Solution to bonus problem 3 (based on the solutions by Yash Bhargava, Ridhi Vyas, Emma Lu and Antonio Canales)

The problem: Consider a square with side length L. Let R be the region inside the square consisting of all points that are closer to the center of the square than to any side of the square. What is the area of R?

Solution: As in the diagram below, put the center of the square at the origin, so that the top side of the square is part of the line $y = \frac{1}{2}L$. The distance from a point (x, y) inside the square to the line $y = \frac{1}{2}L$ is $\frac{1}{2}L - y$, so the point (x, y) inside the square is equidistant from the center of the circle and the top side of the square if

$$x^2 + y^2 = \left(\frac{L}{2} - y\right)^2$$

in other words, if

$$x^2 = \frac{L^2}{4} - Ly$$

or

$$y = \frac{L}{4} - \frac{x^2}{L}.$$

The part of the figure outlined in blue contains one-eighth of the area of R, and it

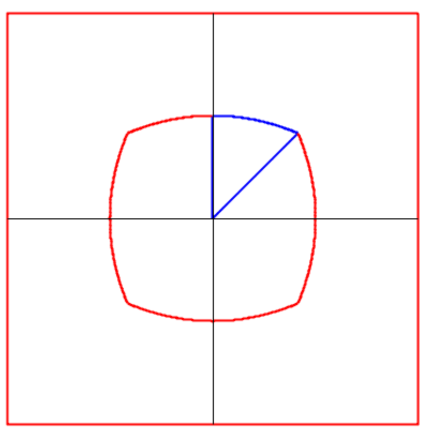

Figure 1: ***The region R is bounded by four parabolic arcs. One-eighth of R is outlined in blue.***

is bounded above by the parabola $y = L/4 - x^2/L$, on the left by the y-axis $x = 0$ and below by the line $y = x$. To find where the line $y = x$ intersects the parabola, we equate their right-hand sides and obtain

$$x^2 + Lx - \frac{L^2}{4} = 0$$

and so, using the quadratic formula, we obtain

$$x = \frac{-L \pm \sqrt{L^2 + L^2}}{2} = \frac{\sqrt{2} - 1}{2}\, L$$

(we can eliminate the root with the minus sign since that will clearly lie outside the square).

So the area of R is

$$\begin{aligned}
A &= 8 \int_0^{\frac{1}{2}(\sqrt{2}-1)L} \frac{L}{4} - \frac{x^2}{L} - x\, dx = \left(2Lx - \frac{8x^3}{3L} - 4x^2\right)\Bigg]_0^{\frac{1}{2}(\sqrt{2}-1)L} \\
&= L^2 \left[(\sqrt{2}-1) - \frac{1}{3}(\sqrt{2}-1)^3 - (\sqrt{2}-1)^2\right] \\
&= L^2(\sqrt{2}-1)\left(1 - \frac{(\sqrt{2}-1)^2}{3} - (\sqrt{2}-1)\right) \\
&= L^2(\sqrt{2}-1)\left(2 - \sqrt{2} - \frac{2 - 2\sqrt{2} + 1}{3}\right) \\
&= L^2(\sqrt{2}-1)\left(1 - \sqrt{2} + \frac{2\sqrt{2}}{3}\right) \\
&= \frac{L^2}{3}(\sqrt{2}-1)(3-\sqrt{2}) = \frac{L^2}{3}(4\sqrt{2}-5)
\end{aligned}$$