Solution outlines for Gressman midterm 1

1. Surface area is given by the integral of $2\pi r\,ds$. Around the y-axis, we have $r = x$, and $ds = \sqrt{1+(y')^2}\,dx$. So

$$SA = \int_0^{\pi/2} 2\pi x\sqrt{1+\sin^2 x}\,dx$$

(D)

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2. Since the density is constant and the region is symmetric about the line $y = x$, we'll have that the centroid lies on that line, so $\overline{y} = \overline{x}$. We'll calculate $\overline{x}$.

The region is a quarter disk of radius 1, so its area is $\frac{1}{4}\pi$ (we don't have to integrate for this). Therefore

$$\overline{x} = \frac{4}{\pi}\int_0^1 x\sqrt{1-x^2}\,dx = \frac{4}{\pi}\int_0^1 \frac{1}{2}\sqrt{u}\,du = \frac{4}{3\pi}u^{3/2}\Big|_0^1 = \frac{4}{3\pi}$$

So the centroid is $\left(\dfrac{4}{3\pi}, \dfrac{4}{3\pi}\right)$. We know $\overline{x} < \frac{1}{2}$ since the region is taller toward the left than toward the right.
(A)

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3. Integrate by parts with $u = \arctan x$ and $dv = x\,dx$:

$$\int_0^1 x\arctan x\,dx = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int 1\frac{1}{1+x^2}\,dx$$
$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}$$

(C)

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4. Energy is the same as work, which is force times distance. The work required to lift something to height h from the ground is mgh, where m is its mass and g is the acceleration of gravity. So we subdivide the sphere into horizontal slabs, and we have to calculate the mass of the slab at each height h, multiply each by gh and add them all together.

The sphere of radius 1/2 centered at $(0, \frac{1}{2})$ has equation $x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$, or $x^2 = y - y^2$. The slab at height y and of thickness dy has volume $\pi x^2\, dy = \pi(y - y^2)\, dy$ and so its mass is $\pi\rho(y - y^2)\, dy$. The work required to raise that slab to height y is thus $\pi\rho g y(y - y^2)\, dy$. Therefore the total amount of work (energy) required to make the sphere is (at least):

$$E = \int_0^1 \pi\rho g(y^2 - y^3)\, dy = \pi\rho g\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi\rho g}{12}.$$

(B)

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5. We'll substitute $u = \sec x$:

$$\int \sec x \tan^3 x\, dx = \int \tan^2 x(\sec x \tan x)\, dx = \int (\sec^2 x - 1) d(\sec x) = \frac{1}{3}\sec^3 x - \sec x + C$$

(E)

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6. Use vertical sections – rotate a vertical section around the x-axis to get a washer with outer radius $y_1 = (1 - x^2)^{1/2}$, inner radius $y_2 = \left(\frac{x}{2}\right)^{1/2}(1 - x)$ and thickness dx. So the volume is

$$\begin{aligned} V &= \int \pi(y_1^2 - y_2^2)\, dx = \int_0^1 \pi\left(1 - x^2 - \frac{x}{2}(1 - x)^2\right)\, dx \\ &= \pi \int_0^1 1 - x^2 - \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right)\, dx \\ &= \pi\left(x - \frac{x^2}{3} - \frac{x^2}{4} + \frac{x^3}{3} - \frac{x^4}{8}\right)\Big|_0^1 \\ &= \pi\left(1 - \frac{1}{4} - \frac{1}{8}\right) = \frac{5\pi}{8} \end{aligned}$$

(E)

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7. Make the substitution $u = \ln|x|$ (so $du = \dfrac{dx}{x}$) to get

$$\int \frac{\cos\ln|x|}{x}\, dx = \int \cos u\, du = \sin u + C = \sin\ln|x| + C$$

(B)

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8. Use vertical sections – rotate a vertical section around the axis $x - 2$ to get a shell with radius $2 - x$, height $y_1 - y_2 = 1 - x$ (ha!) and thickness dx. The volume is then:

$$\begin{aligned} V &= \int 2\pi rht = \int_0^1 2\pi(2-x)(1-x)\,dx = 2\pi \int_0^1 2 - 3x + x^2\,dx \\ &= 2\pi \left(2x - \frac{3x^2}{2} + \frac{x^3}{3}\right)\Bigg|_0^1 = \pi\left(4 - 3 + \frac{2}{3}\right) = \frac{5\pi}{3} \end{aligned}$$

If the axis were $x = -1$, which is just as far from the left edge of the region as $x = 2$ is from the right edge, the volume would decrease, since the taller part of the region would go around smaller circles (and so be multiplied by a smaller radius).
(E)

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9. The form $x^2 + 9$ suggests the trig identity $1 + \tan^2\theta = \sec^2\theta$. Multiply this by 9 to get $9 + 9\tan^2\theta = 9\sec^2\theta$, so we want $x^2 = 9\tan^2\theta$, or $x = 3\tan\theta$. Use this substitution to get:

$$\begin{aligned} \int_0^4 \frac{dx}{(x^2+9)^{3/2}} &= \int_0^{\arctan(4/3)} \frac{3\sec^2\theta\,d\theta}{(9\sec^2\theta)^{3/2}} = \int_0^{\arctan(4/3)} \frac{3\sec^2\theta\,d\theta}{27\sec^3\theta} \\ &= \frac{1}{9}\int_0^{\arctan(4/3)} \cos\theta\,d\theta = \frac{1}{9}\sin\left(\arctan\frac{4}{3}\right) = \frac{1}{9}\cdot\frac{4}{5} = \frac{4}{45} \end{aligned}$$

(F)

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10. First do long division – the quotient is clearly 1 and the remainder is

$$x^2 - 4x + 5 - (x^2 - 3x + 2) = -(x-3)$$

so we have

$$\frac{x^2-4x+5}{x^2-3x+2} = 1 - \frac{x-3}{(x-1)(x-2)} = 1 - \left(\frac{2}{x-1} - \frac{1}{x-2}\right) = 1 - \frac{2}{x-1} + \frac{1}{x+2}$$

(B)

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11. We start with

$$y' = \frac{3x^2+3}{4} - \frac{1}{3(x^2+1}$$

so it's one of those! Therefore

$$L = \frac{x^3+3x}{4} + \frac{1}{3}\arctan x\Big|_0^1 = 1 + \frac{1}{3}\cdot\frac{\pi}{4} = \frac{\pi}{12} + 1$$

(A)

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12. First do partial fractions on the non-logarithm part of the integrand:

$$\frac{x^2+3x+1}{x(x+1)^2} = \frac{1}{x} + \frac{1}{(x+1)^2}.$$

So the integral becomes (we don't need the absolute value signs around the x because in the integral, x is between 1 and e so it is always positive):

$$\int_1^e \frac{(x^2+3x+1)\ln x}{x(x+1)^2}\,dx = \int_1^e \frac{\ln x}{x} + \frac{\ln x}{(x+1)^2}\,dx$$

The integral of the first term is clearly $\frac{1}{2}(\ln x)^2$ (after the substitution $u = \ln x$). For the second, integrate by parts with $u = \ln x$ and $dv = \dfrac{dx}{(x+1)^2}$ and get:

$$\begin{aligned}\int_1^e \frac{\ln x}{x} + \frac{\ln x}{(x+1)^2}\,dx &= \frac{1}{2}(\ln|x|)^2 - \frac{\ln x}{x+1} + \int \frac{dx}{x(x+1)} = \frac{1}{2}(\ln x)^2 - \frac{\ln x}{x+1} + \int \frac{1}{x} - \frac{1}{x+1}\,dx \\ &= \frac{1}{2}(\ln x)^2 - \frac{\ln x}{x+1} + \ln x - \ln(x+1)\Big|_1^e = \frac{1}{2} - \frac{1}{e+1} + 1 - \ln(e+1) + \ln 2 \\ &= \frac{3}{2} - \frac{1}{e+1} - \ln(e+1) + \ln 2\end{aligned}$$

(F)