CHAPTER

3

Differentiation

3.1 Definition of the Derivative

Preliminary Questions

1. What are the two ways of writing the difference quotient?

2. Explain in words what the difference quotient represents.

In Questions 3–5, $f(x)$ is an arbitrary function.

3. What does the following quantity represent in terms of the graph of $f(x)$?

$$\frac{f(8)-f(3)}{8-3}$$

4. For which value of x is

$$\frac{f(x)-f(3)}{x-3}=\frac{f(7)-f(3)}{4}\ ?$$

5. For which value of h is

$$\frac{f(2+h)-f(2)}{h}=\frac{f(4)-f(2)}{4-2}\ ?$$

6. To which derivative is the quantity

$$\frac{\tan(\frac{\pi}{4}+.00001)-1}{.00001}$$

a good approximation?

7. What is the equation of the tangent line to the graph at $x=3$ of a function $f(x)$ such that $f(3)=5$ and $f'(3)=2$?

In Questions 8–10, let $f(x)=x^2$.

8. The expression

$$\frac{f(7) - f(5)}{7 - 5}$$

is the slope of the secant line through two points P and Q on the graph of $f(x)$. What are the coordinates of P and Q?

9. For which value of h is the expression

$$\frac{f(5+h) - f(5)}{h}$$

equal to the slope of the secant line between the points P and Q in Question 8?

10. For which value of h is the expression

$$\frac{f(3+h) - f(3)}{h}$$

equal to the slope of the secant line between the points $(3, 9)$ and $(5, 25)$ on the graph of $f(x)$?

Exercises

1. Which of the following lines is tangent to the curve?

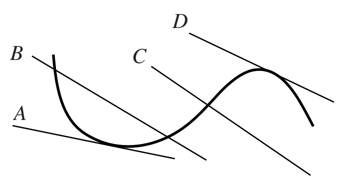


Figure 1

Lines A and D are tangent to the curve.

In Exercises 2–5, use the limit definition to find the derivative of the following linear functions (for any a; the derivative does not depend on a).

3. $f(x) = 2$

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{2-2}{h} = \lim_{h\to 0} 0 = 0.$$

5. $k(z) = 16z + 9$

$$\lim_{h\to 0} \frac{k(a+h) - k(a)}{h} = \lim_{h\to 0} \frac{16(a-h) + 9 - (16a+9)}{h} = \lim_{h\to 0} \frac{16h}{h} = 0.$$

In Exercises 6–9, compute the derivative of the quadratic polynomial at the point indicated using both forms of the derivative definition.

7. $x^2 + 9x;\ a = 2$

Let $f(x) = x^2 + 9x$. Then
$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 + 9(2+h) - 22}{h} = \lim_{h\to 0} \frac{13h + h^2}{h}$$
$$= \lim_{h\to 0}(13 + h) = 13.$$

9. $9 - 3x^2;\ a = 0$

Let $f(x) = 9 - 3x^2$. Then $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{9 - 3h^2 - 9}{h}$
$$= \lim_{h\to 0} \frac{-3h^2}{h} = \lim_{h\to 0}(-3h) = 0.$$

11. R & W Let f be the function shown in Figure 3.

(a) Given a number h, describe in words what the quantity
$$\frac{f(2+h) - f(2)}{h}$$
represents in terms of the graph.

(b) Use the graph to estimate this quantity for $h = .5$ and $h = -.5$.

(c) What does $f'(2)$ represent? Is it larger or smaller than the two numbers computed in (b)? Explain your reasoning.

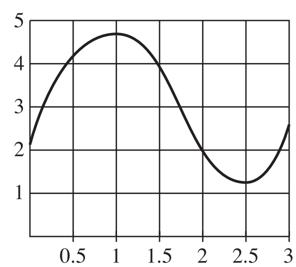


Figure 3

(a) The difference quotient
$$\frac{f(2+h) - f(2)}{h}$$
represents the slope of the secant line through the points $(2, f(2))$ and $(2+h, f(2+h))$ on the curve.

(b) For $h = -.5$, this quantity is $\frac{4-2}{-.5} = -4$. For $h = .5$, this quantity is approximately $\frac{1.2-2}{.5} = -1.6$.

(c) The derivative $f'(2)$ represents the slope of the tangent line to the curve at $x = 2$. It is between the two quantities computed in (b).

13. Let $f(x) = \sqrt{x}$. Let $P = (2, f(2))$ and $Q = (2+h, f(2+h))$

(a) Show that for $h \neq 0$, the slope of the secant line between the points P and Q is equal to $1/(\sqrt{2+h} + \sqrt{2})$.

(b) Use (a) to calculate the slope of the secant line through P and Q for $h = .1, .01, -.1, -.01$.

(c) Describe (in words and a graph) the limiting position of these secant lines as $h \to 0$.

(d) Evaluate $f'(2)$ using the formula in (a).

(a) The slope of the secant line between points $P = (2, f(2))$ and $Q = (2+h, f(2+h))$ is equal to

$$\frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{\sqrt{2+h} - \sqrt{2}}{h} \cdot \frac{\sqrt{2+h} + \sqrt{2}}{\sqrt{2+h} + \sqrt{2}} = 1/\sqrt{2+h} + \sqrt{2}.$$

(b)

h	$-.1$	$-.01$	$.01$	$.1$
$1/(\sqrt{2+h} + \sqrt{2})$	.358	.354	.353	.349

(c) The limiting position of the secant lines is the tangent line at $x = 2$.

(d) $f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}} = \frac{1}{2\sqrt{2}} \approx .354.$

15. Describe what each of the following quantities represents in terms of the graph of $f(x) = \sin x$.

(a) $f'(\frac{\pi}{4})$

(b) $\dfrac{\sin 1.3 - \sin .9}{.4}$

(c) $\sin 3 - \sin 1$

Consider the graph of $y = \sin x$.

(a) The quantity $\sin'(\frac{\pi}{4})$ represents the slope of the tangent line to the graph at $a = \frac{\pi}{4}$.

(b) The quantity $\dfrac{\sin 1.3 - \sin .9}{.4}$ represents the slope of the secant line to the graph between the points $(.9, \sin .9)$ and $(1.3, \sin 1.3)$.

(c) The quantity $\sin 3 - \sin 1$ represents the numerator of the slope of the secant line to the graph between the points $(1, \sin 1)$ and $(3, \sin 3)$.

In Exercises 16–35, compute the derivative of the function at the point indicated using the definition as a limit of difference quotients and find the equation of the tangent line.

17. $3x^2 + 2x, \quad a = -1$

Let $f(x) = 3x^2 + 2x$. Then

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = lim_{h\to 0} \frac{3(1 - 2h + h^2) - 2 + 2h - 1}{h}$$
$$= \lim_{h \to 0} \frac{3(1 - 2h + h^2) - 2 + 2h}{h} = \lim_{h \to 0} \frac{-4h + 3h^2}{h} = -4$$

The tangent line at $a = -1$ is $y = f'(-1)(x+1) + f(-1) = -4(x+1) + 1 = -4x - 3$.

19. $x^3, \quad a = 3$

Let $f(x) = x^3$. Then

$$\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{27 - 27h + 9h^2 + h^3 - 27}{h}$$
$$= \lim_{h\to 0} 27 + 9h + h^2 = 27.$$

At $a = 3$, the tangent line is $y = f'(3)(x-3) + f(3) = 27(x-3) + 27 = 27x - 54$.

21. $3t^3 + 2t, \quad a = 4$

Let $g(t) = 3t^3 + 2t$. Then

$$g'(4) = \lim_{h\to 0} \frac{g(4+h) - g(4)}{h} = \lim_{h\to 0} \frac{3(4+h)^3 + 2(4+h) - 200}{h}$$
$$= \lim_{h\to 0} \frac{3h^3 + 36h^2 + 146h}{h} = \lim_{h\to 0} \left(3h^2 + 36h + 146\right) = 146.$$

The tangent line at $a = 4$ is $y = g(4) + g'(4)(t-4) = 200 + 146(t-4)$, which simplifies to $y = 146t - 384$.

23. $x^{-1}, \quad a = 3$

Let $f(x) = x^-1$. Then

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^-1 - 3^-1}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{3+h} - \left(\frac{1}{3}\right)}{h} = \lim_{h\to 0} \frac{\frac{3-3-h}{3(3+h)}}{h}$$
$$= \lim_{h\to 0} \frac{-h}{(9+3h)h} = -\frac{1}{9}$$

The tangent at $a = 3$ is $y = f'(3)(x-3) + f(3) = \left(\frac{1}{3}\right) - \left(\frac{1}{9}\right)(x-3)$.

25. $x^{-2}, \quad a = -1$

Let $f(x) = x^{-2}$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{1-2h+h^2} - 1}{h}$$
$$= \lim_{h\to 0} \frac{1 - 1 + 2h - h^2}{h(1 - 2h + h^2)} = 2$$

The tangent line at $a = -1$ is $y = f'(-1)(x+1) + f(-1) = 2(x+1) + 1 = 2x + 3$.

27. $\dfrac{2}{1-t}, \quad a = -1$

Let $g(t) = \frac{2}{1-t}$. Then

$$g'(t) = \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h\to 0} \frac{\frac{2}{1-(t+h)} - \frac{2}{1-t}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2}{1-x-h} - \frac{2}{1-t}}{h} = \lim_{h\to 0} \frac{\frac{2-2t-2+2t+2h}{(1-t-h)(1-t)}}{h} = \frac{2}{(1-t)^2}$$

Then $f'(-1) = \frac{2}{(1+1)^2} = \frac{1}{2}$. At $a = -1$, the tangent line is
$y = g'(t)(x+1) + g(t) = \left(\frac{1}{2}\right)(x+1) + 1. = \left(\frac{x}{2}\right) + \frac{3}{2}$

29. $\dfrac{1}{\sqrt{x}}, \quad a = 9$

Let $f(x) = \sqrt{x}$. Then

$$f'(9) = \lim_{h\to 0} \frac{f(9+h) - f(9)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{(9+h)}} - \frac{1}{3}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{3-\sqrt{(9+h)}}{3\sqrt{(9+h)}} \cdot \frac{3+\sqrt{(9+h)}}{3+\sqrt{(9+h)}}}{h} = \lim_{h\to 0} \frac{\frac{9-9-h}{9\sqrt{(9+h)}+3(9+h)}}{h} = -\frac{1}{54}$$

At $a = 9$ the tangent line is $y = f'(9)(x-9) + f(9) = \left(-\frac{1}{54}\right)(x-9) + \frac{1}{3} = \left(-\frac{x}{54}\right) + \left(\frac{1}{2}\right)$.

31. $\dfrac{x+1}{x-1}, \quad a = 3$

Let $f(x) = \dfrac{x+1}{x-1}$. Then

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{4+h}{2+h} - 2}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-h}{2+h}}{h} = \lim_{h\to 0} \frac{-1}{2+h} = -\frac{1}{2}.$$

The tangent line at $a = 3$ is $y = f(3) + f'(3)(x-3) = 2 - \frac{1}{2}(x-3)$ or $y = \frac{7}{2} - \frac{x}{2}$.

33. $t^{-3}, \quad a = 1$

Let $g(t) = \dfrac{1}{t^3}$. Then

$$g'(1) = \lim_{h\to 0} \frac{g(1+h) - g(h)}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^3} - 1}{h}$$
$$= \lim_{h\to 0} \frac{\frac{h\left(3+3h+h^2\right)}{(1+h)^3}}{h} = \lim_{h\to 0} \frac{3+3h+h^2}{(1+h)^3} = -3.$$

The tangent line at $a = 1$ is $y = g(1) + g'(1)(t-1) = 1 - 3(t-1)$ or $y = 4 - 3t$.

35. $(x^2+1)^{-1}, \quad a = 0$

Let $f(x) = \dfrac{1}{x^2+1}$. Then

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{(0+h)^2+1} - 1}{h}$$
$$= \lim_{h\to 0} \frac{\frac{-h^2}{h^2+1}}{h} = \lim_{h\to 0} \frac{-h}{h^2+1} = 0.$$

The tangent line at 0 is $y = f(0) + f'(0)(x-0) = 1 + 0(x-1)$ or $y = 1$.

37. Which of the curves in Figure 6 is the graph of a function with the property that $f'(x) > 0$ for all x? Which has the property that $f'(x) < 0$?

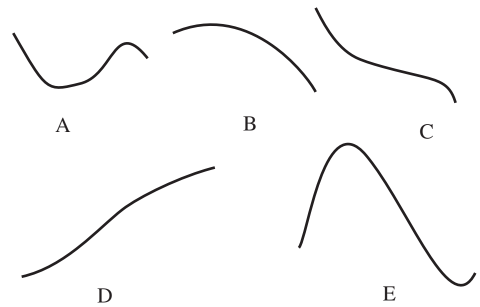


Figure 6

Curve D has $f'(x) > 0$ for all x (its slope is always positive), whereas curve C has $f'(x) < 0$ for all x (its slope is always negative).

Each of the limits in Exercises 39–44 represents a derivative $f'(a)$. Find f and a.

39. $\displaystyle\lim_{h\to 0} \frac{(5+h)^3 - 125}{h}$

$\displaystyle\lim_{h\to 0} \frac{(5+h)^3 - 125}{h}$ represents $f'(a)$ where $f(x) = x^3$ and $a = 5$.

41. $\displaystyle\lim_{h\to 0} \frac{5^{(2+h)} - 25}{h}$

$\displaystyle\lim_{h\to 0} \frac{5^{(2+h)} - 25}{h}$ represents $f'(a)$ where $f(x) = 5^x$ and $a = 2$.

43. $\displaystyle\lim_{h\to 0} \frac{\sin(\frac{\pi}{3} + h) - .5}{h}$

$\displaystyle\lim_{h\to 0} \frac{\sin(\frac{\pi}{6} + h) - .5}{h}$ represents $f'(a)$ where $f(x) = \sin x$ and $a = \frac{\pi}{6}$.

45. Consider the "curve" $y = 2x + 8$ (a straight line!). What is the tangent line at the point $(1, 10)$? Describe the tangent line at an arbitrary point.

Since $y = 2x + 8$ represents a straight line, the tangent line at any point is the line itself, $y = 2x + 8$.

47. Let f be the function whose graph is shown in Figure 8.

(a) Find the values of x such that $f'(x) = 0$.
(b) For which value of x is $f'(x)$ largest?
(c) On which intervals is $f'(x)$ negative?

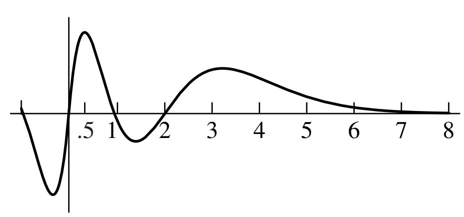


Figure 8

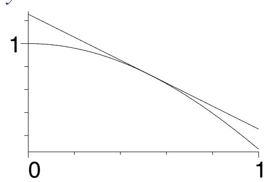
(a) $f'(x) = 0$ at $x \approx -\frac{1}{3}, \frac{1}{3}, 1.4, 3\frac{1}{4}$ since the tangent lines are horizontal there.
(b) $f'(x)$ is largest at $x = 0$ since the tangent line has the steepest slope there.
(c) $f'(0)$ is negative on the (approximate) intervals $(-1, -\frac{1}{3})$, $(\frac{1}{3}, 1.4)$, $(3\frac{1}{4}, 8.2)$.

49. GU Let $f(x) = \cos x$.
(a) Estimate $f'(\frac{\pi}{6})$ to at least three decimal places.
(b) Write an equation for an approximate tangent line at the point $(\frac{\pi}{6}, \frac{\sqrt{3}}{2})$.
(c) Use a CAS to plot f and the approximate tangent line.

(a)

h	$-.01$	$-.001$	$.001$	$.01$
$\dfrac{\cos(\frac{\pi}{6}+h) - \frac{\sqrt{3}}{2}}{h}$	$-.495662$	$-.499567$	$-.500433$	$-.504322$

With $f(x) = \cos x$, we estimate $f'(\frac{\pi}{6}) \approx -.500$.
(b) The tangent line is $y = f(\frac{\pi}{6}) + f'(\frac{\pi}{6})(x - \frac{\pi}{6}) \approx .866 - .500(x - .524)$ or $y \approx 1.128 - .500x$.

(c)


51. R & W Consider the slopes of the secant lines between $x = 1$ and $x = 1 + h$ for each of the graphs in Figure 9. In each case, state whether $f'(1)$ is less than or greater than the slope of the secant line when $h > 0$. Same question for $h < 0$. Explain your reasoning by referring to the graph.

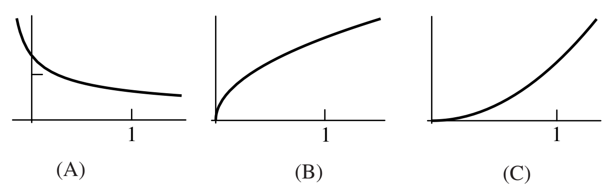


Figure 9

(a) $f'(1)$ is less because it is decreasing at a faster rate.
(b) $f'(1)$ is greater because it is increasing at a faster rate.
(c) $f'(1)$ is less because it is increasing at a slower rate.

53. R & W Let $f(x) = x^{5/2}$.

(a) Use Figure 10 to explain why the following inequalities hold:

$$f'(4) \le \frac{f(4+h) - f(4)}{h} \qquad \text{if } h < 0$$
$$f'(4) \ge \frac{f(4+h) - f(4)}{h} \qquad \text{if } h > 0$$

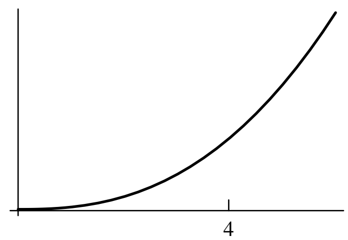


Figure 10 Graph of $f(x) = x^{5/2}$.

(b) Estimate $f'(4)$ to four decimal places.

(a) The slope of the secant line between points $(4, f(4))$ and $(4+h, f(4+h))$ is $\frac{f(4+h)-f(h)}{h}$. $x^{5/2}$ is a smooth curve increasing at a faster rate as $x \to \infty$. Therefore, if $h > 0$, then the slope of the secant line is greater than the slope of the tangent line at $f(4)$, which happens to be $f'(4)$. Likewise, if $h < 0$, the slope of the secant line is less than the slope of the tangent line at $f(4)$, which happens to be $f'(4)$.

(b) We know that $f'(4) = \lim_{h\to 0} \dfrac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \dfrac{(4+h)^{5/2} - 32}{h}$. Creating a table with values of h close to zero:

h	$-.0001$	$-.00001$	$.00001$	$.0001$
$\dfrac{(4+h)^{5/2} - 32}{h}$	19.999625	19.99999	20.0000	20.0000375

$f'(4) \approx 19.999$.

55. R & W Let $f(x) = \sec x$.

(a) Use Figure 11 to explain why the following inequalities hold:

$$f'(1) \le \frac{f(1+h) - f(1)}{h} \qquad \text{if } h < 0$$
$$f'(1) \ge \frac{f(1+h) - f(1)}{h} \qquad \text{if } h > 0$$

(b) Estimate $f'(1)$ to four decimal places.

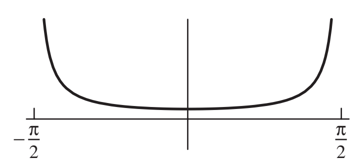


Figure 11 Graph of $f(x) = \sec x$.

(a) The slope of the secant line between points $(1, f(1))$ and $(1+h, f(1+h))$ is $\frac{f(1+h)-f(h)}{h}$. $\sec x$ between the values of 0 and $\frac{\pi}{2}$ is a smooth curve increasing at a faster rate as $x \to \frac{\pi}{2}$. Therefore, if $h > 0$, then the slope of the secant line is greater than the slope of the tangent line at $f(1)$ which happens to be $f'(1)$. Likewise, if $h < 0$, the the slope of the secant line is less than the slope of the tangent line at $f(1)$, which happens to be $f'(1)$.

(b) We know that

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sec(1+h) - \sec(1)}{h}.$$

Create a table with values of h close to zero:

h	$-.001$	$-.0001$	$.0001$	$.001$
$\frac{\sec(1+h)-\sec(1)}{h}$	2.87707	2.88193	2.887899	22.88193

$f'(1) \approx 2.8819$.

57. Let f be the function whose graph is shown in Figure 13 and consider

$$\frac{f(2+h) - f(2)}{h}$$

as a function of h. Is this an increasing or a decreasing function of h? In other words, does it get larger or smaller as h increases?

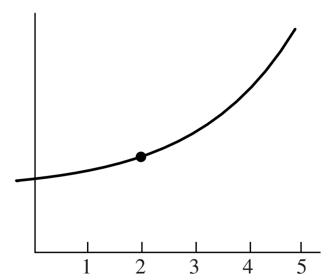


Figure 13

For the depicted graph, the difference quotient $\frac{f(2+h) - f(2)}{h}$ is an increasing function of h since the slopes of the secant lines between $(2, f(2))$ and a nearby point increase as the nearby point moves from left to right.

Further Insights and Challenges

59. For small h, the symmetric difference quotient usually gives a better approximation to the derivative than the ordinary difference quotient. Let $f(x) = 2^x$ and $a = 0$. Compute the symmetric difference quotient with $h = 0.001$ and the ordinary difference quotients with $h = \pm 0.001$. Compare with the actual value of $f'(0)$ which, to 8 decimal places, is 0.69314718.

Let $f(x) = 2^x$ and $a = 0$.

- The ordinary difference quotient for $h = -.001$ is .69290701 and for $h = .001$ is .69338746.
- The symmetric difference quotient for $h = -.001$ is .69314724 and for $h = .001$ is .69314724.
- Clearly the symmetric difference quotient gives a better estimate of the derivative $f'(0) \approx .69314718$.

3.2 The Derivative as a Function

Preliminary Questions

1. What is the slope of the tangent line through the point $(2, f(2))$ if f is a function such that $f'(x) = x^3$?

2. Suppose that f and g are differentiable functions such that $f'(1) = 3$ and $g'(1) = 5$. Which of the following derivatives can be computed using the rules of this section and the information given? Evaluate the derivatives that can be computed.

(a) $(f + g)'(1)$
(b) $(f - g)'(1)$
(c) $(fg)'(1)$
(d) $(3f + 2g)'(1)$
(e) $\left(\frac{f}{g}\right)'(1)$

3. Which of the following functions can be differentiated using the rules covered in this section? Explain.

(a) x^2
(b) 2^x
(c) $\frac{1}{\sqrt{x}}$
(d) $x^{-4/5}$
(e) $\sin x$
(f) $(x + 1)^3$

4. Which algebraic identity is used to prove the Power Rule for positive integer exponents? Explain how it is used.

Exercises

In Exercises 1–6, calculate $f'(x)$ using the limit definition.

1. $4x - 3$

Let $f(x) = 4x - 3$ Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{4(x+h) - 3 - (4x - 3)}{h}$$
$$= \lim_{h\to 0} \frac{4h}{h} = 4.$$

3. $1 - 2x^2$

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{1 - 2(x+h)^2 - (1 - 2x^2)}{h} =$
$\lim_{h\to 0} \dfrac{-4xh - 2h^2}{h} = -4x.$

5. $\sqrt{x}$

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} =$
$\lim_{h\to 0} \dfrac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\dfrac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) = \left(\frac{1}{2}\right) x^{-1/2}.$

7. Let $f(x) = \dfrac{1}{x+1}$.

(a) Show that

$$\frac{f(x+h) - f(x)}{h} = -\frac{1}{(x+1)(x+1+h)}.$$

(b) Find $f'(x)$ by evaluating the limit.

(c) Find the equations of the tangent lines to the graph of f at $x = .5$ and $x = 6$.

Let $f(x) = \dfrac{1}{x+1}$.

(a) Then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\dfrac{1}{x+h+1} - \dfrac{1}{x+1}}{h} = \dfrac{\dfrac{-h}{(x+h+1)(x+1)}}{h} =$
$\dfrac{-1}{(x+h+1)(x+1)}$.

(b) Therefore, $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{-1}{(x+h+1)(x+1)} = -\dfrac{1}{(x+1)^2}$.

(c)
- For $x = \frac{1}{2}$, we have $f'(\frac{1}{2}) = -\frac{4}{9}$. Thus the tangent line to the graph of f is $y = f(\frac{1}{2}) + f'(\frac{1}{2})(x - \frac{1}{2}) = \frac{2}{3} - \frac{4}{9}(x - \frac{1}{2})$ or $y = \frac{8}{9} - \frac{4}{9}x$.
- For $x = 6$, we have $f'(6) = -\frac{1}{49}$. Hence the tangent line to the graph of f is $y = f(6) + f'(6)(x - 6) = \frac{1}{7} - \frac{1}{49}(x - 6)$ or $y = \frac{13}{49} - \frac{x}{49}$.

In Exercises 9–17, use the Power Rule to find the derivatives of the following functions.

9. x^4

$\dfrac{d}{dx}\left(x^4\right) = 4x^3.$

11. $x^{2/3}$

$\dfrac{d}{dx}\left(x^{2/3}\right) = \dfrac{2}{3}x^{-1/3}.$

13. x^{-10}

$\dfrac{d}{dx}\left(x^{-10}\right) = -10x^{-11}.$

15. 5

The derivative of a constant is always 0.

17. $x^{-\pi^2}$

$\frac{d}{dx}(x^{-\pi^2}) = -\pi^2 x^{-\pi^2-1}$

19. Find $f'(x)$ where $f(x) = 6x^4 + 7x^{2/3}$.

Let $f(x) = 6x^4 + 7x^{2/3}$. Then $f'(x) = 24x^3 + \frac{14}{3}x^{-1/3}$.

In Exercises 21–36, calculate the derivative $f'(x)$ of each function.

21. $x^3 + x^2$

$$\frac{d}{dx}\left(x^3 + x^2\right) = 3x^2 + 2x.$$

23. $2x^3 - 10x^{-1}$

$$\frac{d}{dx}\left(2x^3 - 10x^{-1}\right) = 6x^2 + 10x^{-2}.$$

25. $4x^4 + \frac{3}{8}x^2 - 14$

$$\frac{d}{dx}\left(4x^4 + \frac{3}{8}x^2 - 14\right) = 16x^3 + \frac{3}{4}x.$$

27. $8x^{1/4}$

$$\frac{d}{dx}\left(8x^{1/4}\right) = 2x^{-3/4}.$$

29. $7x^{-3} + x^2$

$$\frac{d}{dx}\left(7x^{-3} + x^2\right) = 2x - 21x^{-4}.$$

31. $(5x + 1)^2$

$$\frac{d}{dx}\left((5x + 1)^2\right) = \frac{d}{dx}\left(25x^2 + 10x + 1\right) = 50x + 10.$$

33. $(3x - 1)(2x + 1)$

$$\frac{d}{dx}((3x - 1)(2x + 1)) = \frac{d}{dx}\left(6x^2 + x - 1\right) = 12x + 1.$$

35. $\sqrt{x}(x + 1)$

$$\frac{d}{dx}\left(x^{1/2}(x + 1)\right) = \frac{d}{dx}\left(x^{3/2} + x^{1/2}\right) = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2}.$$

37. Find df/dC where $f(C) = 3C^{2/3}$.

With $f(C) = C^{2/3}$, we have $\dfrac{df}{dC} = \dfrac{2}{3}C^{-1/3}$.

39. Find ds/dt where $s(t) = 4t - 16t^2$.

With $s = 4t - 16t^2$, we have $\frac{ds}{dt} = 4 - 32t$.

41. Find df/dt where $f(t) = 7$.

With $f(t) = 7$, we have $\frac{df}{dx} = 0$.

43. Find df/dy where $f(y) = y^{5.3}$.

Let $f(y) = y^{5.3}$. Then $df/dy = 5.3y^{4.3}$.

In Exercises 45–52, use the limit definition to compute the derivative.

45. $f(x) = \sqrt{x+2}$

Let $f(x) = \sqrt{x+2}$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+2} + \sqrt{x+2})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} \\ &= \frac{1}{2\sqrt{x+2}}. \end{aligned}$$

47. $f(x) = \dfrac{1}{\sqrt{x}}$

Let $f(x) = \dfrac{1}{\sqrt{x}}$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} \\ &= \lim_{h\to 0} \frac{\frac{-h}{(x+h)x}}{h\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} = \lim_{h\to 0} \frac{-1}{(x+h)x\left(\frac{1}{\sqrt{x+h}} + \frac{1}{\sqrt{x}}\right)} \\ &= \frac{-1}{(x^2)\frac{2}{\sqrt{x}}} = -\frac{1}{2}x^{-3/2}. \end{aligned}$$

49. $f(x) = x^{-3/2}$

Let $f(x) = x^{-3/2}$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^{-3/2} - x^{-3/2}}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^{-3} - x^{-3}}{h\left((x+h)^{-3/2} + x^{-3/2}\right)} = \lim_{h\to 0} \frac{\frac{x^3 - (x+h)^3}{x^3(x+h)^3}}{h\left((x+h)^{-3/2} + x^{-3/2}\right)} \\ &= \lim_{h\to 0} \frac{\frac{-3x^2h - 3xh^2 - h^3}{x^3(x+h)^3}}{h\left((x+h)^{-3/2} + x^{-3/2}\right)} = \lim_{h\to 0} \frac{-3x^2 - 3xh - h^2}{x^3(x+h)^3\left((x+h)^{-3/2} + x^{-3/2}\right)} \\ &= \frac{-3x^2}{x^6\left(2x^{-3/2}\right)} = -\frac{3}{2}x^{-5/2}. \end{aligned}$$

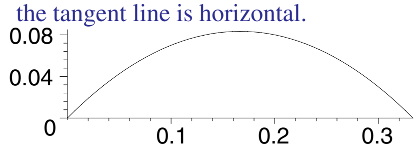
51. $g(t) = \sqrt{t^2 + 1}$

Let $g(t) = \sqrt{t^2 + 1}$. Then,

$$g'(t) = \lim_{h\to 0} \frac{g(t+h) - g(t)}{h} = \lim_{h\to 0} \frac{\sqrt{(t+h)^2+1} - \sqrt{t^2+1}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{(t+h)^2+1} - \sqrt{t^2+1}}{h} \cdot \frac{\sqrt{(t+h)^2+1} + \sqrt{t^2+1}}{\sqrt{(t+h)^2+1} + \sqrt{t^2+1}}$$
$$= \lim_{h\to 0} \frac{2t+h}{\sqrt{(t+h)^2+1} + \sqrt{t^2+1}} = \frac{t}{\sqrt{t^2+1}}.$$

53. Sketch a graph of the function $f(x) = x - 3x^2$. Calculate $f'(x)$ and determine the values of x for which the tangent line is horizontal.

Let $f(x) = x - 3x^2$. Solve $f'(x) = 1 - 6x = 0$ to obtain $x = \frac{1}{6}$. This is the value at which the tangent line is horizontal.



55. Find the points on the curve $y = x^3 + 3x^2 + 1$ at which the slope of the tangent line is equal to 15.

Let $y = x^3 + 3x^2 + 1$. Solving $dy/dx = 3x^2 + 6x = 15$ or $x^2 + 2x - 5 = 0$ yields
$x = \dfrac{-2 \pm \sqrt{4+20}}{2} = -1 \pm \sqrt{6}$.

57. R & W Of the two functions f and g in Figure 1, which is the derivative of the other? Justify your answer.

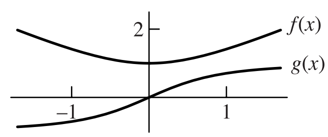
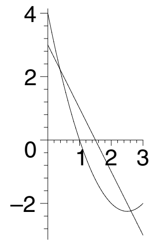


Figure 1

$g(x)$ is the derivative of $f(x)$. For $f(x)$ the slope is negative for negative values of x until $x = 0$, where there is a horizantal tangent, and then the slope is positive for positive values of x. Notice that $g(x)$ is negative for negative values of x, goes through the origin at $x = 0$, and then is positive for positive values of x.

59. **(a)** Graph the functions $f(x) = x^2 - 5x + 4$ and $g(x) = -2x + 3$.
(b) Find the point on the graph of f where the tangent is parallel to the graph of g.



(a)

(b) Let $f(x) = x^2 - 5x + 4$ and $g(x) = 3 - 2x$. Solve $f'(x) = 2x - 5 = -2 = g'(x)$ to obtain $x = \frac{3}{2}$.

61. Determine coefficients a and b such that the polynomial $p(x) = x^2 + ax + b$ satisfies $p(0) = 0$ and $p'(0) = 4$.

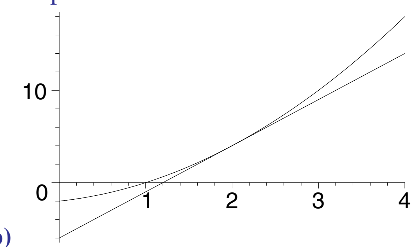
Let $p(x) = x^2 + ax + b$ satisfy $p(0) = 0$ and $p'(0) = 4$. Since $p'(x) = 2x + a$, this implies $0 = p(0) = b$ and $4 = p'(0) = a$; i.e., $a = 4$ and $b = 0$.

63. GU

(a) Determine coefficients a, b such that the tangent line to the graph of $p(x) = x^2 + ax + b$ at $x = 2$ is $y = 5x - 6$.

(b) Check your answer by plotting $p(x)$ and $y = 5x - 6$.

(a) Let $p(x) = x^2 + ax + b$. The tangent line to p at $x = 2$ is $y = p(2) + p'(2)(x - 2) = 4 + 2a + b + (4 + a)(x - 2) = (4 + a)x + b - 4$. In order for the line $y = 5x - 6$ to be tangent to p at $x = 2$, we must have $(4 + a)x + b - 4 = 5x - 6$. Equate coefficients: $4 + a = 5$ and $b - 4 = -6$. This implies $a = 1$ and $b = -2$.

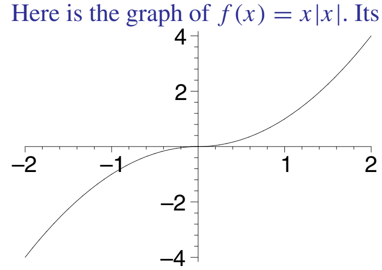


(b)

In Exercises 64–69, find the points c (if any) such that $f'(c)$ does not exist.

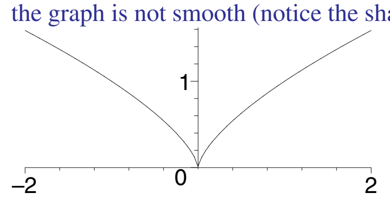
65. $x|x|$

Here is the graph of $f(x) = x|x|$. Its derivative exists everywhere.



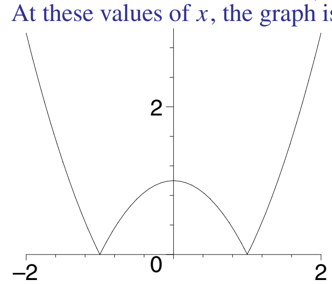
67. $x^{2/3}$

Here is the graph of $f(x) = x^{2/3}$. Its derivative does not exist at $x = 0$. At that value of x, the graph is not smooth (notice the sharp corner or "cusp").



69. $|x^2 - 1|$

Here is the graph of $f(x) = \left|x^2 - 1\right|$. Its derivative does not exist at $x = -1$ or at $x = 1$. At these values of x, the graph is not smooth (notice the sharp corners).



Further Insights and Challenges

71. The parabola, ellipse, and hyperbola were given their names by the Greek mathematician Apollonius of Prega (b. 262 BCE) in his influential treatise on conics. Prove the following Theorem of Apollonius: the tangent to the parabola $y = x^2$ at $x = a$ intersects the x-axis at the midpoint between the origin and $(a, 0)$. Draw a diagram.

Let $f(x) = x^2$. The tangent line to f at $x = a$ is $y = f(a) + f'(a)(x - a) = a^2 + 2a(x - a)$ or $y = 2ax - a^2$. The x-intercept of this line (where $y = 0$) is $\frac{a}{2}$, which is halfway between the origin and the point $(a, 0)$.

73. R & W Let $f(x)$ be a differentiable function and set $g(x) = f(x + c)$, where c is a constant. Recall that the graph of $g(x)$ is obtained by shifting the graph of $f(x)$ to the left c units.

(a) Use the limit definition to show that $g'(x) = f'(x + c)$.

(b) Explain the result of (a) in words based on the graphical interpretation.

(a) Let $g(x) = f(x + c)$. Using the limit definition,

$$g'(x) = \lim_{h\to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h\to 0} \frac{f((x + h) + c) - f(x + c)}{h} = \lim_{h\to 0} \frac{f((x + c) + h) - f(x + c)}{h} = f'(x + c).$$

(b) The graph of $g(x)$ is obtained by shifting $f(x)$ to the left by c units. This implies that $g'(x)$ is equal to $f'(x)$ shifted to the left by c units which happens to be $f'(x + c)$. Therefore, $g'(x) = f'(x + c)$.

75. Negative Exponents Let $f(x) = x^{-n}$, where n is a positive integer.

(a) Show that the difference quotient for f is equal to

$$\frac{-1}{x^n(x+h)^n} \cdot \frac{(x+h)^n - x^n}{h}.$$

(b) Prove that $f'(x) = -x^{-2n}(d/dx)x^n$.

(c) Use the Power Rule for x^n to show that the Power Rule also holds for x^{-n}. Derive the Power Rule for negative exponents.

Let $f(x) = x^{-n}$ where n is a positive integer.

(a) The difference quotient for f is

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^{-n} - x^{-n}}{h} = \frac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h} = \frac{\frac{x^n-(x+h)^n}{x^n(x+h)^n}}{h}$$
$$= \frac{-1}{x^n(x+h)^n}\,\frac{(x+h)^n - x^n}{h}.$$

(b) Therefore,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{-1}{x^n(x+h)^n}\,\frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{-1}{x^n(x+h)^n}\,\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = -x^{-2n}\frac{d}{dx}(x^n).$$

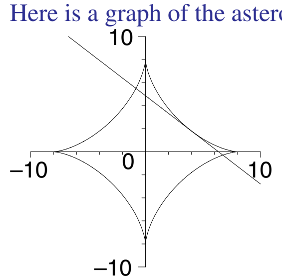
(c) From (b), we continue: $f'(x) = -x^{-2n}\frac{d}{dx}(x^n) = -x^{-2n}\cdot nx^{n-1} = -nx^{-n-1}$. Since n is a positive integer, $k = -n$ is a negative integer and we have $\frac{d}{dx}\left(x^k\right) = \frac{d}{dx}\left(x^{-n}\right) = -nx^{-n-1} = kx^{k-1}$; i.e. $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$ for negative integers k.

77. GU

(a) Use a GU to plot the graph of $f(x) = (4 - x^{2/3})^{3/2}$ (the "asteroid").

(b) Let L be a tangent line to a point on the graph in the first quadrant. Show that the portion of L in the first quadrant has a constant length 8.

(a) Here is a graph of the asteroid.



(b) Let $f(x) = \left(4 - x^{2/3}\right)^{3/2}$. Since we have not yet encountered the Chain Rule, we use Maple throughout this exercise. The tangent line to f at $x = a$ is

$$y = \left(4 - a^{2/3}\right)^{3/2} - \frac{\sqrt{4 - a^{2/3}}(x-a)}{a^{1/3}}$$

The y-intercept of this line is the point $P = \left(0, 4\sqrt{4 - a^{2/3}}\right)$, its x-intercept is the point $Q = \left(4a^{1/3}, 0\right)$, and the distance between P and Q is 8.

79. A Discontinuous Derivative Let

$$g(x) = \begin{cases} x^2 \cdot \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Show, using the limit definition, $f'(0)$ exists.
(b) Calculate $f'(x)$ for $x \neq 0$ and show that $\lim_{x\to 0} f'(x)$ does not exist.
(c) Conclude that $f'(x)$ is *not* continuous at $x = 0$.

(a) $f'(0) = \lim_{h\to 0} \dfrac{f(0+h) - f(h)}{h} = \lim_{h\to 0} \dfrac{f(h) - f(h)}{h} = \lim_{h\to 0} \dfrac{0}{h} = 0.$
(b) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(h)}{h} = \lim_{h\to 0} \dfrac{(x+h)^2 \sin(\frac{1}{x+h}) - h^2 sin(1/h)}{h} = \cdots$ Already we see that we run into a problem with the tern $\sin(1/h)$ since $\sin(1/0)$ is undefined.
(c) This follows from the fact that the limit does not exist as $x \to 0$.

3.3 Product and Quotient Rules

Preliminary Questions

1. Are the following statements true or false? If false, state the correct version.
(a) The notation $f \cdot g$ denotes the function whose value at x is $f(g(x))$.
(b) The notation f/g denotes the function whose value at x is $f(x)/g(x)$.
(c) The derivative of the product is the product of the derivatives.

2. Are the following equations true or false? If false, state the correct version.
(a) $\dfrac{d}{dx}(fg)\Big|_{x=4} = f(4)g'(4) - g(4)f'(4)$
(b) $\dfrac{d}{dx}(f/g)\Big|_{x=4} = [f(4)g'(4) + g(4)f'(4)]/g(4)^2$
(c) $\dfrac{d}{dx}(fg)\Big|_{x=0} = f(0)g'(0) + g(0)f'(0)$

3. Suppose that $f(1) = 0$ and $f'(1) = 2$. What is the value of $g(1)$ if $(fg)'(1) = 10$?

4. What is the value of $(fg)'(a)$, assuming that $f(a)$ and $g(a)$ are 0?

5. What is the derivative of f/g at $x = 1$ if $f(1) = f'(1) = g(1) = 2$, and $g'(1) = 4$?

Exercises

In Exercises 1–4, use the Product Rule to calculate the derivative.

1. $x(x^2 + 1)$

Let $f(x) = x(x^2 + 1)$. Then $f'(x) = (x^2 + 1) + x(2x) = 3x^2 + 1$

3. $(t^2 + 1)(t + 9)$

Let $f(t) = (t^2 + 1)(t + 9)$. Then $f'(t) = 2t(t + 9) + (t^2 + 1) = 3t^2 + 18t + 1$.

In Exercises 5–8, use the Quotient Rule to calculate the derivative.

5. $\dfrac{x}{x-2}$

Let $f(x) = \frac{x}{x-2}$. Then $f'(x) = \frac{(x-2)-x}{(x-2)^2} = \frac{-2}{(x-2)^2}$.

7. $\dfrac{t^2+1}{t^2-1}$

Let $f(t) = \dfrac{t^2+1}{t^2-1}$. Then $f'(t) = \dfrac{\left(t^2-1\right)(2t) - \left(t^2+1\right)(2t)}{\left(t^2-1\right)^2} = -\dfrac{4t}{\left(t^2-1\right)^2}$.

9. Calculate the derivative of $f(t) = (2t+1)(t^2-2)$ in two ways: once using and once not using the Product Rule.

Using the product rule: Let $f(t) = (2t+1)(t^2-2)$. Then, $f'(t) = (2t+1)(2t) + (t^2-2)(2) = 6t^2 + 2t - 4$. Multiplying out first: Let $f(t) = 2t^3 + t^2 - 4t - 2$. Then, $f'(t) = 6t^2 + 2t - 4$.

In Exercises 11–38, calculate the derivative using the appropriate rule or combination of rules.

11. $(x^4-4)(x^2+x+1)$

Let $f(x) = \left(x^4-4\right)\left(x^2+x+1\right)$. Then
$f'(x) = \left(x^4-4\right)(2x+1) + \left(x^2+x+1\right)\left(4x^3\right) = 6x^5 + 5x^4 + 4x^3 - 8x - 4$.

13. $(x^2+x+2)(x^3-7x-1)$

Let $f(x) = \left(x^2+x+2\right)\left(x^3-7x-1\right)$. Then
$f'(x) = \left(x^2+x+2\right)\left(3x^2-7\right) + \left(x^3-7x-1\right)(2x+1)$
$= 5x^4 + 4x^3 - 15x^2 - 16x - 15$.

15. $\dfrac{x+1}{x^{1/2}+1}$

Let $f(x) = \dfrac{x+1}{\sqrt{x}+1}$. Then $f'(x) = \dfrac{\left(\sqrt{x}+1\right)(1) - (x+1)\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}+1\right)^2} = \dfrac{x+2\sqrt{x}-1}{2\sqrt{x}\left(\sqrt{x}+1\right)^2}$.

17. $\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$

Let $f(x) = \dfrac{\sqrt{x}+1}{\sqrt{x}-1}$. Then $f'(x) = \dfrac{\left(\sqrt{x}-1\right)\frac{1}{2\sqrt{x}} - \left(\sqrt{x}+1\right)\frac{1}{2\sqrt{x}}}{\left(\sqrt{x}-1\right)^2} = -\dfrac{1}{\sqrt{x}\left(\sqrt{x}-1\right)^2}$.

19. $(x^7+3x^2-2)(x^5-4x^3)$

Let $f(x) = \left(x^7+3x^2-2\right)\left(x^5-4x^3\right)$. Then
$f'(x) = \left(x^7+3x^2-2\right)\left(5x^4-12x^2\right) + \left(x^5-4x^3\right)\left(7x^6+6x\right)$
$= 12x^{11} - 40x^9 + 21x^6 - 70x^4 + 24x^2$.

21. $\dfrac{5z-9}{4-4z}$

Let $f(z)=\dfrac{5z-9}{4-4z}$. Then

$$f'(z)=\frac{(4-4z)(5)-(5z-9)(-4)}{(4-4z)^2}=-\frac{1}{(z-1)^2}.$$

23. $(x^2+9)(x+x^{-1})$

Let $f(x)=\left(x^2+9\right)\left(x+\dfrac{1}{x}\right)$. Then

$$f'(x)=\left(x^2+9\right)\left(1-\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)(2x)=3x^2+10-\frac{9}{x^2}$$

or $\dfrac{3x^4+10x^2-9}{x^2}$.

25. $3^{1/2}\cdot 5^{1/2}$

Let $f(x)=\sqrt{3}\sqrt{5}$. Then $f'(x)=0$, since $f(x)$ is a *constant* function!

27. $\dfrac{x}{x+x^{-1}}$

Let $f(x)=\dfrac{x}{x+x^{-1}}=\dfrac{x^2}{x^2+1}$, for $x\neq 0$. Then
$f'(x)=\dfrac{\left(x^2+1\right)(2x)-x^2(2x)}{\left(x^2+1\right)^2}=\dfrac{2x}{\left(x^2+1\right)^2}$, for $x\neq 0$.

29. $\pi^2(x-1)$

Let $f(x)=\pi^2(x-1)$. Then $f'(x)=\pi^2$.

31. $(x+3)(x-1)(x-5)$

Let $f(x)=(x+3)(x-1)(x-5)$. Then
$f'(x)=(x+3)\left((x-1)(1)+(x-5)(1)\right)+(x-1)(x-5)(1)=3x^2-6x-13$.
Alternatively, $f(x)=(x+3)\left(x^2-6x+5\right)=x^3-3x^2-13x+15$, whence
$f'(x)=3x^2-6x-13$.

33. $\sqrt{x}(x^2+1)(x^{1/3}-3)$

Let $f(x)=\sqrt{x}\left(x^2+1\right)\left(x^{1/3}-3\right)=x^{17/6}-3x^{5/2}+x^{5/6}-3x^{1/2}$. Then
$f'(x)=\frac{17}{6}x^{11/6}-\frac{15}{2}x^{3/2}+\frac{5}{6}x^{-1/6}+\frac{3}{2}x^{-1/2}$.

35. $\dfrac{z^2+2}{z-1}\dfrac{z^2-1}{z+1}$ *Hint:* simplify first

Let $f(z)=\dfrac{z^2+2}{z-1}\dfrac{z^2-1}{z+1}=z^2+2$, for $z\neq\pm 1$. Then $f'(z)=2z$, for $z\neq\pm 1$.

37. $\dfrac{d}{dx}\left((ax+b)(abx^2+1)\right)$ (a, b constants)

Let $f(x)=(ax+b)\left(abx^2+1\right)$. Then
$f'(x)=(ax+b)(2abx)+\left(abx^2+1\right)(a)=3a^2bx^2+a+2ab^2x$.

39. The curve $y = \dfrac{1}{x^2+1}$ is called the *witch of Agnesi*, after the Italian mathematician Maria Agnesi (1718–1799). This strange name is the result of a mistranslation of the original Italian name *la versiera*, which means "rope that turns a sail." (http://www-groups.dcs.st-andrews.ac.uk/ history/Mathematicians/Agnesi.html) Find the equations of the tangent lines to the curve at $x = \pm 1$.

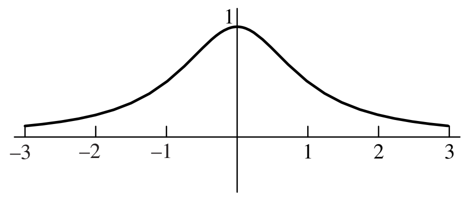


Figure 1 Graph of the *witch of Agnesi*.

Let $f(x) = \dfrac{1}{x^2+1}$. Then $f'(x) = -\dfrac{2x}{(x^2+1)^2}$.

- At $x = -1$, the tangent line is $y = f(-1) + f'(-1)(x+1) = \frac{1}{2} + \frac{1}{2}(x+1) = \frac{1}{2}x + 1$ or $y = \frac{1}{2}x + 1$.
- At $x = 1$, the tangent line is $y = f(1) + f'(1)(x-1) = \frac{1}{2} - \frac{1}{2}(x-1) = 1 - \frac{1}{2}x$ or $y = 1 - \frac{1}{2}x$.

In Exercises 40–43, use the function values

$f(4)$	$f'(4)$	$g(4)$	$g'(4)$
2	−3	5	−1

41. Calculate $F'(4)$, where $F(x) = xf(x)$.

Let $F(x) = xf(x)$. Then $F'(x) = xf'(x) + f(x)$, whence $F'(4) = 4f'(4) + f(4) = 4(-3) + 2 = -10$.

43. Calculate $H'(4)$, where $H(x) = \dfrac{x}{g(x)f(x)}$.

Let $H(x) = \dfrac{x}{g(x)f(x)}$. Then $H'(x) = \dfrac{g(x)f(x)\cdot 1 - x\,(g(x)f'(x) + f(x)g'(x))}{(g(x)f(x))^2}$.
Therefore, $H'(4) = \dfrac{(5)(2) - 4\cdot((5)(-3) + (2)(-1))}{((5)(2))^2} = \dfrac{78}{100} = \dfrac{39}{50}$.

45. Calculate $F'(0)$ where

$$F(x) = (1 + x + x^{1/2} + x^{3/2})\,\frac{3x^5 + 5x^4 + 5x + 1}{8x^9 - 7x^4}$$

Hint: see hint for previous exercise.

The function given is not defined at $x = 0$. Accordingly, its derivative cannot exist there.

47. Use the Product rule to show that $(f^2)' = 2f\cdot f'$.

Let $g = f^2 = ff$. Then $g' = \left(f^2\right)' = (ff)' = ff' + ff' = 2ff'$.

Further Insights and Challenges

49. Let f, g, h be differentiable functions. Show that $(fgh)'(x)$ is equal to

$$f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

Hint: write fgh as $(fg)h$ and apply the Product Rule twice.

Let $p = fgh$. Then $p' = (fgh)' = f\,(gh' + hg') + ghf' = f'gh + fg'h + fgh'$.

51. Derivative of the Reciprocal Let $f(x)$ be a function.

(a) Show that the difference quotient for $1/f(x)$ is equal to

$$\frac{f(x) - f(x+h)}{hf(x)f(x+h)}$$

(b) Show that $1/f(x)$ is differentiable for all x such that $f(x)$ is differentiable and $f(x) \neq 0$, and

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)}$$

(c) Use (b) and the Product Rule to derive the Quotient Rule. *Hint:* Write f/g as $f \cdot (1/g)$.

(d) Show that (b) can also be proved using the Quotient Rule.

Let $f(x)$ be differentiable.

(a) Let $g(x) = 1/f(x)$. The difference quotient for g is

$$\frac{g(x+h) - g(x)}{h} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \frac{\frac{f(x) - f(x+h)}{f(x+h)f(x)}}{h}$$
$$= \frac{f(x) - f(x+h)}{hf(x)f(x+h)} = \frac{-1}{f(x)f(x+h)}\,\frac{f(x+h) - f(x)}{h}$$

(b) Provided $f(x) \neq 0$, then

$$\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\left(\frac{-1}{f(x)f(x+h)}\,\frac{f(x+h) - f(x)}{h}\right) = -\frac{f'(x)}{(f(x))^2}.$$

Because the limit exists, $g(x) = 1/f(x)$ is differentiable for all x.

(c) Let $f/g = f \cdot (1/g)$. From Exercise 51, we have

$$\left(\frac{f}{g}\right)' = \left(f \cdot \left(\frac{1}{g}\right)\right)' = f \cdot \frac{-g'}{g^2} + \frac{1}{g} \cdot f' = \frac{gf' - fg'}{g^2}.$$

(d) Let $g(x) = 1/f(x)$, using the quotient rule, $g'(x) = \dfrac{f'(x) - f(x) \cdot 0}{f^2(x)} = \dfrac{f'(x)}{f^2(x)}$

53. R & W We say that a is a multiple root of a polynomial $f(x)$ if $(x-a)^2$ is a factor of $f(x)$. In other words, $f(x) = (x-a)^2 g(x)$, where $g(x)$ is also a polynomial. Show that if a is a multiple root of f, then $f'(a) = 0$.

Assume that $f(x) = (x-a)^2 g(x)$. Then
$f'(x) = (x-a)^2 g'(x) + g(x)2(x-a) = (x-a)^2 g'(x) + g(x)(2x - 2a)$. Therefore,
$f'(a) = (a-a)^2 g'(a) + g(a)(2a - 2a) = 0$.

The next problem uses the Product Rule to prove the Power Rule for positive integer exponents. It assumes that you are familiar with proof by induction.

55. The Power Rule Revisited Let n be a positive integer.

(a) Show directly that the Power Rule holds for $n = 1$.

(b) Write x^n as $x \cdot x^{n-1}$ and use the Product Rule to show

$$\frac{d}{dx}x^n = x\frac{d}{dx}x^{n-1} + x^{n-1}$$

(c) Use induction to prove that the Power Rule holds for all n.

Let k be a positive integer.

(a) If $k = 1$, then $x^k = x$. Note that $\frac{d}{dx}\left(x^1\right) = \frac{d}{dx}(x) = 1 = 1x^0$. Hence the Power Rule holds for $k = 1$.

(b) For a positive integer $n \geq 2$, observe that
$\frac{d}{dx}(x^n) = \frac{d}{dx}\left(x \cdot x^{n-1}\right) = x\frac{d}{dx}\left(x^{n-1}\right) + x^{n-1}\frac{d}{dx}(x) = x\frac{d}{dx}x^{n-1} + x^{n-1}.$

(c) By (a) the Power Rule holds for $k = 1$. Assume it holds for $k = n$ where $n \geq 2$. Then for $k = n + 1$, we have

$$\begin{aligned}\frac{d}{dx}\left(x^k\right) &= \frac{d}{dx}\left(x^{n+1}\right) = \frac{d}{dx}(x \cdot x^n) = x\frac{d}{dx}(x^n) + x^n\frac{d}{dx}(x)\\ &= x \cdot nx^{n-1} + x^n \cdot 1 = (n+1)x^n = kx^{k-1}\end{aligned}$$

Accordingly, the Power Rule holds for all positive integers by induction.

3.4 Rates of Change

Preliminary Questions

1. Pressure in a tank of water (measured in *atmospheres*) is a function of depth. What units could be used to measure the rate of change of pressure with respect to depth?

2. The concentration of nitrogen dioxide is measured in moles per liter. What units are used to measure the reaction rate, which is the rate of change of concentration?

3. How do we justify the definition of instantaneous rate of change as the derivative?

4. In his *Lectures on Physics*, Nobel laureate Richard Feynmann (1918–1988) uses the following dialogue to make a point about instantaneous velocity:

   Policeman: “My friend, you were going 75 miles an hour.”

   Driver: “That’s impossible, sir, I was traveling for only seven minutes.”

   What is wrong with the driver’s response?

5. Two trains travel from New York to Boston in $3\frac{1}{2}$ hours. The first train travels at a constant velocity of 60 mph, but the velocity of the second train varies. What was the second train’s average velocity during the trip?

6. True or False: the graph of velocity for a falling object is a parabola.

7. Could marginal cost be negative in a real situation?

Exercises

1. The distance (in feet) traveled by a particle at time t (in minutes) is $s(t) = \frac{1}{10}t^4 - \frac{1}{6}t^3 + 1$. Calculate the particle's velocity at $t = 5$.

First compute $s'(t) = \frac{2}{5}t^3 - \frac{1}{2}t^2$, then evaluate $s'(5) = 75/2 = 37.5$ ft/min.

3. A stone is tossed vertically upward with an initial velocity of 25 ft/sec from the top of a 30-ft building.

(a) What is the height of the stone after .25 seconds?

(b) Find the velocity of the stone after 1 second.

(c) When does the stone hit the ground?

We employ Galileo's formula, $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 = 30 + 25t - 16t^2$, where the time t is in seconds (s) and the height s is in feet (ft).

(a) The height of the stone after .25 seconds is $s(.25) = 35.25$ ft.

(b) The velocity at time t is $s'(t) = 25 - 32t$. When $t = 1$, this is -7 ft/s.

(c) When the stone hits the ground, its height is zero. Solve $30 + 25t - 16t^2 = 0$ to obtain $t = \dfrac{25 \pm \sqrt{2545}}{32}$ or $t \approx 2.36$ s.
(The other solution, $t \approx -0.79$, we discard since it represents a time before the stone was thrown.)

5. The velocity (in cm/s) of blood molecules flowing through a capillary of radius .008 cm depends on their distance r from the center of the capillary, according to the formula $v = .001(64 \cdot 10^{-6} - r^2)$. Find the rate of change of velocity as a function of distance when $r = .004$ cm.

The velocity of the blood molecules is $v'(r) = -.002r$. When $r = .004$ cm, this rate is $8 \cdot 10^{-6}\text{cm}^2/\text{s}$.

7. The escape velocity at a distance r from the center of the earth is $v_{\text{esc}} = 2.82 \cdot 10^7 \cdot r^{-1/2}$ m/s. Calculate the rate at which escape velocity changes with respect to distance at the surface of the earth (assume the radius of the earth is $6.77 \cdot 10^6$ m).

The rate that escape velocity changes is $v'_{\text{esc}}(r) = -\frac{1}{2}r^{-3/2}2.82 \cdot 10^7$.
Therefore,$v'_{\text{esc}}(6.77 \cdot 10^6) = -\frac{1}{2}(6.77 \cdot 10^6)^{-3/2}2.82 \cdot 10^7 = -8 \cdot 10^14$.

9. Suppose that the height of a jumping frog is $h(t) = 2.2t - 2.5t^2$.

(a) What is the frog's velocity at $t = .3$?

(b) When is the frog's velocity equal to 0?

(c) What is the maximum height of the frog?

(a) The velocity is $h'(t) = 2.2 - 5t$. Therefore, $h'(.3) = .7$.

(b) Solving for $h'(t) = 2.2 - 5t = 0$, we obtain $t = .44$.

(c) The maximum height the frog reaches is when the velocity is zero, which happens to be at time $t = .44$. Therefore, the maximum height is $h(.44) = 9.196$.

11. By Faraday's law, if a conducting wire of length ℓ meters moves at velocity v m/s in a magnetic field of strength B (in teslas), a voltage of size $V = -B\ell v$ is induced in the wire. Assume that $B = 2$ and $\ell = .5$.

(a) Find the rate of change of V with respect to v.
(b) Find the rate of change of V with respect to time t if $v = 4t + 9$.

(a) Assuming that $B = 2$ and $l = 5$, $V = -2(.5)v$. Therefore, $\frac{dV}{dv} = -2(.5) = -1$.
(b) If $v = 4t + 9$, then $V = -2(.5)(4t + 9)$. Therefore, $\frac{dV}{dt} = -2(.5)(4) = -4$.

13. R & W Table 1 gives total U.S. population during each month of 1999 as determined by the U.S. Department of Commerce.
(a) Estimate $P'(t)$ for each of the months January-November.
(b) Plot these data points for $P'(t)$ and connect the data points by a smooth curve.
(c) Write a newspaper headline describing the information contained in this plot.

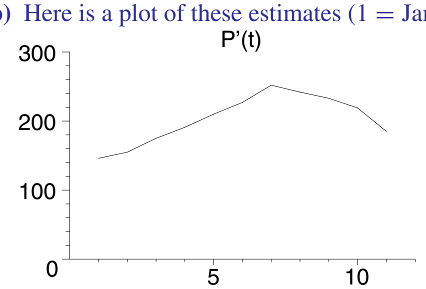
Table 1

t	$P(t)$ in thousands	t	$P(t)$ in thousands
January	271841	July	272945
February	271987	August	273197
March	272142	September	273439
April	272317	October	273672
May	272508	November	273891
June	272718	December	274076

The table in the text gives the growing population $P(t)$ of the United States.
(a) Here are estimates of $P'(t)$ in thousands/month for January–November.

t	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov
$P'(t)$	146	155	175	191	210	227	252	242	233	219	185

(b) Here is a plot of these estimates (1 = Jan, 2 = Feb, etc.)



(c) "U.S. Growth Rate Declines After Midsummer Peak"

15. Suppose that a cylinder has height equal to the radius r of its base. Calculate the rate of change of the volume of the cylinder with respect to r.

The volume of the cylinder is $V = \pi r^2 h = \pi r^3$. Thus $dV/dr = 3\pi r^2$.

17. Find the rate of change of the volume of a cube with respect to its surface area.

The volume of a cube of side length s is $V = s^3$. Its surface area is $A = 6s^2$. Thus $s = \frac{1}{\sqrt{6}}A^{1/2}$ and $V = \left(\frac{1}{\sqrt{6}}A^{1/2}\right)^3 = \frac{1}{6\sqrt{6}}A^{3/2}$. Hence $dV/dA = \frac{1}{4\sqrt{6}}A^{1/2} = \sqrt{6A}/24$.

19. The position of a moving particle during a 5-second trip is $s(t) = t^2 - t + 10$ cm.

(a) What is the average velocity for the entire trip?

(b) Is there a time at which the instantaneous velocity of the particle is equal to this average velocity? If so, find it.

Let $s(t) = t^2 - t + 10$, $0 \le t \le 5$, with s in centimeters (cm) and t in seconds (s).

(a) The average velocity over the t-interval $[0, 5]$ is $\dfrac{s(5) - s(0)}{5 - 0} = \dfrac{30 - 10}{5} = 4$ cm/s.

(b) The (instantaneous) velocity is $v(t) = s'(t) = 2t - 1$. Solving $2t - 1 = 4$ yields $t = \frac{5}{2}$ s, the time at which the instantaneous velocity equals the stated average velocity.

21. What is the velocity of an object dropped from a height of 300 m when it hits the ground?

We employ Galileo's formula, $s(t) = s_0 + v_0 t - \frac{1}{2} g t^2 = 300 - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). When the ball hits the ground its height is 0. Solve $s(t) = 300 - 4.9t^2 = 0$ to obtain $t \approx 7.8246$. (We discard the negative time, which took place before the ball was dropped.) The velocity at impact is $v(7.8246) \approx -76.68$ m/s. This signifies that the ball is *falling* at 76.68 m/s.

23. A ball tossed up vertically from ground level returns to earth four seconds later. What was the initial velocity of the stone?

Galileo's formula gives $s(t) = s_0 + v_0 t - \frac{1}{2} g t^2 = v_0 t - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). When the ball hits the ground after 4 seconds its height is 0. Solve $0 = s(4) = 4v_0 - 4.9(4)^2$ to obtain $v_0 = 19.6$ m/s.

25. Which of the following statements is true for an object falling under the influence of gravity near the surface of the earth?

(a) The object covers equal distance in equal time intervals.

(b) The velocity of a falling object increases by equal amounts in equal time intervals.

(c) The derivative of velocity increases with time.

For an object falling under the influence of gravity, Galileo's formula gives $s(t) = s_0 + v_0 t - \frac{1}{2} g t^2$.

(a) Since the height of the object varies quadratically with respect to time, it is *not* true that the object covers equal distance in equal time intervals.

(b) The velocity is $v(t) = s'(t) = v_0 - gt$. The velocity varies linearly with respect to time. Accordingly, the velocity decreases (becomes more negative) by equal amounts in equal time intervals. Moreover, its *speed* (the magnitude of velocity) increases by equal amounts in equal time intervals.

(c) Acceleration, the derivative of velocity with respect to time, is given by $a(t) = v'(t) = -g$. This is a *constant*; it does not change with time. Hence it is *not* true that acceleration (the derivative of velocity) increases with time.

27. A weight oscillates up and down at the end of a spring. Figure 2 shows the height y of the weight through one cycle of the oscillation. Make a rough sketch of the graph of the velocity as a function of time.

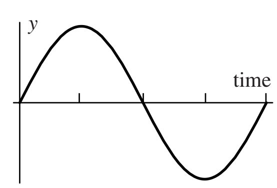
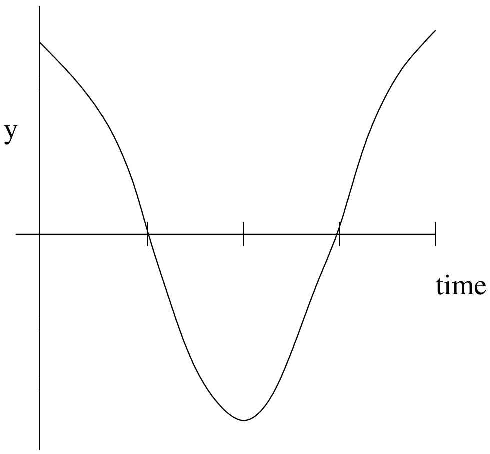


Figure 2

Here is the graph of the velocity of a function of time:



29. The population $P(t)$ of a city (in millions) is given by the function

$$P(t) = (.00005)t^2 + (.01)t + 1,$$

where t denotes the number of years since 1990.

(a) How large is the population in 1996 and how fast is it growing?

(b) When is the population growing at a rate of 12,000 people per year?

Let $P(t) = (.00005)t^2 + (.01)t + 1$ be the population of a city in millions. Here t is the number of years past 1990.

(a) In 1996 ($t = 6$ years after 1990), the population is $P(6) = 1.0618$ million. The rate of growth of population is $P'(t) = .0001t + .01$. In 1996, this corresponds to a growth rate of $P'(6) = .0106$ million per year or 10,600 people per year.

(b) When the growth rate is 12,000 people per year (or .012 million per year), we have $P'(t) = .0001t + .01 = .012$, whence $t = 20$. This corresponds to the year 2010; i.e., 20 years past 1990.

31. The dollar cost of producing x bagels (in units of 1000) is $C(x) = 300 + .25x - 5 \cdot 10^{-10}x^3$.

(a) Find the cost of producing x bagels for $x = 2000$ and $x = 4000$.

(b) *Estimate* the cost of the 2001st bagel (in other words, estimate $C(2001) - C(2000)$). Do the same for $x = 4000$.

(a) $C(2000) = 300 + .25(2000) - 5 \cdot 10^{-10}(2000)^3 = 796$ dollars.
$C(4000) = 300 + .25(4000) - 5 \cdot 10^{-10}(4000)^3 = 1268$ dollars.

(b) $C'(2000) \approx C(2001) - C(2000)$. Therefore, $C'(2000) \approx .244$ dollars and $C'(4000) \approx C(4001) - C(4000) \approx .226$ dollars.

33. Assume that the stopping distance (in feet) is given by the formula $F(s) = 1.1s + .03s^2$, where s is the vehicle's speed in mph (see Example **??**).

(a) Calculate $F(65)$ and $F'(65)$.

(b) Estimate the increase in stopping distance if speed is increased from 65 mph to 66 mph.

Let $F(s) = 1.1s + .03s^2$ be as in Example 4.

(a) Then $F(65) = 198.25$ ft and $F'(65) = 5.00$ ft/mph.

(b) $F'(65) \approx F(66) - F(65)$ is approximately equal to the change in stopping time per 1 mph increase in speed when traveling at 65 mph.

Further Insights and Challenges

35. Let $P(t)$ be the size of a certain animal population in a forest at time t (in months). It is found experimentally that the rate of change satisfies

$$\frac{dP}{dt} = 0.2(300 - P)$$

(a) Is the population growing or shrinking if $P = 250$? If $P = 350$?

(b) Graph dP/dt as a function of P for $0 \le P \le 300$.

(c) Which of the graphs in Figure 4 has the correct shape to be the graph of $P(t)$?

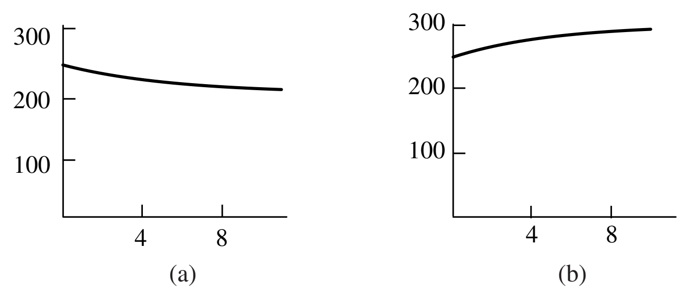
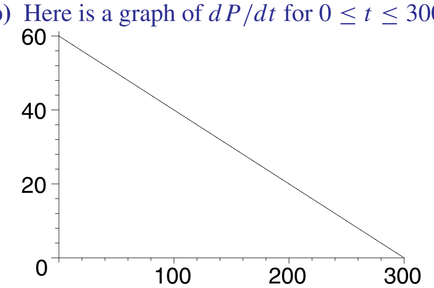


Figure 4

Let $P'(t) = dP/dt = 0.2(300 - P)$.

(a)
- Since $P'(250) = 10$, the population is growing when $P = 250$.
- Since $P'(350) = 10$, the population is shrinking when $P = 350$.

(b) Here is a graph of dP/dt for $0 \le t \le 300$.



(c)
- If $P(0) = 285$, as in graph (a), then $P'(0) = 3 > 0$ and the population is growing, contracting what is depicted in graph ?. Accordingly, graph (a) cannot be the correct shape for $P(t)$.
- If $P(0) = 250$, as in graph (b), then $P'(0) = 10 > 0$ and the population is growing as depicted. Thus graph (b) has the correct shape for $P(t)$.

In Exercises 36–37, define the average cost per unit at production level x as

$$C_{\text{av}}(x) = \frac{C(x)}{x}$$

where $C(x)$ is the cost function. Average cost is a measure of the efficiency *of the production process.*

37. Let $f(x) = \cos x$ and $A(x) = f(x)/x$.

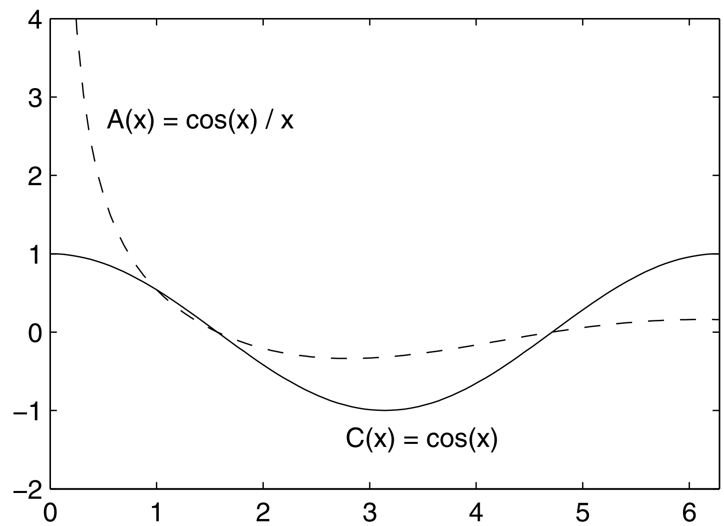
(a) Sketch the graph of $\cos x$ for $0 \le x \le 2\pi$.

(b) Indicate the point where $A(x)$ takes on its smallest value. *Hint:* Place a ruler on the graph so that it goes through the point (0, 1) and points $(x, \cos x)$. Rotate it until it has minimum slope.

(c) What happens if you place the ruler on the graph in (a) and let $x \to 0$?

Let $C(x) = \cos x$ and $A(x) = \dfrac{\cos x}{x}$.

(a) Here is a graph of $C(x)$ and $A(x)$ on the same set of axes.



(b) We place a ruler on the graph so that it goes through the origin and the point $(x, C(x))$ on the graph (as was done in Exercise **??**) and rotate it until it has minimum slope. This occurs around $x_0 = 2.8$.

(c) As we rotate the ruler and let $x \to 0$, the slope in question becomes positively infinite (and hence $A(x) \to \infty$).

3.5 Higher Derivatives

Preliminary Questions

1. An economist who announces that "America's economic growth is slowing" is making a statement about the GNP (gross national product) as a function of time. Is the second derivative of the GNP positive? What about the first derivative?

2. On December 29, 1993, *The New York Times* published an article with the headline "Rise in California Population Slows to Trail Nation's Rate." This statement is poorly worded because it seems to compare a rise, which is the actual change in population, to rate of change. Rewrite the headline as a statement about the appropriate derivative of California's population.

3. Rephrase the following as a statement about the appropriate higher derivative: *the velocity is still increasing, but not as rapidly as before*. Sketch a possible graph of velocity.

4. True or False: the third derivative of position is zero for an object falling to earth under the influence of gravity.

5. Which polynomials have the property that $f''(x) = 0$ for all x?

Exercises

In Exercises 1–6, calculate the second and third derivatives of the function.

1. 14

Let $f(x) = 14$. Then $f'(x) = 0$, $f''(x) = 0$, and $f'''(x) = 0$.

3. $z - \dfrac{1}{z}$

Let $f(z) = z - z^{-1}$. Then $f'(z) = 1 + z^{-2}$, $f''(z) = -2z^{-3}$, and $f'''(z) = 6z^{-1}$.

5. $t^2(t^2 + t)$

Let $g(t) = t^2(t^2 + 1) = t^4 + t^2$. Then $g'(t) = 4t^3 + 2t$, $g''(t) = 12t^2 + 2$, $g'''(t) = 24t$.

7. Find the acceleration at time $t = 5$ minutes of a helicopter whose height (in feet) is $s(t) = -3t^3 + 400t$.

Let $s(t) = 400t - 3t^3$, with t in minutes and s in feet. The velocity is $v(t) = s'(t) = 400 - 9t^2$ and acceleration is $a(t) = -18t$. Thus $a(5) = -90$ ft/min.

In Exercises 9–24, calculate the derivative indicated.

9. $f^{(4)}(1)$; $f(x) = x^4$

Let $f(x) = x^4$. Then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f'''(x) = 24x$, and $f^{(4)} = 24$. Thus $f^{(4)}(1) = 24$.

11. $y'''(1)$; $y = 4t^{-3} + 3t^2 - 9t$

Let $y = 4t^{-3} + 3t^2 - 9t$. Then $y' = -12t^{-4} + 6t - 9$, $y'' = 48t^{-5} + 6$, and $y''' = -240t^{-6}$. Hence $y'''(1) = -240$.

13. $h'''(9)$; $h(x) = \sqrt{x}$

Let $h(x) = \sqrt{x} = x^{1/2}$. Then $h'(x) = \frac{1}{2}x^{-1/2}$, $h''(x) = -\frac{1}{4}x^{-3/2}$, and $h'''(x) = \frac{3}{8}x^{-5/2}$. Thus $h'''(9) = \frac{1}{648}$.

15. $h'''(16)$; $h(x) = x^{-3/4}$

Let $g(x) = x^{-3/4}$. Then $g'(x) = \left(-\frac{4}{3}\right) x^{-7/2}$, $g''(x) = \left(\frac{21}{16}\right) x^{-11/4}$, and $g'''(x) = \left(-\frac{231}{64}\right) x^{-15/4}$. Thus $g'''(16) = (-231/64)16^{-15/4}$.

17. $f''(1)$; $f(t) = \dfrac{1}{t^3 + 1}$

Let $y = \dfrac{1}{t^3 + 1}$. Then

$$f'(t) = \frac{-3t^2}{(t^3+1)^2} \quad \text{and} \quad f''(t) = \frac{\left(t^3+1\right)^2(-6t) - \left(-3t^2\right) \cdot 2\left(t^3+1\right)\left(3t^2\right)}{\left(t^3+1\right)^4}.$$

Thus $f''(1) = \dfrac{-24+36}{16} = \dfrac{3}{4}$. (NOTE: To compute the derivative of $\left(t^3+1\right)^2 = \left(t^3+1\right)\left(t^3+1\right)$, use the Product Rule.)

19. $h''(1)$; $h(x) = \dfrac{1}{\sqrt{x} + 1}$

Let $h(x) = \dfrac{1}{\sqrt{x}+1}$. Then $h'(x) = \dfrac{-\frac{1}{2}x^{-1/2}}{\left(\sqrt{x}+1\right)^2}$ and

$$h''(x) = \frac{\left(\sqrt{x}+1\right)^2\left(\frac{1}{4}x^{-3/2}\right) - \left(-\frac{1}{2}x^{-1/2}\right) \cdot 2\left(\sqrt{x}+1\right)\left(\frac{1}{2}x^{-1/2}\right)}{\left(\sqrt{x}+1\right)^4}.$$

Accordingly, $h''(x)(1) = \frac{1+1}{16} = \frac{1}{8}$. (NOTE: To compute the derivative of $\left(\sqrt{x}+1\right)^2 = \left(\sqrt{x}+1\right)\left(\sqrt{x}+1\right)$, use the Product Rule.)

21. $y'''(1)$; $y = x - \dfrac{1}{x}$

Let $y = x - x^{-1}$. Then $y' = 1 + x^{-2}$, $y'' = -2x^{-3}$, and $y''' = 6x^{-4}$. Hence $y'''(1) = 6$.

23. $y^{(4)}(1)$; $y = x^4 + ax^3 + bx^2 + cx + d$
a, b, c, d constants

Let $y = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are constants. Then $y' = 4x^3 + 3ax^2 + 2bx + c$, $y'' = 12x^2 + 6ax + 2b$, $y''' = 24x + 6a$, and $y^{(4)} = 24$. Hence $y^{(4)}(1) = 24$.

25. Find a polynomial $f(x)$ satisfying the equation $xf''(x) + f(x) = x^2$. *Hint:* try a quadratic polynomial.

Let $f(x) = ax^2 + bx + c$. Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. Substituting into the equation $xf''(x) + f(x) = x^2$ yields $ax^2 + (2a+b)x + c = x^2$, an identity in x. Equating coefficients, we have $a = 1$, $2a + b = 0$, $c = 0$, whence $a = 1$, $b = -2$, and $c = 0$.

27. Find the equation of the tangent to the graph of f' at $x = 3$, where $f(x) = x^4$.

Let $f(x) = x^4$ and $g(x) = f'(x) = 4x^3$. Then $g'(x) = 12x^2$. The tangent line to g at $x = 3$ is given by $y = g(3) + g'(3)(x-3) = 108 + 108(x-3) = 108x - 216$ or $y = 108x - 216$.

29. Find a general formula for the nth derivative of $f(x) = x^{-2}$.

$f'(x) = -2x^{-3}$, $f''(x) = 6x^{-4}$, $f'''(x) = 24x^{-}5$, $f^{(4)}(x) = 5 \cdot 24(x)^{-6}, \ldots$. From this we can conclude that the nth derivative can ve written as $f^{(n)}(x) = (-1)^n(n+1)!x^{-(n+2)}$.

31. Figure 1 shows f, f' and f''. Determine which is which.

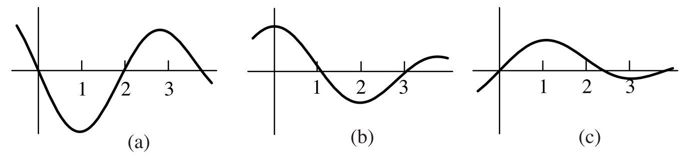


Figure 1

(a) f'' (b) f' (c) f.

33. Figure 3 shows the graph of position as a function of time. Determine the intervals on which the acceleration is positive.

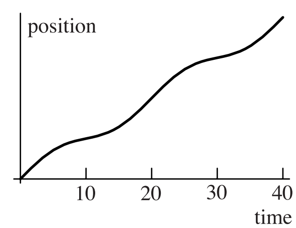


Figure 3

Roughly from time 15 to time 20 and from time 32 on.

35. R & W In a bottle containing p fruit flies, the number of offspring produced by each female is $f(p) = (34 - 0.612p)p^{-0.658}$.

(a) Calculate $f''(10)$.

(b) Give a verbal interpretation of $f''(10)$.

Let $f(p) = (34 - 0.612p)p^{-0.658} = 34p^{-0.658} - 0.612p^{0.342}$. Then
$f'(p) = 22.372p^{-1.658} - 0.209304p^{-0.658}$ and
$f''(p) = 37.092776p^{-2.658} + .137722032p^{-1.658}$.

(a) Therefore, $f(10) \approx 6.1276$, $f'(10) \approx -.5377$, and $f''(10) \approx 0.08455$.

(b) The quantity $f(10) \approx 6.1276$ shows that at Day 10 each female fly is producing roughly 6 offspring per day. The quantity $f'(10) \approx -.5377$ means that the reproduction rate is declining at Day 10. This shows that the flies are becoming less fertile as their population increases (due to overcrowding in the bottle). Finally, $f''(10) \approx .08455$ signifies that the rate at which this decline in reproduction rate occurs is becoming less pronounced; i.e., less negative.

Further Insights and Challenges

37. Show that for any three values A, B, C there exists a unique quadratic polynomial $f(x)$ such that $f(0) = A$, $f'(0) = B$, and $f''(0) = C$.

Let $f(x) = px^2 + qx + r$, where p, q, and r are constants. Then $f'(x) = 2px + q$ and $f''(x) = 2p$. Setting $f(0) = A$, $f'(0) = B$, and $f''(0) = C$ gives $r = A$, $q = B$, and $2p = C$. Hence $p = C/2$, $q = B$, and $r = A$.

39. Use the Product Rule to find a formula for $(fg)'''$ and compare your result with the expansion of $(a+b)^3$. Then try to guess the general formula for $(fg)^{(n)}$.

Continuing from Exercise **??**, we have

$$h''' = f''g' + gf''' + 2(f'g'' + g'f'') + fg''' + g''f' = f'''g + 3f''g' + 3f'g'' + fg'''$$

The binomial theorem gives
$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$ and more generally $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k$, where the binomial coefficients are given by $\binom{n}{k} = \dfrac{k(k-1)\cdots(k-n+1)}{n!}$. Accordingly, the general formula for $(fg)^{(n)}$ is given by $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}g^k$, where $p^{(k)}$ is the kth derivative of p (or p itself when $k = 0$).

3.6 Trigonometric Functions

Preliminary Questions

1. Determine if the following equations are true or false and if false, state the correct version.

(a) $\dfrac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$

(b) $\dfrac{d}{dx}\tan x = \sec x$

(c) $\dfrac{d}{dx}\csc x = -\cot x$

2. Which of the following functions can be differentiated *using the rules we have covered so far*?

(a) $3\cos x \cot x$

(b) $\dfrac{d}{dx}\cos(x^2)$

(c) $\dfrac{d}{dx}x^2\cos x$

3. Compute $d/dx\left(\sin^2 x + \cos^2 x\right)$ without using the derivative formulas for $\sin x$ and $\cos x$.

4. How is the addition formula used in deriving the formula $(\sin x)' = \cos x$?

Exercises

In Exercises 1–4, find the equation of the tangent line at the point indicated.

1. $\sin x; \quad x = \frac{\pi}{4}$

Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and $f'(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$.

3. $\tan x; \quad x = \frac{\pi}{4}$

Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x$ and $f'\left(\frac{\pi}{4}\right) = 2$.

In Exercises 5–26, use the Product and Quotient Rules as necessary to find the derivative of each function.

5. $\sin x \cos x$

Let $f(x) = \sin x \cos x$. Then $f'(x) = -\sin^2 x + \cos^2 x$.

7. $\sin^2 x$

Let $f(x) = \sin^2 x = \sin x \sin x$. Then $f'(x) = 2 \sin x \cos x$.

9. $x^3 \sin x$

Let $f(x) = x^3 \sin x$. Then $f'(x) = x^3 \cos x + 3x^2 \sin x$.

11. $\tan x \sec x$

Let $f(x) = \tan x \sec x$. Then
$f'(x) = \tan x \sec x \tan x + \sec x \sec^2 x = \sec x \tan^2 x + \sec^3 x$ or $\left(\tan^2 x + \sec^2 x\right) \sec x$.

13. $x^2 \sin^2 x$

Let $f(x) = x^2 \sin^2 x$. Then
$f'(x) = x^2 (2 \sin x \cos x) + 2x \sin^2 x = 2x^2 \sin x \cos x + 2x \sin x$. (Here we used the result from Exercise 7.)

15. $(x - x^2) \cot x$

Let $f(x) = (x - x^2) \cot x$. Then $f'(x) = (x - x^2)(-\csc^2 x) + \cot x (1 - 2x)$.

17. $\frac{\sec x}{x^2}$

Let $f(x) = \frac{\sec x}{x^2}$. Then $f'(x) = \frac{\sec x \tan x (x^2) - 2x \sec x}{x^4}$.

19. $\sin t - \frac{2}{\cos t}$

Let $f(t) = \sin t - \frac{2}{\cos t} = \sin t - 2 \sec t$. Then $f'(t) = \cos t - 2 \sec t \tan t$.

21. $\dfrac{x}{\sin x+2}$

Let $f(x)=\dfrac{x}{2+\sin x}$. Then $f'(x)=\dfrac{(2+\sin x)(1)-x\cos x}{(2+\sin x)^2}=\dfrac{2+\sin x-x\cos x}{(2+\sin x)^2}$.

23. $\dfrac{1+\sin x}{1-\sin x}$

Let $f(x)=\dfrac{1+\sin x}{1-\sin x}$. Then
$f'(x)=\dfrac{(1-\sin x)(\cos x)-(1+\sin x)(-\cos x)}{(1-\sin x)^2}=\dfrac{2\cos x}{(1-\sin x)^2}$.

25. $\dfrac{\sec x}{x}$

Let $f(x)=\dfrac{\sec x}{x}$. Then $f'(x)=\dfrac{x\sec x\tan x-(\sec x)(1)}{x^2}=\dfrac{(x\tan x-1)\sec x}{x^2}$.

In Exercises 27–30, calculate the second derivative of the function.

27. $3\sin x+4\cos x$

Let $f(x)=3\sin x+4\cos x$. Then $f'(x)=3\cos x-4\sin x$ and
$f''(x)=-3\sin x-4\cos x$.

29. $\tan x$

Let $f(x)=\tan x$. Then $f'(x)=\sec^2 x=\sec x\sec x$ and
$f''(x)=2(\sec x)(\sec x\tan x)=2\sec^2 x\tan x$.

In Exercises 31–36, find the equation of the tangent line at the point specified.

31. $x^2+\sin x;\qquad x=0$

Let $f(x)=x^2+\sin x$. Then $f'(x)=2x+\cos x$ and $f'(0)=1$. The tangent line at $x=0$ is
$y=f(0)+f'(0)(x-0)=0+1(x-0)$ or $y=x$.

33. $2\sin x+3\cos x;\qquad x=\pi/3$

Let $f(x)=2\sin x+3\cos x$. Then $f'(x)=2\cos x-3\sin x$ and $f'\left(\frac{\pi}{3}\right)=1-\frac{3\sqrt{3}}{2}$. The tangent line at $x=\frac{\pi}{3}$ is
$y=f\left(\frac{\pi}{3}\right)+f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right)=\sqrt{3}+\frac{3}{2}+\left(1-\frac{3\sqrt{3}}{2}\right)\left(x-\frac{\pi}{3}\right)=$
$\left(1-\frac{3\sqrt{3}}{2}\right)x+\sqrt{3}+\frac{3}{2}+\frac{\sqrt{3}}{2}\pi-\frac{\pi}{3}$.

35. $\csc x-\cot x;\qquad x=\pi/4$

Let $f(x)=\csc x-\cot x$. Then $f'(x)=\csc^2 x-\csc x\cot x$ and
$f'\left(\frac{\pi}{4}\right)=2-\sqrt{2}\cdot 1=2-\sqrt{2}$. Hence the tangent line is
$y=f\left(\frac{\pi}{4}\right)+f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)=\left(\sqrt{2}-1\right)+\left(2-\sqrt{2}\right)\left(x-\frac{\pi}{4}\right)$. or
$y=\left(2-\sqrt{2}\right)x+\sqrt{2}-1+\frac{\pi}{4}\left(\sqrt{2}-2\right)$.

37. Use the formulas $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ together with the rules of differentiation to verify the formulas.

(a) $\frac{d}{dx} \cot x = -\csc^2 x$

(b) $\frac{d}{dx} \sec x = \sec x \tan x$

(c) $\frac{d}{dx} \csc x = -\csc x \cot x$

(a) $\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{(\sin x)(-\sin x) - \cos x \cos x}{\sin^2 x}$
$= -\frac{1}{\sin^2 x} = -\csc^2 x.$

(b) $\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = -\frac{-\sin x}{\cos^2 x} = \sec x \tan x.$

(c) $\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\frac{\cos x}{\sin^2 x} = -\csc x \cot x.$

39. Let $f(x) = \cos x$.

(a) Calculate the first five derivatives of f.

(b) What is $f^{(8)}$?

(c) What is $f^{(37)}$?

Let $f(x) = \cos x$.

(a) Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, and $f^{(5)}(x) = -\sin x$.

(b) Accordingly, the successive derivatives of f cycle among $\{-\sin x, -\cos x, \sin x, \cos x\}$ in that order. Since 8 is a multiple of 4, we have $f^{(8)}(x) = \cos x$.

(c) Since 36 is a multiple of 4, we have $f^{(36)}(x) = \cos x$, whence $f^{(37)}(x) = -\sin x$.

41. Find $y^{(157)}$, where $y = \sin x$.

Let $f(x) = \sin x$. Then the successive derivatives of f cycle among $\{\cos x, -\sin x, -\cos x, \sin x\}$ in that order. Since 156 is a multiple of 4, we have $f^{(156)}(x) = \sin x$, whence $f^{(157)}(x) = \cos x$.

43. A weight attached to a spring is oscillating up and down. Its height above ground at time t seconds is $s(t) = 300 + 40 \sin t$ cm. Find the velocity and acceleration of the weight at time $t = \frac{\pi}{3}$ seconds.

Let $h(t) = 300 + 40 \sin t$ be the height. Then the velocity is $v(t) = h'(t) = 40 \cos t$ and the acceleration is
$a(t) = v'(t) = -40 \sin t$. At $t = \frac{\pi}{3}$, the velocity is $v\left(\frac{\pi}{3}\right) = 20$ and the acceleration is $a\left(\frac{\pi}{3}\right) = -20\sqrt{3}$.

Further Insights and Challenges

45. Prove that $d\cos x/dx = -\sin x$ using the limit definition of the derivative and the addition law for the cosine: $\cos(a+b) = \cos a \cos b - \sin a \sin b$.

Let $f(x) = \cos x$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h\to 0} \left((-\sin x)\frac{\sin h}{h} + (\cos x)\frac{\cos h - 1}{h} \right) \\ &= (-\sin x)\cdot 1 + (\cos x)\cdot 0 = -\sin x. \end{aligned}$$

47. This exercise proves that $\sin x$ is not a polynomial function.

(a) Show that the function $y = \sin x$ satisfies $y'' = -y$.

(b) Show that a nonzero polynomial *cannot* satisfy $y'' = -y$. *Hint:* if y is a polynomial, then y'' is a polynomial of lower degree than y.

(c) Does the same reasoning show that $\cos x$ is not a polynomial?

(a) Let $y = \sin x$. Then $y' = \cos x$ and $y'' = -\sin x$. Therefore, $y'' = -y$.

(b) Let p be a nonzero polynomial of degree n and *assume* that p satisfies the differential equation $y'' + y = 0$. Then $p'' + p = 0$ for all x. There are exactly three cases.

- **i.** If $n = 0$, then p is a constant polynomial and thus $p'' = 0$. Hence $0 = p'' + p = p$ or $p \equiv 0$ (i.e., p is equal to 0 for all x or p is identically 0). This is a contradiction, since p is a *non*zero polynomial.
- **ii.** If $n = 1$, then p is a linear polynomial and thus $p'' = 0$. Once again, we have $0 = p'' + p = p$ or $p \equiv 0$, a contradiction since p is a nonzero polynomial.
- **iii.** If $n \geq 2$, then p is a quadratic polynomial and thus p'' is a polynomial of degree $n - 2 \geq 0$. Thus $q = p'' + p$ is a polynomial of degree $n \geq 2$. By assumption, however, $p'' + p = 0$. Thus $q \equiv 0$, a polynomial of degree 0. This is a contradiction, since the degree of q is $n \geq 2$.

CONCLUSION: In all cases, we have reached a contradiction. Therefore the *assumption* that p satisfies the differential equation $y'' + y = 0$ is *false*. Accordingly, a nonzero polynomial *cannot* satisfy the stated differential equation.

(c) If $f(x) = \cos x$ were a polynomial, then $f^{(k)}$, the k^{th} derivative of $\cos x$, would be 0 for some k. This is a contradiction, since the successive derivatives of f cycle among $\{-\sin x, -\cos x, \sin x, \cos x\}$.

49. Use the addition formula for the sine function to show that $\tan(x+h) - \tan x = \sin h \sec x \sec(x+h)$. Then use this identity to verify that $d/dx \tan x = \sec^2 x$.

The addition formula for sine is $\sin(a+b) = \sin a \cos b + \sin b \cos a$ and the addition formula for cosine is $\cos(a+b) = \cos a \cos b - \sin a \sin b$. We rewrite $\tan(x+h) - \tan x$

as :

$$\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}$$
$$= \frac{\sin x \cos h + \sin h \cos x}{\cos x \cos h - \sin x \sin h} - \frac{\sin x}{\cos x}$$
$$= \frac{\sin x \cos h \cos x + \sin h \cos^2 x - \sin x \cos h \cos x + \sin^2 x \sin h}{\cos x(\cos x \cos h - \sin x \sin h)}$$
$$= \frac{\sin h(\cos^2 x + \sin^2 x}{\cos x(\cos x \cos h - \sin x \sin h)}$$
$$= \frac{\sin h}{\cos x(\cos(x+h))}$$
$$= \sin h \sec x \sec(x+h)$$

We verify that $\frac{d}{dx} \tan x = \sec^2 x$ by using the definition of limit:

$$\frac{d}{dx} \tan x = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h} = \lim_{h\to 0} \frac{\sin h \sec h \sec(x+h)}{h}$$
$$= \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \sec x \sec(x+h) = \sec^2 x$$

51. (Exercises 51 and 52 are taken from *Trigonometry*, by I.M. Gelfand and M. Saul, Birkhauser, 2001.) Show that if $\pi/2 < \theta < \pi$, then the distance along the x-axis between θ and the point where the tangent line intersects the x-axis is equal to $|\tan\theta|$.

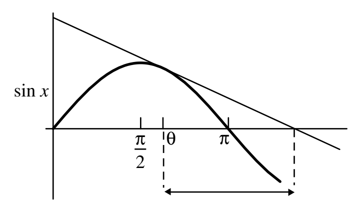


Figure 1

Let $f(x) = \sin x$. Since $f'(x) = \cos x$, this means that the tangent line at $(\theta, \sin\theta)$ is $y = \cos\theta(x - \theta) + \sin\theta$. When $y = 0$, $x = \theta - \tan\theta$. Distance from x to θ is:

$$|\theta - (\theta - \tan\theta)| = |\tan\theta|$$

3.7 The Chain Rule

Preliminary Questions

1. Identify the outside and inside functions for each of the composite functions.

(a) $\sqrt{4x+9x^2}$
(b) $(1+\sec x)^5$
(c) $\frac{\tan x}{\tan x+1}$

2. Which of the following functions can be differentiated easily *without* using the Chain Rule?

$$\tan(7x+2), \quad \frac{x}{x+1}, \quad \sqrt{x}\cdot\sec x, \quad \sqrt{x\cos x}$$

3. What is the derivative of $(f(x))^3$ at $x=0$ if $f(0)=2$ and $f'(0)=3$?

4. What is the derivative of $f(5x)$ at $x=0$ if $f'(0)=3$?

5. How many times must the Chain Rule be used to differentiate the functions?
(a) $\cos(x^2+1)$
(b) $\cos((x^2+1)^4)$
(c) $\sqrt{\cos((x^2+1)^4)}$

6. Suppose that $f'(4)=6$, $g(4)=2$, and $g'(4)=3$. Do we have enough information to compute $G'(4)$ where $G(x)=g(f(x))$? If not, what is missing?

Exercises

1. Fill in a table of the following type

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f\circ g)'$

for the functions:
(a) $f(u)=u^{3/2}$, $g(x)=x^4+1$
(b) $f(u)=\tan u$, $g(x)=x^4$

(a) Let $f(u)=u^{3/2}$ and $g(x)=x^4+1$. Then

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f\circ g)'$
$\left(x^4+1\right)^{3/2}$	$\frac{3}{2}u^{1/2}$	$\frac{3}{2}\left(x^4+1\right)^{1/2}$	$4x^3$	$6x^3\left(x^4+1\right)^{1/2}$

(b) Let $f(u)=\tan u$ and $g(x)=x^4$. Then

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f\circ g)'$
$\tan\left(x^4\right)$	$\sec^2 u$	$\sec^2\left(x^4\right)$	$4x^3$	$4x^3\sec^2\left(x^4\right)$

3. Find functions f and g such that $f(g(x))=(x+\sin x)^4$. Then use the Chain Rule to compute the derivative.

Let $f(x)=x^4$, $g(x)=x+\sin x$, and $h(x)=f(g(x))=(x+\sin x)^4$. Then $h'(x)=f'(g(x))g'(x)=4(x+\sin x)^3(1+\cos x)$.

5. Write the composite function $\cos u$ for each of the following choices of $u(x)$ and calculate $d\cos u/dx$:
(a) $9-x^2$
(b) x^{-1}
(c) $\tan x$

(a) $\cos(u(x)) = \cos(9 - x^2)$. $\dfrac{d}{dx}\cos(u(x)) = -\sin(9 - x^2)(-2x) = 2x\sin(9 - x^2)$.

(b) $\cos(u(x)) = \cos(x^{-1})$. $\dfrac{d}{dx}\cos(u(x)) = \dfrac{-\sin(x^{-1})}{x^2}$.

(c) $\cos(u(x)) = \cos(\tan x)$. $\dfrac{d}{dx}\cos(u(x)) = -\sin(\tan x)(\sec^2 x) = -\sec^2 x\sin(\tan x)$.

In Exercises 7–14, write the composite function $f(g(x))$ and find its derivative.

7. $f(u) = \sin u; \quad g(x) = 2x + 1$

Let $h(x) = f(g(x)) = \sin(2x + 1)$. Then $h'(x) = 2\cos(2x + 1)$.

9. $f(u) = \cos u; \quad g(x) = x^3$

Let $h(x) = f(g(x)) = \cos\left(x^3\right)$. Then $h'(x) = -3x^2\sin\left(x^3\right)$.

11. $f(u) = \sqrt{u}; \quad g(x) = \tan x$

Let $h(x) = f(g(x)) = (\tan x)^{1/2}$. Then $h'(x) = \frac{1}{2}(\tan x)^{-1/2}\sec^2 x = \dfrac{\sec^2 x}{2\sqrt{\tan x}}$.

13. $f(u) = \tan u; \quad g(x) = \sqrt{x}$

Last $h(x) = f(g(x)) = \tan\left(x^{1/2}\right)$. Then $h'(x) = \sec^2\left(x^{1/2}\right) \cdot \dfrac{1}{2}x^{-1/2} = \dfrac{\sec^2\left(\sqrt{x}\right)}{2\sqrt{x}}$.

In Exercises 15–18, write both composite functions $f(g(x))$ and $g(f(x))$, and calculate their derivatives.

15. $f(u) = \cos u + \sin u; \quad g(t) = t^{1/2}$

Let $f(u) = \cos u + \sin u$, $g(t) = t^{1/2}$, $h(x) = \cos\left(x^{1/2}\right) + \sin\left(x^{1/2}\right)$, and $H(x) = (\cos x + \sin x)^{1/2}$.

Then $h'(x) = -\sin\left(x^{1/2}\right) \cdot \dfrac{1}{2}x^{-1/2} + \cos\left(x^{1/2}\right) \cdot \dfrac{1}{2}x^{-1/2} = \dfrac{\cos\left(\sqrt{x}\right) - \sin\left(\sqrt{x}\right)}{2\sqrt{x}}$ and

$H'(x) = \dfrac{1}{2}(\cos x + \sin x)^{-1/2}(\cos x - \sin x) = \dfrac{\cos x - \sin x}{2\sqrt{\cos x + \sin x}}$.

17. $f(u) = \dfrac{1}{u}; \quad g(x) = x^2 + 3x + 1$

Let $f(u) = 1/u$, $g(x) = x^2 + 3x + 1$, $h(x) = \left(x^2 + 3x + 1\right)^{-1}$, and $H(x) = x^{-2} + 3x^{-1} + 1$.

Then $h'(x) = -\left(x^2 + 3x + 1\right)^{-2}(2x + 3) = -\dfrac{2x + 3}{(x^2 + 3x + 1)^2}$ and

$H'(x) = -2x^{-3} - 3x^{-2}$.

In Exercises 19–36, use the Chain Rule to find the derivative.

19. $\cos(x^2)$

Let $f(x) = \cos\left(x^2\right)$. Then $f'(x) = -\sin\left(x^2\right) \cdot 2x = -2x\sin\left(x^2\right)$.

21. $\cot(4x^2+9)$

Let $f(x) = \cot\left(4x^2+9\right)$. Then $f'(x) = -\csc^2\left(4x^2+9\right)\cdot 8x = -8x\csc^2\left(4x^2+9\right)$

23. $\tan(x+\cos x)$

Let $f(x) = \tan(x+\cos x)$. Then
$f'(x) = \sec^2(x+\cos x)\cdot(1-\sin x) = (1-\sin x)\sec^2(x+\cos x)$.

25. $\left(\dfrac{x+1}{x-1}\right)^4$

Let $f(x) = \left(\dfrac{x+1}{x-1}\right)^4$. Then
$f'(x) = 4\left(\dfrac{x+1}{x-1}\right)^3 \cdot \dfrac{(x-1)\cdot 1-(x+1)\cdot 1}{(x-1)^2} = -\dfrac{8(x+1)^3}{(x-1)^5}$ or $\dfrac{8(1+x)^3}{(1-x)^5}$.

27. $(t^2+3t+1)^{1/2}$

Let $f(t) = \left(t^2+3t+1\right)^{1/2}$. Then $f'(t) = \dfrac{1}{2}\left(t^2+3t+1\right)^{-1/2}(2t+3) = \dfrac{2t+3}{2\sqrt{t^2+3t+1}}$.

29. $(t^4+3t^2-9)^{1/3}$

Let $f(t) = \left(t^4+3t^2-9\right)^{1/3}$. Then
$f'(t) = \dfrac{1}{3}\left(t^4+3t^2-9\right)^{-2/3}\left(4t^3+6t\right) = \dfrac{2\left(2t^3+3t\right)}{3\left(t^4+3t-9\right)^{2/3}}$.

31. $\cot(\sin\sqrt{x})$

Let $f(x) = \cot\left(\sin\left(x^{1/2}\right)\right)$. Then
$f'(x) = -\csc^2\left(\sin\left(x^{1/2}\right)\right)\cdot\cos\left(x^{1/2}\right)\cdot\dfrac{1}{2}x^{-1/2} = -\dfrac{\cos\left(\sqrt{x}\right)\csc^2\left(\sin\left(\sqrt{x}\right)\right)}{2\sqrt{x}}$.

33. $(\csc(1-x)+x)^5$

Let $f(x) = (x+\csc(1-x))^5$. Then
$f'(x) = 5(x+\csc(1-x))^4\cdot(1-\csc(1-x)\cot(1-x)\cdot(-1))$
$= 5(x+\csc(1-x))^4(1+\csc(1-x)\cot(1-x))$.

35. $(\sqrt{x+1}-1)^{3/2}$

Let $f(x) = \left((x+1)^{1/2}-1\right)^{3/2}$. Then
$f'(x) = \dfrac{3}{2}\left((x+1)^{1/2}-1\right)^{1/2}\cdot\left(\dfrac{1}{2}(x+1)^{-1/2}\cdot 1\right) = \dfrac{3\sqrt{\sqrt{x+1}-1}}{4\sqrt{x+1}}$.

In Exercises 37–63, find the derivative using the appropriate rule or combination of rules.

37. $x\cos(1-3x)$

Let $f(x) = x\cos(1-3x)$. Then
$f'(x) = x(-\sin(1-3x))\cdot(-3)+\cos(1-3x)\cdot 1 = 3x\sin(1-3x)+\cos(1-3x)$.

39. $\sin(\cos x)$

Let $f(x) = \sin(\cos x)$. Then $f'(x) = \cos(\cos x) \cdot (-\sin x) = -\sin x \cos(\cos x)$.

41. $\sin(\cos(\sin x))$

Let $f(x) = \sin(\cos(\sin x))$. Then
$f'(x) = \cos(\cos(\sin x)) \cdot (-\sin(\sin x)) \cdot \cos x = -\cos x \sin(\sin x) \cos(\cos(\sin x))$.

43. $\sqrt{x^3 + 2x}$

Let $f(x) = \left(x^3 + 2x\right)^{1/2}$. Then $f'(x) = \frac{1}{2}\left(x^3 + 2x\right)^{-1/2}\left(3x^2 + 2\right) = \dfrac{3x^2 + 2}{2\sqrt{x^3 + x}}$.

45. $7 \cos 4t$

Let $f(t) = 7 \cos 4t$. Then $f'(t) = 7(-\sin 4t) \cdot 4 = -28 \sin 4t$.

47. $(\cos 6x + \sin x^2)^{1/2}$

Let $f(x) = \left(\cos 6x + \sin\left(x^2\right)\right)^{1/2}$. Then

$$f'(x) = \frac{1}{2}\left(\cos 6x + \sin\left(x^2\right)\right)^{-1/2}\left(-\sin 6x \cdot 6 + \cos\left(x^2\right) \cdot 2x\right)$$
$$= \frac{x \cos\left(x^2\right) - 3 \sin 6x}{\sqrt{\cos 6x + \sin\left(x^2\right)}}.$$

49. $\tan^3 x + \tan(x^3)$

Let $f(x) = \tan^3 x + \tan\left(x^3\right) = (\tan x)^3 + \tan\left(x^3\right)$. Then
$f'(x) = 3(\tan x)^2 \sec^2 x + \sec^2\left(x^3\right) \cdot 3x^2 = 3\left(x^2 \sec^2\left(x^3\right) + \sec^2 x \tan^2 x\right)$.

51. $\sqrt{\dfrac{x+1}{x-1}}$

Let $f(x) = \left(\dfrac{x+1}{x-1}\right)^{1/2}$. Then $f'(x) = \dfrac{1}{2}\left(\dfrac{x+1}{x-1}\right)^{-1/2} \cdot \dfrac{(x-1) \cdot 1 - (x+1) \cdot 1}{(x-1)^2}$

$$= \frac{-1}{\sqrt{x+1}\,(x-1)^{3/2}} = \frac{1}{(1-x)\sqrt{x^2-1}}.$$

53. $\left(1 + \cot^5(x^4 + 1)\right)^9$

Let $f(x) = \left(1 + \cot^5\left(x^4 + 1\right)\right)^9$. Then

$$f'(x) = 9\left(1 + \cot^5\left(x^4 + 1\right)\right)^8 \cdot 5 \cot^4\left(x^4 + 1\right) \cdot \left(-\csc^2\left(x^4 + 1\right)\right) \cdot 4x^3$$
$$= -180x^3 \cot^4\left(x^4 + 1\right) \csc^2\left(x^4 + 1\right)\left(1 + \cot^5\left(x^4 + 1\right)\right)^8.$$

55. $\cot^7(x^5)$

Let $f(x) = \cot^7\left(x^5\right)$. Then

$$f'(x) = 7 \cot^6\left(x^5\right) \cdot \left(-\csc^2\left(x^5\right)\right) \cdot 5x^4$$
$$= -35x^4 \cot^6\left(x^5\right) \csc^2\left(x^5\right).$$

57. $\sqrt{1+\sqrt{1+\sqrt{x}}}$

Let $f(x) = \left(1 + \left(1 + x^{1/2}\right)^{1/2}\right)^{1/2}$. Then
$f'(x) = \frac{1}{2}\left(1 + \left(1 + x^{1/2}\right)^{1/2}\right)^{-1/2} \cdot \frac{1}{2}\left(1 + x^{1/2}\right)^{-1/2} \cdot \frac{1}{2}x^{-1/2}$
$= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}$.

59. $\sqrt{\sqrt{\cos x + 1} + 1}$

Let $f(x) = \left(1 + (1 + \cos x)^{1/2}\right)^{1/2}$. Then
$f'(x) = \frac{1}{2}\left(1 + (1 + \cos x)^{1/2}\right)^{-1/2} \cdot \frac{1}{2}(1 + \cos x)^{-1/2}(-\sin x)$
$= -\frac{\sin x}{4\sqrt{1+\cos x}\sqrt{1+\sqrt{1+\cos x}}}$.

61. $\sqrt{kx+b}$; k and b any constants.

Let $f(x) = (kx + b)^{1/2}$, where b and k are constants. Then
$f'(x) = \frac{1}{2}(kx + b)^{-1/2} \cdot k = \frac{k}{2\sqrt{kx+b}}$.

63. $\frac{1}{\sqrt{kt^4+b}}$; k and b any constants.

Let $f(t) = \left(kt^4 + b\right)^{-1/2}$, where b and k are constants. Then
$f'(t) = -\frac{1}{2}\left(kt^4 + b\right)^{-3/2} \cdot 4kt^3 = -\frac{2kt^3}{(kt^4+b)^{3/2}}$.

65. Write the function $(1 + \cos x)^3$ as a composite function $f \circ g$ in two different ways. Apply the Chain Rule both ways and check that you get the same answer in both cases.

Let $h(x) = (1 + \cos x)^3$.

- Let $f(x) = x^3$ and $g(x) = 1 + \cos x$. Then $h(x) = f(g(x))$ and
  $h'(x) = f'(g(x))g'(x) = 3(1 + \cos x)^2 \cdot (-\sin x)$
  $= -3\sin x\,(1 + \cos x)^2$.
- Let $f(x) = (1 + x)^3$ and $g(x) = \cos x$. Then $h(x) = f(g(x))$ and
  $h'(x) = f'(g(x))g'(x) = 3(1 + \cos x)^2 \cdot (-\sin x)$
  $= -3\sin x\,(1 + \cos x)^2$.

In Exercises 67–71, use the Chain Rule to compute higher derivatives.

67. Calculate $f''(x)$ where $f(x) = \sin(x^2)$.

Let $f(x) = \sin\left(x^2\right)$. Then $f'(x) = 2x\cos\left(x^2\right)$ and
$f''(x) = 2x\left(-\sin\left(x^2\right) \cdot 2x\right) + 2\cos\left(x^2\right) = 2\cos\left(x^2\right) - 4x^2\sin\left(x^2\right)$.

69. Calculate $f'''(x)$ where $f(x) = (3x + 9)^{11}$.

Let $f(x) = (3x + 9)^{11}$. Then $f'(x) = 11\,(3x + 9)^{10} \cdot 3 = 33\,(3x + 9)^{10}$,
$f''(x) = 330\,(3x + 9)^{9} \cdot 3 = 990\,(3x + 9)^{9}$, and
$f'''(x) = 8910\,(3x + 9)^{8} \cdot 3 = 26730\,(3x + 9)^{8}$.

71. Use the Chain Rule to express the second derivative of $f(g(x))$ in terms of the derivatives of f and g.

Let $h(x) = f(g(x))$. Then $h'(x) = f'(g(x))g'(x)$ and
$h''(x) = f'(g(x))g''(x) + g'(x)f''(g(x))g'(x)$
$= f'(g(x))g''(x) + f''(g(x))\,(g'(x))^2$.

In Exercises 72–76, use the following values to calculate the derivative.

x	1	4	6
$f(x)$	4	0	6
$f'(x)$	5	7	4
$g(x)$	4	1	6
$g'(x)$	5	$\frac{1}{2}$	3

73. $d/dx\,\sin(f(x))$ at $x = 4$.

Let $h(x) = \sin(f(x))$. Then $d/dx\,\sin(f(x)) = h'(x) = \cos(f(x)) \cdot f'(x)$ and
$h'(4) = \cos(f(4)) \cdot f'(4) = \cos(0) \cdot 7 = 7$.

75. $d/dx\,\sin(f(x)g(x))$ at $x = 4$.

Let $h(x) = \sin(f(x)g(x))$. Then
$d/dx\,\sin(f(x)g(x)) = h'(x) = \cos(f(x)g(x)) \cdot (f(x)g'(x) + g(x)f'(x))$ and
$h'(4) = \cos(f(4)g(4)) \cdot (f(4)g'(4) + g(4)f'(4)) = \cos(0 \times 1) \cdot \left(0 \cdot \frac{1}{2} + 1 \cdot 7\right) = 7$.

77. Compute the derivative of $h(\sin x)$ at $x = \frac{\pi}{6}$, assuming that $h'(.5) = 10$.

Let $u = \sin x$ and suppose that $h'(.5) = 10$. Then $\dfrac{d}{dx}(h(u)) = \dfrac{dh}{du}\dfrac{du}{dx} = \dfrac{dh}{du}\cos x$. When $x = \frac{\pi}{6}$, we have $u = .5$. Accordingly, the derivative of $h(\sin x)$ at $x = \frac{\pi}{6}$ is
$10\cos\left(\frac{\pi}{6}\right) = 5\sqrt{3}$.

79. Compute the derivative of $\tan(h(x))$ at $x = 2$, assuming that $h(2) = \frac{\pi}{4}$ and $h'(2) = 3$.

Let $f(x) = \tan(h(x))$. Then $f'(x) = \sec^2(h(x)) \cdot h'(x)$ and
$f'(2) = \sec^2(h(2)) \cdot h'(2) = \sec^2\left(\frac{\pi}{4}\right) \cdot 3 = 6$.

81. An expanding sphere has radius $r = .4t$ cm at time t (in seconds).
(a) Use the formula $V = (\frac{4}{3})\pi r^3$ to compute the rate of change of volume.
(b) Use the Chain Rule to compute the rate of change of volume with respect to time.
(c) Find dV/dt at time $t = 3$.

Let $r = .4t$, where t is in seconds (s) and r is in centimeters (cm).
(a) With $V = \frac{4}{3}\pi r^3$, we have $\dfrac{dV}{dr} = 4\pi r^2$.
(b) Thus $\dfrac{dV}{dt} = \dfrac{dV}{dr}\dfrac{dr}{dt} = 4\pi r^2 \cdot (.4) = 1.6\pi r^2$.
(c) When $t = 3$, we have $r = 1.2$. Hence $dV/dt = 1.6\pi\,(1.2)^2 \approx 7.24$ cm/s.

83. Conservation of Energy The position at time t of a weight of mass m oscillating back and forth at the end of a spring is given by the formula $x = x(t) = L\sin(2\pi\omega t)$. Here $x = 0$ is the natural equilibrium position of the spring, L is the maximum length of the spring, and ω is the frequency (number of oscillations per second).

(a) By Hooke's Law, the spring exerts a force $F = -kx$, where k is the *spring constant*. Calculate the acceleration $a = a(t)$ of the weight and use Newton's Law $F = ma$ to show that the frequency is given by $\omega = \sqrt{k/m}$.

(b) Physicists define the kinetic energy of the spring $K = \frac{1}{2}mv^2$ (where v is the velocity) and the potential energy $U = \frac{1}{2}kx^2$. Use the Chain Rule together with Newton's Law $F = ma$ (where $a = dv/dt$) to prove that the total energy $T = K + U$ is conserved, that is, show that $dT/dt = 0$.

(a) $a(t) = x''(t)$, $x'(t) = 2\pi\omega L\cos(2\pi\omega t)$ and $x''(t) = -(2\pi\omega)^2 L\sin(2\pi\omega t)$. Given that $F = -kx = ma(t)$, $-k(L\sin(2\pi\omega t)) = m(-(2\pi\omega)^2 L\sin(2\pi\omega t))$ if and only if $k = m(2\pi\omega)^2$ if and only if $\frac{k}{m} = (2\pi\omega)^2$ if and only if $m = \frac{k}{(2\pi\omega)^2}$ if and only if $\sqrt{\frac{k}{m}} = 2\pi\omega$.

(b) Velocity is $x'(t) = 2\pi\omega L\cos(2\pi\omega t)$. If total energy $T = PE + KE = (1/2)kx^2 + (1/2)mv^2 = (1/2)k(L\sin(2\pi\omega t)^2) + (1/2)m(2\pi\omega L\cos(2\pi\omega t))^2$, then $\frac{dT}{dt} = k(L\sin(2\pi\omega t))(2\pi\omega t)L\cos(2\pi\omega t)(2\pi\omega)^2 L\sin(2\pi\omega t)$. From part (a), $m = \frac{m}{(2\pi\omega)^2}$ so $\frac{dT}{dt} =$ $k(2\pi\omega)L^2\sin(2\pi\omega t)\cos(2\pi\omega t) - \frac{k}{(2\pi\omega)^2}(2\pi\omega)(2\pi\omega)^2 L^2\cos(2\pi\omega t)\sin(2\pi\omega t) =$ $k(2\pi\omega)L^2\sin(2\pi\omega t)\cos(2\pi\omega t) - k(2\pi\omega)L^2\sin(2\pi\omega t)\cos(2\pi\omega t) = 0$.

Further Insights and Challenges

85. R & W Recall that a function is called *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$.

(a) Sketch a graph of any even function and explain graphically why its derivative is an odd function.

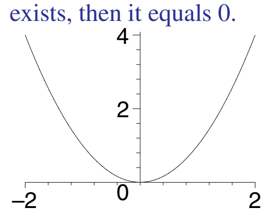
(b) Use the Chain Rule to show that if f is even, then f' is odd. *Hint:* differentiate the function $f(-x)$ using the Chain Rule.

(c) Similarly, show that if f is odd, then f' is even.

(d) Suppose that f' is even. Is f necessarily odd? *Hint:* the derivative of a linear function is always even.

A function is *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$.

(a) The graph of an even function is symmetric with respect to the y-axis. Accordingly, its image in the left half-plane is a mirror reflection of that in the right half-plane through the y-axis. If at $x = a \geq 0$, the slope of f exists and is equal to m, then by reflection its slope at $x = -a \leq 0$ is $-m$. That is, $f'(-a) = -f'(a)$. *Note:* This means that if $f'(0)$ exists, then it equals 0.



(b) If f is even, then $f(-z) = f(z)$. Suppose that f is differentiable at $x = a \geq 0$; say $f'(a) = m$. Then $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m$. Using the fact that f is even, we have

$$f'(-a) = \lim_{x \to -a} \frac{f(x) - f(-a)}{x - (-a)} = \lim_{t \to a} \frac{f(-t) - f(-a)}{-t + a}$$
$$= \lim_{t \to a} \left(-\frac{f(t) - f(a)}{t - a} \right) = -m$$

In other words, $f'(-a) = -f'(a)$, whence f' is an odd function.

(c) If f is odd, then $f(-z) = -f(z)$. Suppose that f is differentiable at $x = a \geq 0$; say $f'(a) = m$. Then $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m$. Using the fact that f is odd, we have

$$f'(-a) = \lim_{x \to -a} \frac{f(x) - f(-a)}{x - (-a)} = \lim_{t \to a} \frac{f(-t) - f(-a)}{-t + a}$$
$$= \lim_{t \to a} \frac{-f(t) + f(a)}{-t + a} = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} = m$$

In other words, $f'(-a) = f'(a)$, whence f' is an even function.

(d) Suppose that f' is even. Then f is not necessarily odd. Let $f(x) = 4x + 7$. Then $f'(x) = 4$, an even function. But f is not odd. For example, $f(2) = 15$, $f(-2) = -1$, but $f(-2) \neq -f(2)$.

In Exercises 87–89, use the following fact that will be introduced in Chapter 4: if a differentiable function f satisfies $f'(x) = 0$ for all x, then f is a constant function.

87. Differential Equation of Sine and Cosine Suppose that f is a function satisfying

$$f''(x) = -f(x) \qquad (1)$$

This is an example of a *differential equation*.

(a) Use the Chain Rule to show that the derivative of $f(x)^2 + f'(x)^2$ is zero for all x.

(b) Conclude that $f(x)^2 + f'(x)^2 = C$, where C is the constant $f(0)^2 + f'(0)^2$.

(c) Show that $\sin x$ and $\cos x$ satisfy Eq. (1).

(d) Use (a) to prove that $\sin^2 x + \cos^2 x = 1$.

(a) Let $g(x) = f(x)^2 + f'(x)^2$. Then $g'(x) = 2f(x)f'(x) + 2f'(x)f''(x)$. Given that $f''(x) = -f(x)$; so substituting for $f''(x)$, we have $g'(x) = 2f(x)f'(x) + 2f'(x)(-f(x)) = 2f(x)f'(x) - 2f'(x)f(x) = 0$.

(b) Because $g'(x) = 0$ for all x, that implies that $g(x) = f(x)^2 + f'(x)^2$ Graphically $g(x) = C$ is a horizontal line going through the point $(0, C)$. The value of C is determined by $g(0) = f(0)^2 + f'(0)^2$.

(c) Let $f(x) = \sin(x)$. Then $f'(x) = \cos(x)$ and $f''(x) = -\sin(x)$. $f''(s) = -\sin x = -f(x)$

(d) Let $\sin(x) = f(x)$. Then $f(x^2 + f'(x)^2 = \sin^2(x) + (\cos x)^2 = \sin^2(x) + \cos^2(x)$. Let $g(x) = \sin^2(x) + \cos^2(x)$. Then $g'(x) = 2\sin(x)\cos(x) - 2\cos(x)\sin(x) = 0$ for all x. Therefore, by part (a) and (b), $C = \sin^2 x + \cos^2 x = \sin^2(0) + \cos(0)^2 = 0 + 1 = 1$.

89. Conservation of Energy Let $h(t)$ be the height at time t of an object released at the surface of earth with a vertical velocity v_0. The object's potential energy is $PE = mgh(t)$ and its kinetic energy is $\frac{1}{2}mv(t)^2$, where $v(t) = h'(t)$ is the velocity.

(a) Show that the total energy $E(t) = mgh(t) + \frac{1}{2}mv(t)^2$ is a constant C. *Hint:* Use the Chain Rule and the fact that $h''(t) = -g$ is the acceleration due to gravity.

(b) Show $C = mgh_{\max} = \frac{1}{2}mv_0^2$, where $h_{\max}$ is the object's maximum height, and conclude that $h_{\max} = \frac{1}{2}\frac{v_0^2}{g}$.

(a) Equivalently, we show that $E'(t) = 0$. $E'(t) = mgh'(t + \left(\frac{1}{2}\right)m2v(t)v'(t) = mgh'(t) + \left(\frac{1}{2}\right)m2h'(t)(h''(t)) = mgh'(t) + mh'(t)(-g) = mgh'(t) - mgh'(t) = 0$.

(b) From part (a) $C = mgh(t) + \left(\frac{1}{2}\right)mv(t)^2$. $h_{\max} = h(0)$ because the object is being dropped. Since the total energy is conserved, potential energy at $t = 0$ will equal C. Therefore, $C = mgh_{\max}$. This is also equal to the kinetic energy of the object the instant it is dropped with a vertical velocity of v_0, so $C = \left(\frac{1}{2}\right)mv_0^2$. $h_{\max} = \left(\frac{1}{2}\right)mv_0^2$ since $C = mgh_{\max} = \frac{1}{2}mv_0^2$ and it is not accelerating at time 0.

3.8 Implicit Differentiation

Preliminary Questions

1. Which differentiation rule is used to show $d/dx \sin y = y' \cos y$?

2. Which of the following equations is *incorrect* and why?

(a) $\dfrac{d}{dy}\sin(y^2) = 2y\cos(y^2)$

(b) $\dfrac{d}{dx}\sin(x^2) = 2x\cos(x^2)$

(c) $\dfrac{d}{dx}\sin(y^2) = 2y\cos(y^2)$

3. Which of the following equations is *incorrect* and why?

(a) $\dfrac{d}{dt}\sqrt{x+y} = \dfrac{1}{2}(x+y)^{-1/2}(1+y')$

(b) $\dfrac{d}{dt}\sqrt{x+y} = \dfrac{1}{2}(x+y)^{-1/2}\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$

4. On an exam, Joe was asked to differentiate the equation

$$x^2 + 2xy + y^3 = 7$$

Find the errors in Joe's answer:

$$2x + 2xy' + 3y^2 = 0$$

Exercises

In Exercises 1–5, calculate the derivative of the expression with respect to x.

1. x^2y^3

Assuming that y depends on x, then $\frac{d}{dx}\left(x^2y^3\right) = x^2 \cdot 3y^2y' + y^3 \cdot 2x = 3x^2y^2y' + 2xy^3$.

3. $\frac{y^3}{x}$

Assuming that y depends on x, then $\frac{d}{dx}\left(\frac{y^3}{x}\right) = \frac{-y^3}{x^2} + \frac{3y^2y'}{x}$.

5. $(x^2 + y^2)^{3/2}$

Assuming that y depends on x, then
$\frac{d}{dx}\left(\left(x^2 + y^2\right)^{3/2}\right) = \frac{3}{2}\left(x^2 + y^2\right)^{1/2}\left(2x + 2yy'\right) = 3\left(x + yy'\right)\sqrt{x^2 + y^2}$.

Calculate dy/dx using implicit differentiation in Exercises 6–9.

7. $x^2y + 2xy^2 = x + y$

Let $x^2y + 2xy^2 = x + y$. Then $x^2y' + 2xy + 2x \cdot 2yy' + 2y^2 = 1 + y'$, whence
$y' = \frac{1 - 2xy - 2y^2}{x^2 + 4xy - 1}$.

9. $\sqrt{x + y} = \frac{1}{x} + \frac{1}{y}$

Let $(x + y)^{1/2} = x^{-1} + y^{-1}$. Then $\frac{1}{2}(x + y)^{-1/2}(1 + y') = -x^{-2} - y^{-2}y'$ or
$x^2y^2(1 + y') = -2y^2\sqrt{x + y} - 2x^2y'\sqrt{x + y}$, whence $y' = -\frac{y^2\left(x^2 + 2\sqrt{x + y}\right)}{x^2\left(y^2 + 2\sqrt{x + y}\right)}$.

11. Calculate the time derivative dy/dt in terms of dx/dt when the variables are related by the equation $xy = 1$.

Let $xy = 1$. Then $x\frac{dy}{dt} + y\frac{dx}{dt} = 0$, whence $\frac{dy}{dt} = -\frac{y}{x}\frac{dx}{dt}$.

In Exercises 12–26, calculate dy/dx using implicit differentiation.

13. $3x^2y^4 + x + y = 0$

Let $3x^2y^4 + x + y = 0$. Then $3x^2 \cdot 4y^3y' + y^4 \cdot 6x + 1 + y' = 0$, whence
$y' = -\frac{6xy^4 + 1}{12x^2y^3 + 1}$.

15. $x^3y^5 = 1$

Let $x^3y^5 = 1$. Then $x^3 \cdot 5y^4y' + y^5 \cdot 3x^2 = 0$, whence $y' = -\frac{3y}{5x}$.

17. $\sqrt{x+y}+\sqrt{y}=2x$

Let $(x+y)^{1/2}+y^{1/2}=2x$. Then $\frac{1}{2}(x+y)^{-1/2}(1+y')+\frac{1}{2}y^{-1/2}y'=2$ or $\sqrt{y}(1+y')+y'\sqrt{x+y}=4\sqrt{y}\sqrt{x+y}$, whence $y'=\dfrac{\left(4\sqrt{x+y}-1\right)\sqrt{y}}{\sqrt{x+y}+\sqrt{y}}$.

19. $x^{1/2}+y^{2/3}=x+y$

Let $x^{1/2}+y^{2/3}=x+y$. Then $\frac{1}{2}x^{-1/2}+\frac{2}{3}y^{-1/3}y'=1+y'$, whence $y'=\dfrac{1-\frac{1}{2}x^{-1/2}}{\frac{2}{3}y^{-1/3}-1}=\dfrac{3y^{1/3}\left(2x^{1/2}-1\right)}{2x^{1/2}\left(2-3y^{1/3}\right)}$.

21. $\dfrac{x}{y}+\dfrac{y^2}{x+1}=0$

Let $\dfrac{x}{y}+\dfrac{y^2}{x+1}=0$. Then $\dfrac{y\cdot 1-xy'}{y^2}+\dfrac{(x+1)\cdot 2yy'-y^2\cdot 1}{(x+1)^2}=0$ or $(x+1)^2y-x(x+1)^2y'+2y^3(x+1)y'-y^4=0$. Therefore, $y'=\dfrac{y^4-y(x+1)^2}{2y^3(x+1)-x(x+1)^2}$.

23. $\sin(xy)=xy$

Let $\sin(xy)=xy$. Then $\cos(xy)\cdot(xy'+y)=xy'+y$, whence $y'=\dfrac{y-y\cos(xy)}{x\cos(xy)-x}=-\dfrac{y}{x}$.

25. $\cos x-\sin y=1$

Let $\cos x-\sin y=1$. Then $-\sin x-y'\cos y=0$, whence $y'=-\dfrac{\sin x}{\cos y}$.

27. Find the equation of the tangent line to the graph of $x^2y-y^3x+x=1$ at $P=(1,0)$.

Let $x^2y-y^3x+x=1$. Then $x^2y'+2xy-y^3\cdot 1+x\cdot\left(-3y^2y'\right)+1=0$, whence $y'=\dfrac{y^3-2xy-1}{x^2-3xy^2}=-1$ at $P\,(1,0)$. The tangent line at P is therefore $y=0+(-1)(x-1)$ or $y=1-x$.

In Exercises 29–30, use the given equation to calculate dy/dt in terms of dx/dt.

29. $x^2-y^2=1$

Let $x^2-y^2=1$. Then $2x\dfrac{dx}{dt}-2y\dfrac{dy}{dt}=0$, whence $\dfrac{dy}{dt}=\dfrac{x}{y}\dfrac{dx}{dt}$.

31. **(a)** Show that $2yy'=3x^2-3$ if y' is the derivative dy/dx for the equation $y^2=x^3-3x+1$.

(b) Do not solve for y'. Rather, set $y'=0$ and solve for x (this gives two values of x).

(c) Show that the positive value of x does not correspond to a point on the graph.

(d) Find the coordinates of the two points where the tangent is horizontal.

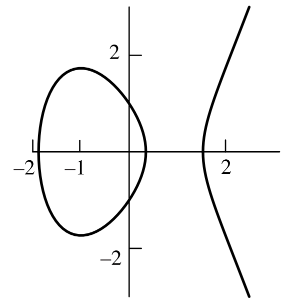


Figure 1 Graph of $y^2 = x^3 - 3x + 1$.

(a) Applying implicit differentiation to $y^2 = x^3 - 3x + 1$, we have $2y\dfrac{dy}{dx} = 3x^2 - 3$if and only if$\dfrac{dy}{dx} = \dfrac{3x^2 - 3}{2y}$. Therefore, $2y(y') = 2y(\dfrac{dx}{dy}) = 2y(\dfrac{3x^2 - 3}{2y}) = 3x^2 - 3$.

(b) Setting $y' = 0$ we have $0 = 3x^2 - 3$, so $x = 1$ or $x = -1$.

(c) By looking at the graph you see that $x = 1$ does not have a value. $y^2 = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$, so $y = \sqrt{3}$ or $-\sqrt{3}$. The tangent is horizontal at the points $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$.

33. Figure 2 shows the graph of the equation

$$y^4 + xy = x^3 - x + 2$$

(a) Find the slope of the tangent lines at the two points on the graph with x-coordinate 0.

(b) Find the equation of the tangent line to the graph at the point $(-1, -1)$.

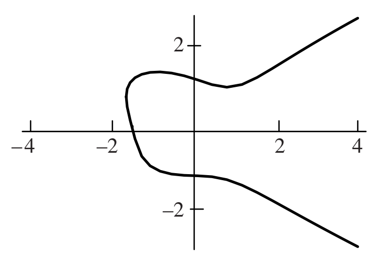


Figure 2 Graph of $y^4 + xy = x^3 - x + 2$.

Consider the equation $y^4 + xy = x^3 - x + 2$. Then $4y^3y' + xy' + y = 3x^2 - 1$, whence $y' = \dfrac{3x^2 - y - 1}{x + 4y^3}$.

(a) Substituting $x = 0$ into $y^4 + xy = x^3 - x + 2$ gives $y^4 = 2$, which has two real solutions, $y = \pm 2^{1/4}$. When $y = 2^{1/4}$, we have $y' = \dfrac{-2^{1/4} - 1}{4\,(2^{3/4})} = -\dfrac{\sqrt{2} + \sqrt[4]{2}}{8} \approx -.3254$. When $y = -2^{1/4}$, we have $y' = \dfrac{2^{1/4} - 1}{-4\,(2^{3/4})} = -\dfrac{\sqrt{2} - \sqrt[4]{2}}{8} \approx -.02813$.

(b)
- The point $(-1, 1)$ is *not* on the curve, lest $0 = 2$ in the curve's equation by substitution.
- At the point $(-1, -1)$, which *is* on the curve, we have $y' = -\frac{3}{5}$. An equation of the tangent line is then
  $y = -1 - \frac{3}{5}(x + 1)$ or $y = -\frac{3}{5}x - \frac{8}{5}$.
- At the point $(1, 1)$, which also happens to be on the curve, we have $y' = \frac{1}{5}$. At this point the tangent line is
  $y = 1 + \frac{1}{5}(x - 1)$ or $y = \frac{1}{5}x + \frac{4}{5}$.

35. If the derivative dx/dy exists at a point and $dx/dy = 0$, then the tangent line is vertical. Calculate dx/dy for the equation $y^4 + 1 = y^2 - x^2$ and find the points on the graph where the tangent line is vertical.

There are *no* points on the graph of the equation $y^4 + 1 = y^2 - x^2$. To see this, rewrite the equation as $y^4 - y^2 + 1 = -x^2$ or (*).

- If $y = 0$, then plugging into (*) gives $1 = -x^2$ or $x^2 = -1$, which has no real solutions.
- If $0 < |y| < 1$, then the left-hand side of (*) is $y^4 - y^2 + 1 = y^2\left(y^2 - 1\right) + 1 > 0$, since $y^2\left(y^2 - 1\right) > -1$. Accordingly, for real values of x, the left-hand side of (*) is positive, whereas its right-hand side is nonnegative, which cannot occur.
- If $|y| \geq 1$, then the left-hand side of (*) is $y^4 - y^2 + 1 = y^2\left(y^2 - 1\right) + 1 \geq 1 > 0$. Once again, for real values of x, the left-hand side of (*) is positive, whereas its right-hand side is nonnegative, which can't happen.

Some Classical Curves

Many curves with interesting shapes were discovered by geometers prior to the invention of calculus. They were then investigated using the tools of calculus in the 17th and 18th centuries.

37. The *folium of Descartes* is the graph of the equation

$$x^3 + y^3 = 3xy$$

The French philosopher-mathematician René Descartes, who discussed the curve in 1638, chose the name *folium*, which means leaf-shaped. Descartes' scientific colleague Gilles de Roberval called it the *jasmine flower*. The curve is indeed shaped like a leaf in the positive quadrant, but both men believed incorrectly that the same shape was repeated ain each quadrant, giving it the appearance of four petals of a flower. Find the equation of the tangent line to this graph at the point $(\frac{2}{3}, \frac{4}{3})$.

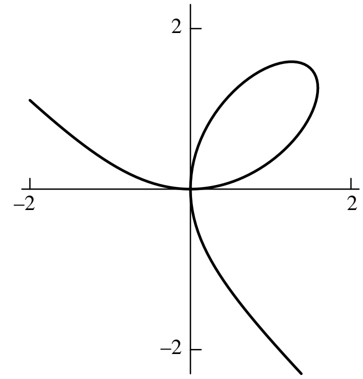


Figure 4 Folium of Descartes: $x^3 + y^3 = 3xy$.

Let $x^3 + y^3 = 3xy$. Then $3x^2 + 3y^2y' = 3xy' + 3y$, whence
$y' = \dfrac{x^2 - y}{x - y^2} = \dfrac{\frac{4}{9} - \frac{4}{3}}{\frac{2}{3} - \frac{16}{9}} = \dfrac{-\frac{8}{9}}{-\frac{10}{9}} = \frac{4}{5}$ at the point $P\left(\frac{2}{3}, \frac{4}{3}\right)$. The tangent line at P is thus
$y = \frac{4}{3} + \frac{4}{5}\left(x - \frac{2}{3}\right)$ or $y = \frac{4}{5}x + \frac{4}{5}$.

39. The curve defined by

$$xy = x^3 - 5x^2 + 2x - 1$$

is an example of a *trident* curve, so named by Isaac Newton in his treatise on curves published in 1710. Find the points where the tangent to the trident is horizontal as follows.

(a) Show that $xy' + y = 3x^2 - 10x + 2$.

(b) Set $y' = 0$ in (a), replace y by $x^{-1}(x^3 - 5x^2 + 2x - 1)$, and solve the resulting equation for x.

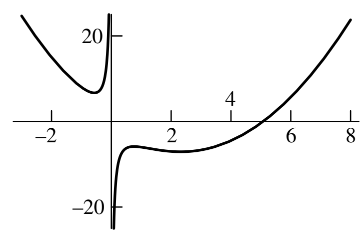


Figure 6 Trident curve: $xy = x^3 - 5x^2 + 2x - 1$.

Consider the equation of a trident curve: $xy = x^3 - 5x^2 + 2x - 1$ or $y = x^{-1}(x^3 - 5x^2 + 2x - 1)$.

(a) Let $y = x^2 - 5x + 2$, then $y' = 2x - 5$. Taking the derivative of $xy = x^3 - 5x^2 + 2x - 1$ and plugging in y', we have $x(2x - 5) + x^2 - 5x + 2 = 2x^2 - 5x + x^2 - 5x + 2 = 3x^2 - 10x + 2$.

(b) Setting $y' = 0$ in (a) gives $y = 3x^2 - 10x + 2$. Thus, we have $x^{-1}(x^3 - 5x^2 + 2x) = 3x^2 - 10x + 2$. Collecting like terms and setting to zero, we have $0 = 4x^3 - 5x^2 + 1$ where $x = 1, \frac{1 \pm \sqrt{17}}{8}$.

Implicit differentiation can be used to calculate higher derivatives. This is illustrated in Exercises 41–44.

41. Consider the equation $y^3 - \frac{3}{2}x^2 = 1$.

(a) Use implicit differentiation to show that $y' = (x/y^2)$.

(b) Differentiate the expression for y' in (a) to show that

$$y'' = \frac{y^2 - 2xyy'}{y^4}$$

(c) Express y'' in terms of x and y using the result of (a).

(d) Calculate y'' at the point (0, 1).

Let $y^3 - \frac{3}{2}x^2 = 1$.

(a) Then $3y^2y' - 3x = 0$, whence $y' = x/y^2$.

(b) Therefore, $y'' = \dfrac{y^2 \cdot 1 - x \cdot 2yy'}{y^4} = \dfrac{y^2 - 2xyy'}{y^4}$.

(c) Substituting the expression for y' in (a) into the result from (b) gives $y'' = \dfrac{y^2 - 2xy\left(x/y^2\right)}{y^4} = \dfrac{y^3 - 2x^2}{y^5}$.

(d) At $(x, y) = (0, 1)$, we have $y'' = 1$.

43. Calculate y'' at the point (1, 1) on the equation $xy^2 + y - 2 = 0$ by the following steps.

(a) Find y' by implicit differentiation and calculate y' at the point (1, 1).

(b) Differentiate the expression for y' in (a) but do not express y'' in terms of x and y. Rather, obtain a numerical formula for y'' at (1, 1) by substituting $x = 1$, $y = 1$, and the value of y' found in (a).

Let $xy^2 + y - 2 = 0$.

(a) Then $x \cdot 2yy' + y^2 \cdot 1 + y' = 0$, whence $y' = -\dfrac{y^2}{2xy + 1} = -\dfrac{1}{3}$ at $(x, y) = (1, 1)$.

(b) Therefore,

$$\begin{aligned} y'' &= -\frac{(2xy+1)(2yy') - y^2(2xy' + 2y)}{(2xy+1)^2} \\ &= -\frac{(3)\left(-\frac{2}{3}\right) - (1)\left(-\frac{2}{3} + 2\right)}{3^2} = -\frac{-6 + 2 - 6}{27} = \frac{10}{27} \end{aligned}$$

given that $(x, y) = (1, 1)$ and $y' = -\frac{1}{3}$.

Further Insights and Challenges

45. The curve with equation

$$(x^2 + y^2)^2 = 4(x^2 - y^2)$$

was discovered by Jacob Bernoulli in 1694. He noted that it is “shaped like a figure 8, or a knot, or the bow of a ribbon,” and called it the *lemniscate curve* from the Latin word *lemniscus* meaning a pendant ribbon.

(a) Calculate dy/dx.

(b) Find the coordinates of the four points at which the tangent line is horizontal.

Hint: Show that the coordinates of these four points satisfy $x^2 + y^2 = 2$. Then use this relation together with the equation of the lemniscate to solve for x and y. Note also that the formula for the derivative is not valid at the origin $(0, 0)$ since it involves division by 0.

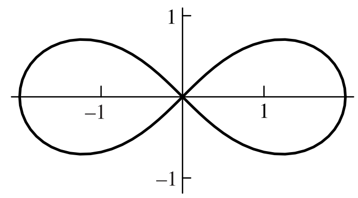


Figure 8 Lemniscate curve: $(x^2 + y^2)^2 = 4(x^2 - y^2)$.

Consider the equation of a lemniscate curve: $\left(x^2 + y^2\right)^2 = 4\left(x^2 - y^2\right)$.

(a) We have $2\left(x^2 + y^2\right)(2x + 2yy') = 4\,(2x - 2yy')$, whence

$$y' = \frac{8x - 4x\left(x^2 + y^2\right)}{8y + 4y\,(x^2 + y^2)} = -\frac{\left(x^2 + y^2 - 2\right)x}{(x^2 + y^2 + 2)\,y}.$$

(b) If $y' = 0$, then either $x = 0$ or $x^2 + y^2 = 2$.

- If $x = 0$ in the lemniscate curve, then $y^4 = -4y^2$or $y^2\left(y^2 + 4\right) = 0$. If y is real, then $y = 0$. The formula for y' in (a) is not defined at the origin $(0/0)$. An alternative parametric analysis shows that the slopes of the tangent lines to the curve at the origin are ± 1.
- If $x^2 + y^2 = 2$ or $y^2 = 2 - x^2$, then plugging this fact into the lemniscate equation gives $4 = 4\left(2x^2 - 2\right)$ which yields $x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$. Thus $y = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$. Accordingly, the four points at which the tangent lines to the lemniscate curve are horizontal are $\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{6}}{2}, -\frac{\sqrt{2}}{2}\right)$, and $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$.

3.9 Rates of Change

Preliminary Questions

1. Assign variables and restate (but do not solve) the following problem in terms of known and unknown derivatives: how fast is the volume of a cube increasing if its side is increasing at a rate of .5 cm/sec?

Questions 2–4 refer to the following set-up: water is poured into a cylindrical glass of radius 4 *cm. The variables V and h denote the volume and water level in the glass at time t, respectively.*

2. Restate the following problem in terms of the derivatives dV/dt and dh/dt: How fast is the water level rising if water is poured in at a rate of 2 cm^3/min?

3. Do the same for the problem: At what rate is water pouring in if the water level is rising at a rate of 1 cm/min?

4. Does dV/dh depend on the rate at which the water is poured? *Hint:* compute dV/dh.

5. Let V, S, and r denote the volume, surface area, and radius of an expanding spherical balloon at time t. Recall that $V = (\frac{4}{3})\pi r^3$ and $S = 4\pi r^2$. What is the relation between dV/dt and dr/dt?

6. Restate the following problem in terms of the derivatives: How fast is the volume increasing if the surface area is increasing at a rate of 8 in^2/min?

Exercises

1. Water is pouring into a bathtub with an 18-ft^2 rectangular base at a rate of .7 ft^3/min. At what rate is the water level rising?

Let h be the height of the water in the tub and V be the volume of the water. Then $V = 18h$ and $\frac{dV}{dt} = 18\frac{dh}{dt}$. Thus $\frac{dh}{dt} = \frac{1}{18}\frac{dV}{dt} = \frac{1}{18}(.7) \approx .039$ ft/min.

3. The radius of a circular oil slick expands at a rate of 2 m/min.

(a) How fast is the area of the oil slick increasing when the radius is 25 m?

(b) If the radius is 0 at time $t = 0$, how fast is the area increasing after 3 minutes?

Let r be the radius of the oil slick and A its area.

(a) Then $A = \pi r^2$ and $\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2\pi\,(25)\,(2) = 100\pi \approx 314.16\ \text{m}^2/\text{min}$.

(b) After 3 minutes, we have $\frac{dA}{dt} = 2\pi\,(3\cdot 2)\,(2) = 24\pi \approx 75.40\ \text{m}^2/\text{min}$.

5. Let ℓ be the length of the hypotenuse of a right triangle whose other two sides have length ℓ and s.

(a) Find a formula for $d\ell/dt$ in terms of s and ds/dt.

(b) A road perpendicular to the highway leads to a farmhouse located 1 mile away. An automobile travels past the farmhouse at a speed of 60 mph. How fast is the distance between the automobile and the farmhouse increasing when the automobile is 3 miles past the intersection of the highway and the road?

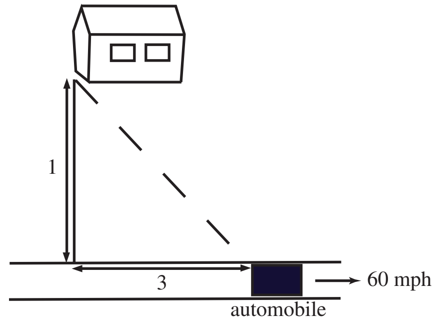


Figure 1

Let h be the length of the hypotenuse of the right triangle and L and s the lengths of its two sides.

(a) Then $h^2 = L^2 + s^2$ and $2h\dfrac{dh}{dt} = 2L\dfrac{dL}{dt} + 2s\dfrac{ds}{dt}$, whence

$$\frac{dh}{dt} = \frac{L\frac{dL}{dt} + s\frac{ds}{dt}}{h} = \frac{L\frac{dL}{dt} + s\frac{ds}{dt}}{\sqrt{L^2+s^2}}.$$

(b) When the auto is 3 miles past the intersection, we have

$$\frac{dh}{dt} = \frac{1\cdot 0 + 3\cdot 60}{\sqrt{1^2+3^2}} = \frac{180}{\sqrt{10}} = 18\sqrt{10} \approx 56.92 \text{ mph}.$$

7. Use the same set-up as Exercise 6 but assume that Sonya begins traveling 1 minute after Isaac takes off. At what rate are they separating 12 minutes after Isaac takes off?

With Isaac x miles east of the center of the lake and Sonya y miles south of its center, let h be the distance between them.

- After 12 minutes or $\frac{12}{60} = \frac{1}{5}$ hour, Isaac has traveled $\frac{1}{5} \times 27 = \frac{27}{5}$ miles. After 11 minutes or $\frac{11}{60}$ hour, Sonya has traveled $\frac{11}{60} \times 32 = \frac{88}{15}$ miles.
- We have $h^2 = x^2 + y^2$ and $2h\dfrac{dh}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$, whence

$$\frac{dh}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{h} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}} = \frac{\left(\frac{27}{5}\right)(27) + \left(\frac{88}{15}\right)(32)}{\sqrt{\left(\frac{27}{5}\right)^2 + \left(\frac{88}{15}\right)^2}}$$

$$= \frac{5003}{\sqrt{14305}} \approx 41.83 \text{ mph}.$$

9. Use the same set-up as Exercise 8 but assume that the water level is rising at a rate of .3 m/min when the water level is 2 m. At what rate is water flowing in?

Consider the cone of water in the tank at a certain instant. Let r be the radius of its (inverted) base, h its height, and V its volume. By similar triangles, $\frac{r}{h} = \frac{2}{3}$ or $r = \frac{2}{3}h$ and thus $V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$. Accordingly, $\dfrac{dV}{dt} = \dfrac{4}{9}\pi h^2 \dfrac{dh}{dt} = \dfrac{4}{9}\pi\,(2)^2\,(.3) \approx 1.68 \text{ m}^3/\text{s}$.

11. At a given moment, a plane passes directly above a radar station at an altitude of 6 miles.

(a) If the plane's speed is 500 mph, how fast is the distance between the plane and the station changing half an hour later?

(b) How fast is the distance between the plane and the station changing when the plane is directly above the station?

Let x be the distance of the plane from the station along the ground and h the distance through the air.

(a) By the Pythagorean Theorem, we have $h^2 = x^2 + 6^2 = x^2 + 36$. After an half hour, $x = \frac{1}{2} \times 500 = 250$ miles. Thus $2h\frac{dh}{dt} = 2x\frac{dx}{dt}$, whence $\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt} = \frac{250}{\sqrt{250^2+36}} \times 500 = \frac{31250\sqrt{15634}}{7817} \approx 499.86$ mph.

(b) When the plane is directly above the station, the distance between the plane and the station is not changing, for at this instant we have $\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt} = \frac{0}{6} \times 500 = 0$ mph.

Exercises 13–16 refer to a 16-foot ladder sliding down the wall as in Figure **??**. *The variables h and x denote the height of the ladder's top and the distance (in ft) from the wall to the ladder's bottom at time t (in seconds), respectively.*

13. Assume the bottom is sliding away from the wall at a rate of 3 ft/s. Find the velocity of the top of the ladder at time $t = 2$ assuming that the bottom is located 5 feet from the wall at time $t = 0$.

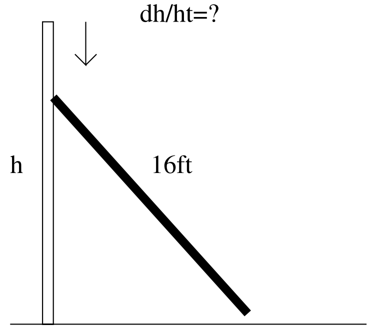


fig. CCFchapter3.9.13

Figure 3 Ladder leaning on wall.

We let x equal the distance that the base of the ladder is away from the wall, and h is the height of the top of the ladder from the floor. The ladder is 16 ft. Therefore, we know that $h^2 + x^2 = 16^2$. At any time t, $x = 5 + 3t$. Therefore, at time $t = 2$, the base is $5 + 3(2) = 11$ ft from the wall. To find the rate at which the top of the ladder is moving, we solve for $\frac{dh}{dt}$ in the equation $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$ and thus we obtain $\frac{dh}{dt} = \frac{-11}{\sqrt{15}}$.

15. Suppose that $h(0) = 12$ and the top is sliding down the wall at a rate of 4 ft/sec. Calculate x and dx/dt at $t = 2$ seconds.

Let h and x be the height of the ladder's top and the distance from the wall of the ladder's bottom, respectively. After 2 seconds, $h = 12 + 2(-4) = 4$ ft. Since $h^2 + x^2 = 16^2$, we have $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$, whence

$$\frac{dx}{dt} = -\frac{h}{x}\frac{dh}{dt} = -\frac{4}{\sqrt{16^2-4^2}}(-4) = \frac{4}{\sqrt{15}}$$

or approximately 1.03 ft/s.

17. A particle moves counterclockwise around the ellipse $9x^2 + 16y^2 = 25$.

(a) In which of the four quadrants is the derivative dx/dt positive? Explain your answer.

(b) Find a relation between the two derivatives dx/dt and dy/dt.

(c) At what rate is the x-coordinate changing when the particle passes the point $(1, 1)$, assuming that its y-coordinate is decreasing at a rate of 6 ft/s?

(d) What is dy/dt when the particle is at the top and bottom of the ellipse (the points $(0, \frac{5}{4})$, $(0, -\frac{5}{4})$)?

A particle moves counterclockwise around the ellipse with equation $9x^2 + 16y^2 = 25$.

(a) The derivative dx/dt is positive in quadrants 3 and 4 since the particle is moving to the right.

(b) From $9x^2 + 16y^2 = 25$ we have $18x\dfrac{dx}{dt} + 32y\dfrac{dy}{dt} = 0$.

(c) From (b), we have $\dfrac{dx}{dt} = -\dfrac{16y}{9x}\dfrac{dy}{dt} = -\dfrac{16 \cdot 1}{9 \cdot 1}(6) = -\dfrac{32}{3}$ ft/s.

(d) From (b), we have $\dfrac{dy}{dt} = -\dfrac{9x}{16y}\dfrac{dx}{dt} = 0$ when $(x, y) = \left(0, \pm\dfrac{5}{4}\right)$.

19. At what rate is the diagonal of a cube increasing if its edges are increasing at a rate of 2 cm/s?

Let s be the length of an edge of the cube and q the length of its diagonal. Two applications of the Pythagorean Theorem (or the distance formula) yield $q = \sqrt{3}s$. Thus $\dfrac{dq}{dt} = \sqrt{3}\dfrac{ds}{dt} = \sqrt{3} \times 2 = 2\sqrt{3} \approx 3.46$ cm/s.

21. A rocket is traveling vertically at a speed of 800 mph. The rocket is tracked through a telescope by an observer located 10 miles from the launching pad. Find the rate at which the angle between the telescope and the ground is increasing 3 minutes after lift-off.

Let y be the height of the rocket and θ the angle between the telescope and the ground. Using trigonometry, we have $\tan\theta = \frac{y}{10}$. After the rocket has traveled for 3 minutes (or $\frac{1}{20}$ hour), its height is $\frac{1}{20} \times 800 = 40$ miles. At this instant, $\tan\theta = 40/10$ and thus $\cos\theta = \dfrac{10}{\sqrt{40^2 + 10^2}} = \dfrac{1}{\sqrt{17}}$. Therefore, $\sec^2\theta \cdot \dfrac{d\theta}{dt} = \dfrac{1}{10}\dfrac{dy}{dt}$, whence

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{10}\frac{dy}{dt} = \frac{1/17}{10}(800) = \frac{80}{17} \approx 4.71 \text{ rad/hr}.$$

23. Calculate the rate (in cm^2/sec) at which area is swept out by the second hand of a circular clock as a function of the clock's radius.

Let r be the radius of the circular clock in centimeters. In 60 seconds, the second hand sweeps out the full area of the circular clock face, $A = \pi r^2$. Therefore, the constant rate at which area is swept out by the second hand is $\dfrac{\pi r^2}{60}$ cm^2/s.

25. A car travels down a highway at a speed of 55 mph. An observer is standing 500 feet from the highway.

(a) How fast is the distance between the observer and the car increasing at the moment the car passes in front of the observer? Can you justify the answer without relying on any calculations?

(b) How fast is the distance between the observer and the car increasing 1 minute later?

Let x be the distance (in feet) along the road that the car has traveled and h be the distance (in feet) between the car and the observer.

(a) By the Pythagorean Theorem, we have $h^2 = x^2 + 500^2$. Thus $2h\frac{dh}{dt} = 2x\frac{dx}{dt}$, whence $\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt} = 0$ mph, since $x = 0$ at the moment the car passes in front of the observer. This is obvious for the following reason. Before the car passes the observer, we have $dh/dt < 0$; after it passes, we have $dh/dt > 0$. So at the instant it passes we have $dh/dt = 0$, given that dh/dt varies continously since the car travels at a constant velocity.

(b) The car travels at 55 mph $= 80\frac{2}{3}$ ft/s. After 1 minute (60 seconds), the car has traveled $x = 60 \times 80\frac{2}{3} = 4840$ ft. Therefore,
$$\frac{dh}{dt} = \frac{4840}{\sqrt{4840^2 + 500^2}}\left(80\frac{2}{3}\right) \approx 80.24 \text{ ft/s} \approx 54.71 \text{ mph}.$$

In Exercises 26–27, we consider a gas that is expanding adiabatically (without heat gain or loss). A relation of the form $PV^k = C$ holds, where P is pressure (measured in units of kilopascals), V is the volume (measured in units of cm^3), and k and C are constants.

27. (Adapted from *Calculus Problems for a New Century*.) Find the value of k assuming that $P = 25$ kPa, $dP/dt = 12$ kPa/min, $V = 100\,\text{cm}^2$, and $dV/dt = 20\,\text{cm}^3$/min.

Let $PV^k = C$. Then $PkV^{k-1}\frac{dV}{dt} + V^k\frac{dP}{dt} = 0$, whence $k = -\frac{V}{P}\frac{dP/dt}{dV/dt} = -\frac{100}{25} \times \frac{12}{20} = -\frac{12}{5}$. (*Note:* If instead we have $d\frac{P}{dt} = -12$ kPa/min, then $k = \frac{12}{5}$.)

29. Suppose that the base x of the right triangle in Figure 4 increases at a rate of 5 cm/sec while the height remains constant at $h = 20$. How fast is the angle θ changing when $x = 20$?

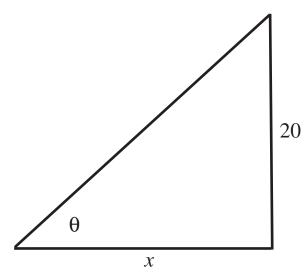


Figure 4

We have $\cot\theta = \frac{x}{20}$, from which $-\csc^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$ and thus
$$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{20}\frac{dx}{dt} = -\frac{\sin^2\left(\frac{\pi}{4}\right)}{20}(5) = -\frac{1}{8} \text{ rad/s}.$$

31. Suppose that the radius r of a circular cone of fixed height $h = 20$ cm is increasing at a rate of 2 cm/sec. How fast is the volume of the cone increasing when $r = 10$?

Consider the cone of water in the tank at a certain instant. Let r be the radius of its (inverted) base, h its height, and V its volume. Then $V = \frac{1}{3}\pi r^2 h = \frac{20}{3}\pi r^2$ and thus
$$\frac{dV}{dt} = \frac{40}{3}\pi r\frac{dr}{dt} = \frac{40}{3}\pi(10)(2) = \frac{800\pi}{3} \approx 837.76 \text{ cm}^3/\text{s}.$$

Further Insights and Challenges

33. (This problem is due to Kay Dundas.) Two parallel paths 50 feet apart run through the woods. Shirley jogs east on one path at 6 mph while Sam walks west on the other path at 4 mph. They pass each other at time $t = 0$.

(a) How far apart are they 3 seconds later?

(b) How fast is the distance between them changing at $t = 3$?

Shirley jogs at 6 mph $= \frac{44}{5}$ ft/s and Sam walks at 4 mph $= \frac{88}{15}$ ft/s. At time zero, consider Shirley to be at the origin (0, 0) and (without loss of generality) Sam to be at (0, 50); i.e., due north of Shirley. Then at time t, the position of Shirley is $\left(\frac{44}{5}t, 0\right)$ and that of Sam is $\left(-\frac{88}{15}t, 50\right)$. The distance between them is
$L = \sqrt{\left(\frac{44}{5}t + \frac{88}{15}t\right)^2 + (50)^2} = \left(\left(\frac{44}{3}t\right)^2 + 50^2\right)^{1/2}$.

(a) When $t = 3$ seconds, the distance between them is $L = \sqrt{44^2 + 50^2} = 2\sqrt{1109} \approx 66.60$ ft.

(b) When $t = 3$ seconds, the distance between them is changing at

$$\frac{dL}{dt} = \frac{1}{2}\left(\left(\frac{44}{3}t\right)^2 + 50^2\right)^{-1/2}\left(2\left(\frac{44}{3}t\right)\frac{44}{3}\right)$$
$$= \frac{\frac{2}{3}(44)^2}{2\sqrt{44^2 + 50^2}} = \frac{968\sqrt{1109}}{3327} \approx 9.69 \text{ ft/s}$$

35. A man is walking away from a 12-ft lamppost. Suppose that the tip of his shadow is moving twice as fast as he is. What is the man's height?

Let h be the height of the man, x his distance from the base of the lamppost, and s the length of his shadow. By similar triangles, we have $\frac{s}{h} = \frac{x + s}{12}$. Thus $\frac{1}{h}\frac{ds}{dt} = \frac{1}{12}\left(\frac{dx}{dt} + \frac{ds}{dt}\right)$, whence $h = \frac{12\frac{ds}{dt}}{\frac{dx}{dt} + \frac{ds}{dt}} = \frac{12\left(2\frac{dx}{dt}\right)}{\frac{dx}{dt} + 2\frac{dx}{dt}} = \frac{24}{3} = 8$ ft. He must be a basketball player and a large one at that!

37. Using a telescope, you track a rocket that was launched 2 miles away. You record the angle θ between the telescope and the ground at half-second intervals. Estimate the velocity of the rocket, if $\theta(10) = .205363$ and $\theta(10.5) = .225241$. *Hint:* relate the rocket's velocity to $\theta'(t)$ in rad/sec and use the approximation $\theta'(t) \approx (\theta(t + h) - \theta(t))/h$.

Let h be the height of the vertically ascending rocket. Using trigonometry, $\tan\theta = \frac{h}{2}$, whence $\frac{dh}{dt} = 2\sec^2\theta \cdot \frac{d\theta}{dt}$. We use the table in the text to estimate $d\theta/dt$ using a forward difference quotient, then compute dh/dt.

t	10	10.5	11	11.5	12	12.5
$d\theta/dt$	.039756	.039400	.039014	.038606	.038176	.037727
dh/dt	.082962	.082937	.082903	.082874	.082845	.082820

The rocket's velocity is approximately .083 mi/s or roughly 300 mph or 480 km/h.

39. (Exercises 39 and 41 are adapted from *Calculus Problems for a New Century*.) A baseball player runs from home plate toward first base at 20 ft/sec. How fast is the player's distance from second base changing when the player is halfway to first base?

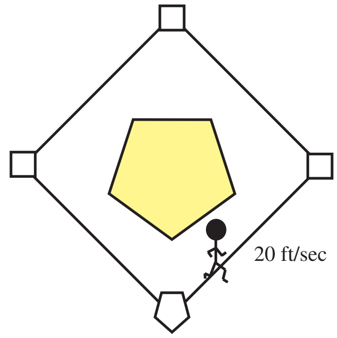


Figure 7 Baseball diamond.

In baseball, the distance between bases is 90 feet. Let x be the distance of the player from home plate and h the player's distance from second base. Using trigonometry, we have $h^2 = 90^2 + (90 - x)^2$. Therefore, $2h\frac{dh}{dt} = 2(90 - x)\left(-\frac{dx}{dt}\right)$, whence $\frac{dh}{dt} = -\frac{90 - x}{h}\frac{dx}{dt} = -\frac{45}{\sqrt{90^2 + 45^2}}(20) = -4\sqrt{5} \approx -8.94$ ft/s when the player is halfway to first base.

41. A spectator is seated 300 m away from the center of a circular track of radius 100 m, watching an athlete run laps at a speed of 5 m/sec. How fast is the distance between the spectator and athlete changing when the runner is approaching the spectator and the distance between them is 250 m? *Hint:* The diagram for this problem is similar to Figure 8 with $r = 100$ and $x = 300$. Eq. (1) still holds but now x is constant and y is changing. Differentiate (1) to find dy/dt.

From the diagram, the coordinates of P are $(r\cos\theta, r\sin\theta)$ and those of Q are $(x, 0)$.

- The distance formula gives $y = \sqrt{(x - r\cos\theta)^2 + (-r\sin\theta)^2}$, whence $y^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta$. Note that x (the distance of the spectator from the center of the track) and r (the radius of the track) are constants.
- Differentiating with respect to t gives $2y\frac{dy}{dt} = 2(x - r\cos\theta)\, r\sin\theta\frac{d\theta}{dt} + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt}$, whence
$$\frac{dy}{dt} = \frac{rx}{y}\sin\theta\frac{d\theta}{dt}.$$
- Recall the relation between arc length s and angle θ, namely $s = r\theta$. Thus $\frac{d\theta}{dt} = \frac{1}{r}\frac{ds}{dt} = \frac{1}{100}(-5) = -\frac{1}{20}$ rad/s. (*Note:* In this scenario, the runner traverses the track in a *clockwise* fashion and approaches the spectator from Quadrant 1.)
- Next, the Law of Cosines gives $y^2 = r^2 + x^2 - 2rx\cos\theta$, whence $\cos\theta = \frac{r^2 + x^2 - y^2}{2rx} = \frac{100^2 + 300^2 - 250^2}{2(100)(300)} = \frac{5}{8}$. Accordingly, $\sin\theta = \sqrt{1 - \left(\frac{5}{8}\right)^2} = \frac{\sqrt{39}}{8}$.

- Finally

$$\frac{dy}{dt} = \frac{rx}{y}\sin\theta\frac{d\theta}{dt} = \frac{(300)\,(100)}{250}\left(\frac{\sqrt{39}}{8}\right)\left(-\frac{1}{20}\right)$$
$$= -\frac{3\sqrt{39}}{4} \approx -4.68 \text{ m/s}.$$