Solution outlines for Gressman midterm 2

1. Start by completing the square in the expression under the radical:

$$4x^2 - 8x + 13 = 4x^2 - 8x + 4 + 9 = (2x-2)^2 + 9$$

so we will make the substitution $u = 2x - 2$ to use the "fact":

$$\int_1^3 x - \sqrt{4x^2 - 8x + 13}\, dx = \left.\frac{x^2}{2}\right|_1^3 - \int_1^3 \sqrt{(2x-2)^2 + 9}\, dx$$
$$= 4 - \frac{1}{2}\int_0^4 \sqrt{u^2+9}\, du = 4 - \frac{20 + 9\ln 3}{4} = -\frac{4 + 9\ln 3}{4}$$

(B)

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2. The formula in the box at the bottom of page 498 of the textbook says that

$$|E_T| < \frac{M(b-a)^3}{12n^2}$$

where $M = \max|f''|$on $[a,b]$ and for this integral $a = -1$ and $b = 1$ and $f = e^{x^2-1} = \frac{1}{e} e^{x^2}$. So

$$f' = \frac{2x}{e} e^{x^2} \quad \text{and} \quad f'' = \frac{2_4 x^2}{e} e^{x^2}.$$

Since $2 + 4x^2 \leq 6$ on $[-1,1]$ and $e^{x^2} \leq e$ on $[-1,1]$, we have $|f''| < 6$ So the error formula tells us that

$$|E_T| < \frac{6 \cdot 2^3}{12n^2}$$

For this to be less than 10^{-6} we'll need $10^6 \cdot 2^2 \leq n^2$, i.e., $n > 2000$.
(D)

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3. I. We have

$$\int_0^\infty \frac{dt}{2e^t + 1} < \int_0^\infty \frac{dt}{2e^t} = \int_0^\infty \frac{1}{2} e^{-t}\, dt = \frac{1}{2}$$

so this integral is convergent.

II. Evaluate:

$$\int_0^1 \frac{dx}{x^2} = -\frac{1}{x}\bigg|_0^1$$

which diverges at $x = 0$.
(C)

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4. The condition $P(t \le X < \infty) = e^{-t/2}$ means that

$$\int_t^\infty f(s)\,ds = e^{-t/2}$$

Differentiate both sides to get $-f(t) = -\frac{1}{2}e^{-t/2}$ or $f(t) = \frac{1}{2}e^{-t/2}$. This is a probability distribution on $[0, \infty)$ because

$$\int_0^\infty \frac{1}{2}e^{-t/2}\,dt = e^{-t/2}\bigg|_0^\infty = 1.$$

For the mean we'll need to integrate by parts with $u = t$ and $dv = \frac{1}{2}e^{-t/2}dt$. We'll get

$$\mu = \int_0^\infty \frac{t}{2}e^{-t/2}\,dt = \; -te^{-t/2} - 2e^{-t/2}\bigg|_0^\infty = 2$$

The distribution is "memoryless" because the amount of time the center expects to have to wait is independent of how long it has already waited.
(B)

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5. We say that $\lim\limits_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$ there is an N such that for every $n > N$ we have $|a_n - L| < \varepsilon$. We expect that

$$\lim_{n\to\infty} \frac{n^2}{2n^2+1} = \frac{1}{2}$$

so we need find out for which n the quantity

$$\frac{1}{2} - \frac{n^2}{2n^2+1} = \frac{1}{2(2n^2+1)}$$

is less than $\frac{1}{402}$. For this we should have $402 < 2(2n^2+1)$ or $201 < 2n^2+1$, or $100 < n^2$ or $10 < n$. So $N = 10$ is the best we can do, so we much choose (E).

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6. Do this one in pieces:

$$\lim_{n\to\infty} \frac{n+1}{2n+\sin n} = \lim_{n\to\infty} \frac{1+\dfrac{1}{n}}{2+\dfrac{\sin n}{n}} = \frac{1}{2}$$

(after dividing numerator and denominator by n) For the second term use top ten limit number 9:

$$\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n = e^a$$

to conclude

$$\lim_{n\to\infty} \left(1-\frac{\ln 2}{n}\right)^n = e^{-\ln 2} = \frac{1}{2}$$

Therefore

$$\lim_{n\to\infty} \left(\frac{n+1}{2n+\sin n} - \left(1-\frac{\ln 2}{n}\right)^n\right) = \frac{1}{2} - \frac{1}{2} = 0$$

(C)

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7. Using the substitution $u = \ln x$,

$$\int_2^\infty \frac{1}{x\ln x}\,dx = \int_{\ln 2}^\infty \frac{du}{u} = \ln(\ln x)\Big|_2^\infty = \infty.$$

Since the integral diverges, so does the series.

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8. I. We'll limit-compare this series to $\sum e^{-n}$:

$$\lim_{n\to\infty} \frac{\dfrac{1}{e^n - e^{-n}}}{e^{-n}} = \lim_{n\to\infty} \frac{e^n}{e^n - e^{-n}} = \lim_{n\to\infty} \frac{1}{1-e^{-2n}} = 1$$

(after dividing numerator and denominator by e^n). Since $\sum e^{-n}$ is a convergent geometric series, we have that the given series converges as well.

II. From part I, this series converges absolutely.

So both series converge.

(A)

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9. The function $f(x)$ is certainly positive on $[1, 100]$, so we integrate:

$$\int_1^{100} \frac{1}{(\ln 100)x}\, dx = \left.\frac{\ln x}{\ln 100}\right|_1^{100} = 1.$$

For the median M, we need to find M so that

$$\int_1^{M} \frac{1}{(\ln 100)x}\, dx = \frac{1}{2}$$

which gives us

$$\frac{\ln M}{\ln 100} = \frac{1}{2} \qquad \text{or} \qquad 2\ln M = \ln 100$$

so $M = 10$. For the mean,

$$\mu = \int_1^{1} 00\frac{x}{\ln(100)x}\, dx = \frac{99}{\ln 100}$$

and since $\ln(100) < 9$ we have $\mu > 10$ so the mean is greater than the median.
(C)

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10. If the limit exists and equals L, then $L = \frac{1}{2}L(l+1)$, i.e., $L^2 - L = 0$ so either $L = 0$ or $L = 1$. Now, observe that if $0 < a_n < 1$, then

$$a_{n+1} = \frac{1}{2}a_n(a_n + 1) < \frac{1}{2}a_n(1+1) = a_n$$

so the sequence will be decreasing (and bounded below by 0), so it must converge to 0.

If $b_n = 1$, then $b_{n+1} = \frac{1}{2} \cdot 1 \cdot (1+1) = 1$. So the limit will be 1.

If $c_n > 1$, then

$$c_{n+1} = \frac{1}{2}c_n(c_n + 1) > \frac{1}{2}c_n(1+1) = c_n.$$

So c_n is an increasing sequence and can't have limit 0 or 1. Therefore $c_n \to \infty$.
(A)

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11. Since $0 < x < 1$, we have $|x^2 - 1| = 1 - x^2$ We can write:

$$\int_0^1 \ln|x^2 - 1|\, dx = \int_0^1 \ln(1 - x^2)\, dx = \int_0^1 \ln(1 - x)\, dx + \int_0^1 \ln(1 + x)\, dx$$

The second of these integrals is not improper and is thus finite. For the first, make the substitution $u = 1 - x$ and integrate by parts to see that

$$\int_0^1 \ln(1-x)\,dx = \int_0^1 \ln u\,du = u\ln u - u\Big|_0^1 = -1$$

so the entire integral converges. I'm not sure which of the (interestingly labeled) choices this implies. Probably the second (A).

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12. Since $\dfrac{n+1}{n} \to 1$, this series diverges by the nth term test.
(F)