Solution to bonus problem 2 (based on the elegant solutions by Antonio Canales, Arun Kirubarajan, Arianna Sarnet, Manqing Liu and Yash Bhargava)

The problem: Let ABC be a right-angled triangle with angle $ABC = 90$ degrees (so the right angle is at vertex B), and let D be the point on side AB such that $AD = 2\,DB$. What is the maximum possible value of angle ACD? (*Hint*: This is a max-min problem, and the first hard thing is to figure out what the variables are!)

Solution: Let α be angle BCD, let β be angle ACD and let θ be angle ACB, so $\theta = \alpha + \beta$. The problem is to maximize β, where θ ranges over all angles between 0 and $\pi/2$.

Since we know that $AD = 2\,DB$, it is true that $AB = 3\,DB$. Therefore $\tan(\theta) = 3\tan(\alpha)$. And $\beta = \theta - \alpha$, so we need to maximize the function

$$\beta(\theta) = \theta - \arctan\left(\frac{\tan\theta}{3}\right).$$

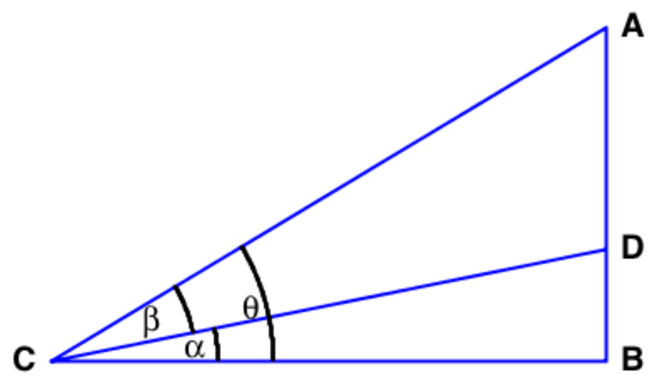

Figure 1: ***The triangle ABC with the point D***

Now we can calculate:

$$\frac{d\beta}{d\theta} = 1 - \frac{\frac{1}{3}\sec^2\theta}{1 + \frac{1}{9}\tan^2\theta} = 1 - \frac{\frac{1}{3} + \frac{1}{3}\tan^2\theta}{1 + \frac{1}{9}\tan^2\theta}.$$

If we set this derivative equal to zero, and multiply both sides by $1 + \frac{1}{9}\tan^2\theta$, we get the equation

$$0 = \frac{2}{3} - \frac{2}{9}\tan^2\theta$$

and so $\tan\theta = \pm\sqrt{3}$. We reject $-\sqrt{3}$ since we want $0 < \theta < \pi/2$, and so $\theta = \pi/3$.

The problem actually asks for the maximum value of β, so we have

$$\beta = \theta - \arctan\left(\frac{\tan\theta}{3}\right) = \frac{\pi}{3} - \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

Technically, since we have found only one critical point, we need to check the endpoints of the θ interval. Clearly $\beta(0) = 0$, but we have to take a limit to see what happens as $\theta \to \pi/2$ (since $\tan\theta \to \infty$ as $\theta \to \pi/2$). But since $\tan\theta \to \infty$ as $\theta \to \pi/2$, certainly $\frac{1}{3}\tan\theta \to \infty$ as well, and as its argument approaches infinity, the arctangent approaches $\pi/2$. Thus $\beta \to 0$ as $\theta \to \pi/2$, and we have completed the proof that $\pi/6$ is the maximum possible value of β, and so is the answer to the problem.