Solution to bonus problem 5 (based on the solutions by Manqing Liu and Sima Parekh)

The problem: Let $I = 3\sqrt{2}\int_0^x \frac{\sqrt{1+\cos t}}{17-8\cos t}\,dt$. If $0 < x < \pi$ and $\tan I = \frac{2}{\sqrt{3}}$, what is x?

Solution: Begin with the substitution $u = \cos t$, or $t = \arccos u$, so that $dt = -\frac{du}{\sqrt{1-u^2}}$. Then we have

$$\begin{aligned} I &= 3\sqrt{2}\int_0^x \frac{\sqrt{1+\cos t}}{17-8\cos t}\,dt = -3\sqrt{2}\int_1^{\cos x} \frac{\sqrt{1+u}}{17-8u}\,\frac{du}{\sqrt{1-u^2}} \\ &= -3\sqrt{2}\int_1^{\cos x} \frac{du}{(17-8u)\sqrt{1-u}} \end{aligned}$$

Note that we have converted the original integral, which was not improper, into an improper integral (but this is not going to cause any problem). Next, make the substitution $v = \sqrt{1-u}$, so $u = 1 - v^2$ and $-du = 2v\,dv$.Therefore:

$$\begin{aligned} I &= 6\sqrt{2}\int_0^{\sqrt{1-\cos x}} \frac{v\,dv}{(9+8v^2)v} = 6\sqrt{2}\int_0^{\sqrt{1-\cos x}} \frac{dv}{9+8v^2} \\ &= \frac{2\sqrt{2}}{3}\int_0^{\sqrt{1-\cos x}} \frac{dv}{1+\frac{8}{9}v^2} = \int_0^{\arctan\left(\frac{2\sqrt{2}}{3}\sqrt{1-\cos x}\right)} d\theta \\ &= \arctan\left(\frac{2\sqrt{2}}{3}\sqrt{1-\cos x}\right) \end{aligned}$$

after the trig substitution $\tan\theta = \frac{2\sqrt{2}\,v}{3}$, or $v = v = \frac{3}{2\sqrt{2}}\tan\theta$, so that $dv = \frac{3}{2\sqrt{2}}\sec^2\theta$. The reverse of the substitution (to get the limits of integration) is $\theta = \arctan\left(\frac{2\sqrt{2}\,v}{3}\right)$. So now we now the value of I. And so

$$\tan I = \frac{2\sqrt{2}}{3}\sqrt{1-\cos x} = \frac{2}{\sqrt{3}}$$

or

$$\sqrt{1-\cos x} = \frac{\sqrt{3}}{\sqrt{2}}$$

or

$$1 - \cos x = \frac{3}{2}.$$

Finally, we conclude that $\cos x = -\frac{1}{2}$, and since $0 < x < \pi$ we have that $x = \frac{2\pi}{3}$.