Solution to bonus problem 1 (based on solutions by Arun Kirubarajan and Manqing Liu, and some partial solutions by others, and with some additional commentary)

There are two situations to consider, the first (and easier) is where one of the numbers is zero and the other two are negatives of one another. The normal line to the parabola at $x = 0$ is just the y-axis, and the other two normal lines will meet on the y-axis by symmetry, proving the result in this case.

So we can assume that none of the three normal lines meet the parabola at $(0, 0)$. We'll be using this assumption without comment as we divide by the x-coordinates of the points.

First, calculate the equation of the normal line to the parabola at an arbitrary point, say the point has $x = a$ (with $a \neq 0$), so $y = a^2$.

Since the derivative of x^2 at $x = a$ is $2a$, the slope of the normal line at (a, a^2) is $-\dfrac{1}{2a}$. So the equation of the normal line at (a, a^2) is

$$y = a^2 - \frac{1}{2a}(x - a) = a^2 + \frac{1}{2} - \frac{x}{2a}.$$

The normal line through a second point (b, b^2) is thus

$$y = b^2 + \frac{1}{2} - \frac{x}{2b}$$

and to find the intersection of these two lines we solve

$$a^2 + \frac{1}{2} - \frac{x}{2a} = b^2 + \frac{1}{2} - \frac{x}{2b}$$

as follows: put the x terms on one side and the constants on the other:

$$\left(\frac{1}{2b} - \frac{1}{2a}\right) x = b^2 - a^2$$

then divide to get

$$x = \frac{b^2 - a^2}{\frac{1}{2b} - \frac{1}{2a}} = \frac{2ab(b^2 - a^2)}{a - b} = -2ab(a + b).$$

This is the x-coordinate of the intersection, and the y coordinate of the intersection is

$$y = b^2 + \frac{1}{2} - \frac{-2ab(a + b)}{2b} = b^2 + a^2 + ab + \frac{1}{2}.$$

Just as a check, we could substitute this point $(-2ab(a+b)\,,\, b^2 + a^2 + ab + \frac{1}{2})$ into the equation of the $x = a$ normal line, and we would get:

$$b^2 + a^2 + ab + \frac{1}{2} = a^2 + \frac{1}{2} - \frac{-2ab(a + b)}{2a} = a^2 + \frac{1}{2} + ab + b^2,$$

so the y-coordinate of the point of intersection, which we calculated by using the $x = b$ normal line, is correct.

Now, we want a third normal line, say at the point (c, c^2) to intersect the other two. This means we have to derive the condition on the value of c so that the the point of intersection of the $x = a$ and $x = b$ normal lines, namely

$$(-2ab(a+b)\,,\, b^2 + a^2 + ab + \frac{1}{2})$$

is on the normal line at $x = c$, namely

$$y = c^2 + \frac{1}{2} - \frac{x}{2c}.$$

This will be true if

$$b^2 + a^2 + ab + \frac{1}{2} = c^2 + \frac{1}{2} - \frac{-2ab(a+b)}{2c}$$

and we have to solve this last equation for c. Multiply by c to clear the denominators and obtain:

$$(a^2 + b^2 + ab)c = c^3 + ab(a+b).$$

This is a cubic equation for c, and we already know two of its roots, namely $c = a$ and $c = b$, because certainly the intersection point of the normal lines at $x = a$ and $x = b$ is on the normal lines at $x = a$ and $x = b$ (duh). But this means that $c - a$ and $c - b$ are factors of the cubic polynomial

$$c^3 - (a^2 + b^2 + ab)c + ab(a+b)$$

and we have to find the other factor.

Now from the statement of the problem, we suspect that that other root of the polynomial is $-(a+b)$, so that the other factor will be $(a+b+c)$, and we can check this:

$$\begin{aligned}(c-a)(c-b)(a+b+c) &= (c^2 - (a+b)c + ab)(a+b+c)\\ &= (c^2 - (a+b)c + ab)(c + (a+b))\\ &= c^3 + (a+b)c^2 - (a+b)c^2 - (a+b)^2c + abc + ab(a+b)\\ &= c^3 - (a^2 + b^2 + ab)c + ab(a+b)\end{aligned}$$

which verifies that the x-coordinates of the three points must add up to zero (as long as the three points are distinct). We could also have obtained the third factor by doing long division of the cubic polynomial $c^3 - (a^2 + b^2 + ab)c + ab(a+b)$ by the quadratic $(c-a)(c-b)$.

Since factorization of polynomials is unique, we have the statement is if and only if, in other words, the normals at three different points of the parabola all intersect at the same point if and only if the x-coordinates of the three points add up to zero.