A standard substitution

This week, we looked at the method of substitution (or u-substitution) for calculating integrals. Recall that the idea of subsitution is to "reverse" the chain rule in order to simplify integrals of certain products.

For instance, to calculate $\int x^3 \sin 3x^4\,dx$, you notice first that the integrand is a product, and that there are two factors: a simpler one, x^3, and a more complicated one, $\sin 3x^4$. Moreover, the simple one is (a constant times) the derivative of part of the complicated one: the derivative of $3x^4$ is $12x^3$. So we let $u = 3x^4$, and calculate its differential: $du = 12x^3\,dx$. In the integral, we can then substitute u for $3x^4$ in the sine function, and for $x^3\,dx$ we can substitute $\frac{1}{12}\,du$. The result is

$$\int x^3 \sin 3x^4\,dx = \int \frac{1}{12} \sin u\,du = -\frac{1}{12}\cos u + C = -\cos 3x^4 + C,$$

once we "un-substitute" $3x^4$ for u.

The standard substitution: The purpose of this note is to highlight a specific kind of substitution that comes up so often that it should become "second nature" to you. It occurs when you are trying to integrate a function (like exponential, sine or cosine, logarithm, square root, etc...) whose argument is a *linear* function of the variable, like $ax+b$. If you substitute $u = ax+b$ in this case, you will simplify the integral and *always* end up with a factor of $1/a$. Let's do a couple of examples to illustrate:

Example 1: $\int \cos 2x\,dx$. — The integrand is a function of $2x$, which is of the form $ax+b$ for $a = 2$ and $b = 0$. Make the substitution $u = 2x$. Then $du = 2dx$, or equivalently $dx = \frac{1}{2}du$. If we do the substitution we get that

$$\int \cos 2x\,dx = \int \frac{1}{2}\cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$$

— as promised, there is a factor of $1/a = 1/2$ in the answer.

Example 2: $\int e^{3x}\,dx$. — The integrand is a function of $3x$, so we're expecting a $1/3$ in the answer. Make the substitution $u = 3x$, Then $du = 3\,dx$ or equivalently $dx = \frac{1}{3}\,du$. Do the substitution and get

$$\int e^{3x}\,dx = \int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C.$$

Example 3: One more: $\int \sqrt{6x-3}\,dx$ – The integrand is a function of $6x-3$ – so we expect a factor of $1/6$. Make the substitution $u = 6x-3$, so we get $du = 6\,dx$, or equivalently $dx = \frac{1}{6} du$. Do the substitution and get

$$\int \sqrt{6x-3}\, dx = \int \frac{1}{6}\sqrt{u}\, du = \int \frac{1}{6} u^{1/2}\, du = \frac{1}{6} u^{3/2} \cdot \frac{2}{3} + C = \frac{1}{9} u^{3/2} + C = \frac{1}{9}(6x-3)^{3/2} + C.$$

Now you should try a couple: Calculate: $\int e^{-2x}\, dx$, $\int \sin\left(5x - \frac{\pi}{2}\right)\, dx$, $\int (4x-2)^{10}\, dx$.

So that you can check your understanding, I got $-\frac{1}{2}e^{-2x} + C$, $-\frac{1}{5}\cos(5x - \pi/2) + C$, and $\frac{1}{44}(4x-12)^{11} + C$ as the answers.