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$C = 60$, so we have

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mentation that, in certain
d into another substance,

the time rate of change of the amount x of unconverted substance is proportional to x .

Let the amount of unconverted substance be known at some specified time; that is, let $x = x_0$ at $t = 0$. Then the amount x at any time $t > 0$ is determined by the differential equation

$$\frac{dx}{dt} = -kx \quad (1)$$

and the condition that $x = x_0$ when $t = 0$. Since the amount x is decreasing as time increases, the constant of proportionality in equation (1) is taken to be $(-k)$.

From equation (1) it follows that

$$x = C e^{-kt}.$$

But $x = x_0$ when $t = 0$. Hence $C = x_0$. Thus we have the result

$$x = x_0 e^{-kt}. \quad (2)$$

Let us now add another condition, which will enable us to determine k . Suppose it is known that at the end of half a minute, at $t = 30$ (sec), two-thirds of the original amount x_0 has already been converted. Let us determine how much unconverted substance remains at $t = 60$ (sec).

When two-thirds of the substance has been converted, one-third remains unconverted. Hence $x = \frac{1}{3}x_0$ when $t = 30$. Equation (2) now yields the relation

$$\frac{1}{3}x_0 = x_0 e^{-30k}$$

from which k is easily found to be $\frac{1}{30} \ln 3$. Then with t measured in seconds, the amount of unconverted substance is given by the equation

$$x = x_0 \exp\left(-\frac{1}{30}t \ln 3\right). \quad (3)$$

At $t = 60$,

$$x = x_0 \exp(-2 \ln 3) = x_0(3)^{-2} = \frac{1}{9}x_0.$$

Exercises

1. The radius of the moon is roughly 1080 miles. The acceleration of gravity at the surface of the moon is about $0.165g$, where g is the acceleration of gravity at the surface of the earth. Determine the velocity of escape for the moon.

ANS. 1.5 miles/sec.

2. Determine, to two significant figures, the velocity of escape for each of the celestial bodies listed below. The data given are rough and g may be taken to be $6.1(10)^{-3}$ miles/sec².

	<i>Accel. of gravity at surface</i>	<i>Radius in miles</i>	<i>Ans. in miles/sec</i>
Venus	0.85g	3,800	6.3
Mars	0.38g	2,100	3.1
Jupiter	2.6g	43,000	37
Sun	28g	432,000	380
Ganymede	0.12g	1,780	1.6

3. A thermometer reading 18°F is brought into a room where the temperature is 70°F ; 1 min later the thermometer reading is 31°F . Determine the temperature reading as a function of time and, in particular, find the temperature reading 5 min after the thermometer is first brought into the room.
ANS. $u = 70 - 52 \exp(-0.29t)$; when $t = 5$, $u = 58$.
4. A thermometer reading 75°F is taken out where the temperature is 20°F . The reading is 30°F 4 min later. Find (a) the thermometer reading 7 min after the thermometer was brought outside, and (b) the time taken for the reading to drop from 75°F to within a half degree of the air temperature. ANS. (a) 23°F ; (b) 11.5 min.
5. At 1:00 P.M., a thermometer reading 70°F is taken outside where the air temperature is -10°F (ten below zero). At 1:02 P.M., the reading is 26°F . At 1:05 P.M., the thermometer is taken back indoors where the air is at 70°F . What is the thermometer reading at 1:09 P.M.?
6. At 9 A.M., a thermometer reading 70°F is taken outdoors where the temperature is 15°F . At 9:05 A.M., the thermometer reading is 45°F . At 9:10 A.M., the thermometer is taken back indoors where the temperature is fixed at 70°F . Find (a) the reading at 9:20 A.M. and (b) when the reading, to the nearest degree, will show the correct (70°F) indoor temperature. ANS. (a) 58°F ; (b) 9:46 A.M.
7. At 2:00 P.M., a thermometer reading 80°F is taken outside where the air temperature is 20°F . At 2:03 P.M., the temperature reading yielded by the thermometer is 42°F . Later, the thermometer is brought inside where the air is at 80°F . At 2:10 P.M., the reading is 71°F . When was the thermometer brought indoors? ANS. At 2:05 P.M.
8. Suppose that a chemical reaction proceeds according to the law given in Section 15 above. If half the substance A has been converted at the end of 10 sec, find when nine-tenths of the substance will have been converted. ANS. 33 sec.
9. The conversion of a substance B follows the law used in Section 15 above. If only a fourth of the substance has been converted at the end of 10 sec, find when nine-tenths of the substance will have been converted. ANS. 80 sec.
10. For a substance C , the time rate of conversion is proportional to the square of the amount x of unconverted substance. Let k be the numerical value of the constant of proportionality and let the amount of unconverted substance be x_0 at time $t = 0$. Determine x for all $t \geq 0$. ANS. $x = x_0/(1 + x_0kt)$.
11. Two substances, A and B , are being converted into a single compound C . In the laboratory it has been shown that, for these substances, the following law of conversion holds: the time rate of change of the amount x of compound C is proportional to the product of the amounts of unconverted substances A and B . Assume

Initial temperature (F)	Ans. in miles/sec
800	6.3
700	3.1
600	37
500	380
480	1.6

room where the temperature is 70°F. Determine the temperature reading 5 min after the temperature reading 5 min

(-0.29t); when $t = 5$, $u = 58$. the temperature is 20°F. The thermometer reading 7 min after the thermometer for the reading to drop from 23°F to 11.5 min.

outside where the air temperature is 26°F. At 1:05 P.M., the thermometer is 0°F. What is the thermometer

readings where the temperature is 70°F. At 9:10 A.M., the thermometer is at 70°F. Find (a) the reading at 10:00 A.M., (b) the reading at 10:05 A.M., (c) the reading at 10:10 A.M.

inside where the air temperature is 42°F. The thermometer is at 42°F. The air is at 80°F. At 2:10 P.M., the thermometer is at 42°F. At what time will the thermometer be at 42°F? ANS. At 2:05 P.M.

According to the law given in Section 15 above, find when the temperature is 42°F. ANS. 33 sec.

Find in Section 15 above. If only 10 sec, find when the temperature is 42°F. ANS. 80 sec.

proportional to the square of the numerical value of the constant of proportionality. Let x_0 be the initial amount of substance A at time $t = 0$. ANS. $x = x_0/(1 + x_0kt)$.

the units of measure so chosen that one unit of compound C is formed from the combination of one unit of A with one unit of B . If at time $t = 0$ there are a units of substance A , b units of substance B , and none of compound C present, show that the law of conversion may be expressed by the equation

$$\frac{dx}{dt} = k(a - x)(b - x).$$

Solve this equation with the given initial condition.

$$\text{ANS. If } b \neq a, x = \frac{ab[\exp(b - a)kt - 1]}{b \exp(b - a)kt - a}; \text{ if } b = a, x = \frac{a^2kt}{akt + 1}.$$

12. In the solution of exercise 11 above, assume that $k > 0$ and investigate the behavior of x as $t \rightarrow \infty$. ANS. If $b \geq a$, $x \rightarrow a$; if $b \leq a$, $x \rightarrow b$.

13. Radium decomposes at a rate proportional to the quantity of radium present. Suppose that it is found that in 25 years approximately 1.1% of a certain quantity of radium has decomposed. Determine approximately how long it will take for one-half the original amount of radium to decompose. ANS. 1600 years.

14. A certain radioactive substance has a half-life of 38 hr. Find how long it takes for 90% of the radioactivity to be dissipated. ANS. 126 hr.

15. A bacterial population B is known to have a rate of growth proportional to B itself. If between noon and 2 P.M. the population triples, at what time, no controls being exerted, should B become 100 times what it was at noon? ANS. About 8:22 P.M.

16. In the motion of an object through a certain medium (air at certain pressures is an example), the medium furnishes a resisting force proportional to the square of the velocity of the moving object. Suppose a body falls, due to the action of gravity, through such a medium. Let t represent time, and v represent velocity, positive downward. Let g be the usual constant acceleration of gravity and let w be the weight of the body. Use Newton's law, force equals mass times acceleration, to conclude that the differential equation of the motion is

$$\frac{w}{g} \frac{dv}{dt} = w - kv^2,$$

where kv^2 is the magnitude of the resisting force furnished by the medium.

17. Solve the differential equation of exercise 16 with the initial condition that $v = v_0$ when $t = 0$. Introduce the constant $a^2 = w/k$ to simplify the formulas.

$$\text{ANS. } \frac{a + v}{a - v} = \frac{a + v_0}{a - v_0} \exp\left(\frac{2gt}{a}\right).$$

18. List a consistent set of units for the dimensions of the variables and parameters of exercises 16 and 17 above.

$$\text{ANS. } \begin{array}{ll} t \text{ in sec} & g \text{ in ft/sec}^2 \\ v \text{ in ft/sec} & k \text{ in (lb)(sec}^2\text{)/ft}^2 \\ w \text{ in lb} & a \text{ in ft/sec} \end{array}$$

19. There are mediums that resist motion through them with a force proportional to the first power of the velocity. For such a medium, state and solve problems analogous to exercises 16 through 18 above, except that for convenience a constant

$b = w/k$ may be introduced to replace the a^2 of exercise 17. Show that b has the dimensions of a velocity.

$$\text{ANS. } v = b + (v_0 - b) \exp\left(-\frac{gt}{b}\right).$$

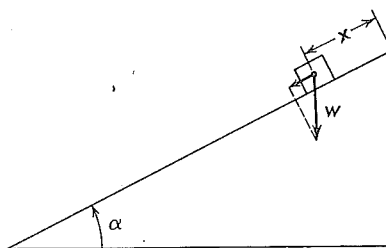


FIGURE 9

20. Figure 9 shows a weight, w pounds (lb), sliding down an inclined plane which makes an angle α with the horizontal. Assume that no force other than gravity is acting on the weight; that is, there is no friction, no air resistance, and so on. At time $t = 0$, let $x = x_0$ and let the initial velocity be v_0 . Determine x for $t > 0$.

$$\text{ANS. } x = \frac{1}{2}gt^2 \sin \alpha + v_0 t + x_0.$$

21. A long, very smooth board is inclined at an angle of 10° with the horizontal. A weight starts from rest 10 ft from the bottom of the board and slides downward under the action of gravity alone. Find how long it will take the weight to reach the bottom of the board and determine the terminal speed.

$$\text{ANS. } 1.9 \text{ sec and } 10.5 \text{ ft/sec.}$$

22. Add to the conditions of exercise 20 above a retarding force of magnitude kv , where v is the velocity. Determine v and x under the assumption that the weight starts from rest with $x = x_0$. Use the notation $a = kg/w$.

$$\text{ANS. } v = a^{-1}g \sin \alpha (1 - e^{-at}); x = x_0 + a^{-2}g \sin \alpha (-1 + e^{-at} + at).$$

23. A man, standing at O in Figure 10, holds a rope of length a to which a weight is attached, initially at W_0 . The man walks to the right dragging the weight after him. When the man is at M , the weight is at W . Find the differential equation of the path (called the tractrix) of the weight and solve the equation.

$$\text{ANS. } x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}.$$

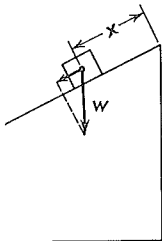
24. A tank contains 80 gallons (gal) of pure water. A brine solution with 2 lb/gal of salt enters at 2 gal/min, and the well-stirred mixture leaves at the same rate. Find (a) the amount of salt in the tank at any time, and (b) the time at which the brine leaving will contain 1 lb/gal of salt.

$$\text{ANS. } (a) s = 160[1 - \exp(-t/40)]; (b) t = 40 \ln 2 \text{ min.}$$

25. For the tank in the previous exercise, determine the limiting value for the amount of salt in the tank after a long time. How much time must pass before the amount of salt in the tank reaches 80% of this limiting value?

a^2 of exercise 17. Show that b has the

ANS. $v = b + (v_0 - b) \exp\left(-\frac{gt}{b}\right)$.



9

ing down an inclined plane which makes an angle of 10° with the horizontal. A weight of w slides downward along the plane. Assume that the weight is acted upon by a retarding force of magnitude kv , where v is the velocity. Determine x for $t > 0$.

ANS. $x = \frac{1}{2}gt^2 \sin \alpha + v_0 t + x_0$.
 A weight of w is attached to a rope of length a to which a weight is attached at the other end. The weight is released from the top of the rope. Find the differential equation of the motion of the weight W . Solve the equation.

ANS. 1.9 sec and 10.5 ft/sec.

over a retarding force of magnitude kv , where v is the velocity. Assume that the weight is acted upon by a retarding force of magnitude kv , where v is the velocity. Assume that the weight is acted upon by a retarding force of magnitude kv , where v is the velocity.

ANS. $x = x_0 + a^{-2}g \sin \alpha (-1 + e^{-at} + at)$.
 A rope of length a to which a weight is attached at the other end. The weight is released from the top of the rope. Find the differential equation of the motion of the weight W . Solve the equation.

$$x = a \ln \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$$

ter. A brine solution with 2 lb/gal of salt leaves at the same rate. Find (a) the time at which the brine leaving

ANS. $50[1 - \exp(-t/40)]$; (b) $t = 40 \ln 2$ min.
 Determine the limiting value for the amount of salt in the tank after a long time. How much time must pass before the amount of salt in the tank is within 1% of the limiting value?

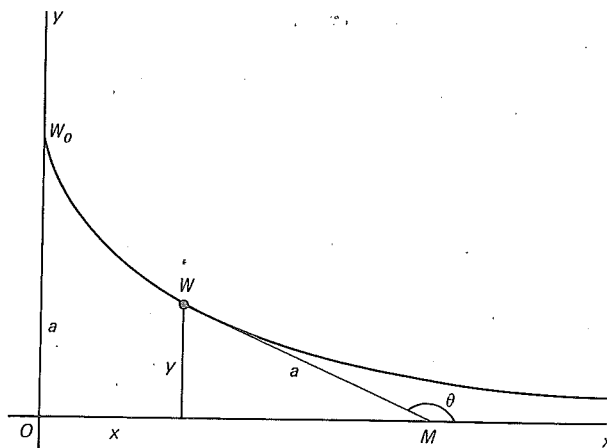


FIGURE 10

26. A certain sum of money P draws interest compounded continuously. If at a certain time there is P_0 dollars in the account, determine the time when the principal attains the value $2P_0$ dollars, if the annual interest rate is (a) 2%, or (b) 4%.

ANS. (a) $50 \ln 2$ years; (b) $25 \ln 2$ years.

27. A bank offers 5% interest compounded continuously in a savings account. Determine (a) the amount of interest earned in 1 year on a deposit of \$100 and (b) the equivalent rate if the compounding were done annually.

ANS. (a) \$5.13; (b) 5.13%.

16. Orthogonal trajectories; rectangular coordinates

Suppose that we have a family of curves given by

$$f(x, y, c) = 0, \tag{1}$$

one curve corresponding to each c in some range of values of the parameter c . In certain applications it is found desirable to know what curves have the property of intersecting a curve of the family (1) at right angles whenever they do intersect.

That is, we wish to determine a family of curves with equations

$$g(x, y, k) = 0 \tag{2}$$

such that, at any intersection of a curve of the family (2) with a curve of the family (1), the tangents to the two curves are perpendicular. The families (1) and (2) are then said to be *orthogonal trajectories** of each other.

* The word orthogonal comes from the Greek *orthos* (right) and *gonia* (angle); the word trajectory comes from the Latin *trajectus* (cut across). Hence a curve that cuts across certain others at right angles is called an orthogonal trajectory of those others.