(1) = 0$ tells us the $C = -1$, and so $y = x^3 - x^2$, and $y(3) = 27 - 9 = 18$.
(F)

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8. Partial fractions:

$$\int \frac{2}{y(y^2+1)}\, dy = \int \frac{2}{y} - \frac{2y}{y^2+1}\, dy = 2\ln y - \ln(y^2+1) + C = \ln\frac{y^2}{y^2+1} + C$$

(B)

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9. For the family of functions $y = Ce^{-\arctan x}$, we have $y' = -\dfrac{Ce^{-\arctan x}}{1+x^2}$. Divide the second of these equations by the first to eliminate C:

$$\frac{y'}{y} = -\frac{1}{1+x^2}$$

So

$$y' = -\frac{y}{1+x^2}$$

This is the differential equation for the original curves. The orthogonal trajectories have slopes (derivatives) that are the negative reciprocals of the originals:

$$y' = \frac{1+x^2}{y}$$

for the orthogonal trajectories, or

$$y\,dy = (1+x^2)\,dx$$

Integrate this to get

$$\frac{1}{2}y^2 = x + \frac{1}{3}x^3 + C$$

for the formula of the orthogonal trajectories.
(E) (although in some sense (B) is a formula for the O.T.s).

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10. Use the substitution $x = \sec\theta$ to write

$$\int_1^{\sqrt{2}} \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/4} \frac{\sec\theta\tan^2\theta}{\sec\theta}\,d\theta = \int_0^{\pi/4} \sec^2\theta - 1\,d\theta = \tan\theta - \theta\Big|_0^{\pi/4} = 1 - \frac{\pi}{4}$$

(A)

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11. Both series have positive terms so either they converge absolutely or else they diverge.

I. Limit comparison with (divergent) harmonic series:

$$\lim_{n\to\infty} \frac{1}{n+n^2e^{-n}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{1+ne^{-n}} = 1$$

so series I diverges.

II. Since $\dfrac{1}{n+n^2e^n} < \dfrac{1}{e^n}$, this series is less than the convergent geometric series $\sum e^{-n}$, hence converges absolutely.
(D)

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12. Around the y-axis, the radius is x. So we have

$$SA = \int 2\pi r\, ds = \int_1^2 2\pi x\sqrt{1+(y')^2}\, dx = \int_1^2 2\pi x\sqrt{1+\frac{1}{x^2}}\, dx = \int_1^2 2\pi\sqrt{x^2+1}\, dx$$

(A)