a hydrodynamic exercise in free probability: free Euler equations

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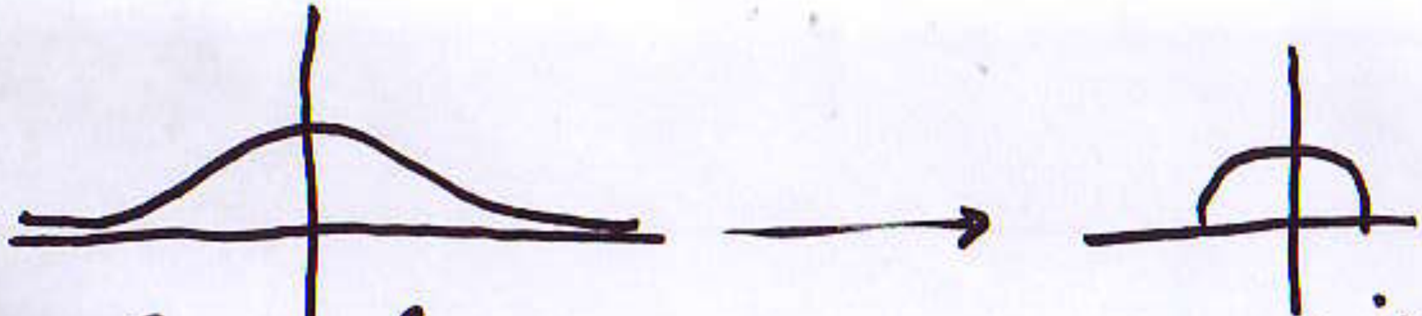
research supported in part
by a Grant from the National Science Foundation

Free Probability

probabilistic framework for quantities with highest degree of noncommutativity

random variables are quantum mechanical i.e. operators on Hilbert space

free independence
(a modification of the definition of independence)



Gauss law semicircle law

connects with: v. Neumann algebras, combinatorics, random matrices, asymptotics of groups representations

Classical / Free parallel
goes unexpectedly far, and still growing

Classical

Free

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this talk: adding to the parallel
an instance of Euler equations
(equations of the flow of inviscid fluid)

Bourret–Frisch "Parastochastics"
precursor, fluids and noncommutative
analogues of Gaussians in another context.

Arnold approach to Euler equations

$\mathfrak{g}$ Lie algebra, $\langle\ ,\ \rangle$ non-degenerate scalar product

$B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ bilinear

$$\langle [a,b], c \rangle = \langle B(c,a), b \rangle$$

$$[0,T) \ni t \longrightarrow v(t) \in \mathfrak{g}$$

$$\dot{v}(t) = -B(v(t), v(t))$$

Classical example

$G = \mathcal{D}iff(\mathbb{R}^n \mid \gamma_n)$ diffeomorphism of $\mathbb{R}^n$ preserving γ_n Gaussian measure

$\mathfrak{g} = \mathcal{V}ect(\mathbb{R}^n \mid \gamma_n)$ vector fields on $\mathbb{R}^n$ preserve γ_n infinitesimally

$$\langle (u_j)_{1\le j\le n}, (v_j)_{1\le j\le n} \rangle = \int \sum_j u_j v_j \, d\gamma_n$$

vector fields

Probabilistic translation

coordinate functions $X_1, \cdots, X_n$ on $(\mathbb{R}^n, \gamma_n)$

$\sim$ n-tuple of centered i.i.d. Gaussian random variables $\tilde{X}_1, \cdots, \tilde{X}_n$

$\mathcal{V}ect(\mathbb{R}^n \mid \gamma_n)$

$\sim$ n-tuples $v_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ so that $\left(\tilde{X}_j + \varepsilon v_j(\tilde{X}_1, \cdots, \tilde{X}_n)\right)_{1 \le j \le n}$ has same distribution as $\tilde{X}_1, \cdots \tilde{X}_n$ up to $O(\varepsilon^2)$

i.e. moments.

Free probability analogy

$L^2(\mathbb{R}^n, \gamma_n) \longrightarrow \mathcal{T}(\mathbb{C}^n) = \bigoplus_{k \geq 0} (\mathbb{C}^n)^{\otimes k}$ full Fock space

$(\mathbb{C}^n)^{\otimes 0} \simeq \mathbb{C}\,\mathbb{1}$ – vaccuum vector

$\tilde{X}_1, \dots, \tilde{X}_n \longrightarrow s_1 = l_1 + l_1^*, \dots, s_n = l_n + l_n^*$

$l_j \xi = e_j \otimes \xi$ left creation

expectation (w.r.t. γ_n) $\longrightarrow \langle \cdot\, 1, 1 \rangle$

$\mathrm{Vect}(\mathbb{R}^n | \gamma_n) \longrightarrow P_j$ noncommutative polynomials (provisionally) $1 \leq j \leq n$

$(s_j + \varepsilon P_j(s_1, \dots s_n))_{1 \leq j \leq n}$ has up to $O(\varepsilon^2)$ same distribution as $(s_i)_{1 \leq i \leq n}$.

Free difference quotients and cyclic derivatives

here $s_1, \dots, s_n$ indeterminates $\in M$ unital algebra

$\mathbb{C}\langle n\rangle$ noncommutative polynomials in $s_1, \dots, s_n$

$\partial_j : \mathbb{C}\langle n\rangle \to \mathbb{C}\langle n\rangle \otimes \mathbb{C}\langle n\rangle$ partial difference quotient

$$\partial_j s_{i_1} \cdots s_{i_p} = \sum_{\{k \mid i_k = j\}} s_{i_1} \cdots s_{i_{k-1}} \otimes s_{i_{k+1}} \cdots s_{i_p}$$

$\delta_j : \mathbb{C}\langle n\rangle \to \mathbb{C}\langle n\rangle$ partial cyclic derivative

$$\delta_j s_{i_1} \cdots s_{i_p} = \sum_{\{k \mid i_k = j\}} s_{i_{k+1}} \cdots s_{i_p} s_{i_1} \cdots s_{i_{k-1}}$$

$\mathcal{E}$ $\mathbb{C}\langle n\rangle$-bimodule (could be M), $b \in \mathcal{E}$

$m_b : \mathbb{C}\langle n\rangle \otimes \mathbb{C}\langle n\rangle \ni \sum \xi \otimes \eta \to \sum \xi b \eta \in \mathcal{E}$

$(b_1, \dots, b_n) \in \mathcal{E}^n$

$D_{(b_1,\dots,b_n)} = \sum_j m_{b_j} \circ \partial_j : \mathbb{C}\langle n\rangle \to \mathcal{E}$ derivation

$(b_1,\dots,b_n) \in M$, $P \in \mathbb{C}\langle n\rangle$

$\frac{d}{d\varepsilon} P(s_1+\varepsilon b_1, \dots, s_n+\varepsilon b_n)\Big|_{\varepsilon=0} = D_{(b_1,\dots,b_n)} P$

$\tau : M \to \mathbb{C}$ trace, $\tau(ab) = \tau(ba)$

$\frac{d}{d\varepsilon} \tau(P(s_1+\varepsilon b_1, \dots, s_n+\varepsilon b_n))\Big|_{\varepsilon=0} = \tau\left(\sum_j b_j \delta_j P\right)$

$$\mathrm{Vect}\,\mathbb{C}_{\langle n\rangle} = (\mathbb{C}_{\langle n\rangle})^n$$

$$[P, Q] = (D_P Q_j - D_Q P_j)_{1\le j\le n}$$

Lie algebra of noncommutative vector fields

$$\mathrm{Vect}\,\mathbb{C}_{\langle n|\tau\rangle} = \{P \in \mathrm{Vect}_{\mathbb{C}\langle n\rangle} \mid \sum_j \tau(P_j\,\delta_j R) = 0, \forall R \in \mathbb{C}_{\langle n\rangle}\}$$

trace-preserving vector-fields
analogue of divergence-free vector-fields

$$\delta : \mathbb{C}_{\langle n\rangle} \ni P \longrightarrow (\delta_j P)_{1\le j\le n} \in (\mathbb{C}_{\langle n\rangle})^n$$

cyclic gradient

scalar product $\langle P, Q\rangle = \sum_j \tau(P_j Q_j)$ on $\mathcal{V}ect\, \mathbb{C}_{\langle n\rangle}$

$$\mathcal{V}ect\, \mathbb{C}_{\langle n|\tau\rangle} = \delta\mathbb{C}_{\langle n\rangle}^{\perp}$$

Exact sequence for cyclic gradients

$$0 \longrightarrow \mathbb{C}1 + [\mathbb{C}_{\langle n\rangle}, \mathbb{C}_{\langle n\rangle}] \hookrightarrow \mathbb{C}_{\langle n\rangle} \xrightarrow{\delta} (\mathbb{C}_{\langle n\rangle})^n \xrightarrow{\theta} \mathbb{C}_{\langle n\rangle}$$

$$\theta(P_1 \oplus \cdots \oplus P_n) = \sum_j [X_j, P_j]$$

$$\mathrm{Im}\,\theta + \mathbb{C}1 = \mathrm{Ker}\,\delta = \mathrm{Ker}\, C$$

$$C : \mathbb{C}_{\langle n\rangle} \to \mathbb{C}_{\langle n\rangle}, \quad C\, s_{i_1} \cdots s_{i_p} = \sum_{1\le j\le p} s_{i_{j+1}} \cdots s_{i_p} s_{i_1} \cdots s_{i_j}$$

Semicircular case

$s_j = \ell_j + \ell_j^*$ on $\mathcal{T}(\mathbb{C}^n)$, $M =$ v. Neumann alg $(s_1, \dots s_n)$

$\tau = \langle \cdot 1, 1 \rangle$ trace, $M \simeq L(F_n)$ free group II_1 factor

$L^2(M, \tau) \ni \xi \xrightarrow{\sim} \xi 1 \in \mathcal{T}(\mathbb{C}^n)$, $\langle a, b \rangle_{L^2} = \tau(ab^*)$

$P_{k_1}(s_{i_1}) \cdots P_{k_m}(s_{i_m}) 1 = e_{i_1}^{\otimes k_1} \cdots e_{i_m}^{\otimes k_m}$

$i_j \neq i_{j+1}$, $k_j > 0$, P_k Chebyshev polynomials

$\mathrm{Vect}\, \mathbb{C}_{\langle n \rangle} \hookrightarrow (L^2(M, \tau))^n$ dense

Π Leray projection, $\overline{\mathrm{Vect}\, \mathbb{C}_{\langle n \rangle}}$ onto $\overline{\mathrm{Vect}\, \mathbb{C}_{\langle n | \sigma \rangle}}$

$$\mathcal{V}\mathrm{ect}\,\mathbb{C}_{\langle n|\tau\rangle} = \sum_{k\geq 0} \mathcal{X}_k, \quad \mathcal{X}_k \subset ((\mathbb{C}^n)^{\otimes k})^n$$

$$\delta\,\mathbb{C}_{\langle n\rangle} = \sum_{k\geq 0} \mathcal{Y}_k, \quad \mathcal{Y}_k \subset ((\mathbb{C}^n)^{\otimes k})^n$$

$$\mathcal{X}_k + \mathcal{Y}_k = ((\mathbb{C}^n)^{\otimes k})^n$$

$$\mathcal{X}_k = \{((\ell_j^* - r_j^*)\xi)_{1\leq j\leq n} \mid \xi \in (\mathbb{C}^n)^{\otimes(k+1)}\}$$

($r_j\eta = \eta \otimes e_j$ right creation)

$$\Pi\ \mathcal{V}\mathrm{ect}\,\mathbb{C}_{\langle n\rangle} = \mathbb{C}_{\langle n|\tau\rangle}$$

$$(I-\Pi)\ \mathcal{V}\mathrm{ect}\,\mathbb{C}_{\langle n\rangle} = \delta\,\mathbb{C}_{\langle n\rangle}$$

$\mathfrak{X}_h^{sa}$, $\mathcal{Y}_h^{sa}$, $\mathcal{V}ect\,\mathbb{C}_{\langle n\rangle}^{sa}$, $\mathcal{V}ect\,\mathbb{C}_{\langle n|\tau\rangle}^{sa}$

real form, real Lie algebra ...

$\mathcal{V}ect\,\mathbb{C}_{\langle n|\tau\rangle}^{sa}$ exponentiate to one-parameter groups of automorphisms of M

. . . .

Euler equations on $\mathrm{Vect}\,\mathbb{C}^{sa}_{\langle n|\tau\rangle}$

$$\langle (a_j)_{1\le j\le n}, (b_j)_{1\le j\le n} \rangle = \sum_j \tau(a_j^* b_j)$$

$$\langle [a,b], c \rangle = \langle B(c,a), b \rangle$$

$$B(c,a) = \Pi\Big(-D_a c_k - \sum_j m_{c_j}(\sim \partial_k a_j)\Big)_{1\le k\le n}$$

$$B(a,a) = \Pi(-D_a a_k)_{1\le k\le n}$$

$$\dot v = \Pi(D_v v_k)_{1\le k\le n} \quad \text{or}$$

$$\dot v_k = D_v v_k + \delta_k p \quad 1\le k\le n\,.$$

The subalgebra $\mathcal{B}_{\infty,1}$

$$\mathcal{B}_{\infty,1} = \{ a \in M \mid [a, r_j - r_j^*] \in \mathcal{C}_1 \}$$

$$|||a||| = \|a\| + \max_{1 \le j \le n} \| [a, r_j - r_j^*] \|_1$$

(note that $[s_k, r_j - r_j^*] = 2\delta_{jk} \langle \cdot , 1 \rangle 1$)

$b = (b_j)_{1 \le j \le n} \in M^n$

Derivation $D_b : \mathbb{C}_{\langle n \rangle} \to M$ has extension

$\mathcal{D}_b : \mathcal{B}_{\infty,1} \to L^1(M, \tau)$, $|\mathcal{D}_b a|_1 \le \|b\| \cdot |||a|||$.

cyclic derivatives also have extensions

$\tilde{\delta}_j : \mathcal{B}_{\infty,1} \longrightarrow L^1(M,\tau)$, $1 \le j \le n$

and C has an extension $\tilde{C}$ since

$C a = \sum_j s_j \delta_j(a)$.

Also $|\tilde{\delta}_j(a)|_1 \le |||a|||$.

Properties of $\mathcal{D}_b$, $\tilde{\delta}_j$, $\tilde{C}$ on $\mathbb{C}\langle n\rangle$

extend by continuity to

$\mathcal{C}_{\infty,1}$ closure of $\mathbb{C}\langle n\rangle$ in $\mathcal{B}_{\infty,1}$

Free Euler equations in $\mathcal{B}_{\infty,1}$

one can make sense of the free Euler equations

$$\dot{v}(t) = \Pi\left(D_{v(t)} v_k(t)\right)_{1\le k\le n}, \quad v = \Pi v$$

when $v : [0,T) \longrightarrow (\mathcal{B}^{sa}_{\infty,1})^n$. Leray projection defined on $(L^2(M,\tau))^n$ so $v = \Pi v$ makes sense. Instead of $D_{v(t)}$ will use $\mathcal{D}_{v(t)}$ extension to $\mathcal{B}_{\infty,1}$. Then $\left(\mathcal{D}_{v(t)} v_k(t)\right)_{1\le k\le n} \in (L^1(M,\tau))^n$.
Π probably not defined on L^1, but $\Pi \sim \sum_{k\ge 0} \Pi_k$
Π_k defined on L^1 and we get equations

$$\Pi_k \dot{v}(t) = \Pi_k\left(\mathcal{D}_{v(t)} v_k(t)\right)_{1\le k\le n}.$$

Cyclic vorticity

classical vorticity: apply curl and pressure term, being a gradient, will disappear

in free Euler, pressure term is a cyclic gradient, map θ from cyclic gradients exact sequence should replace curl

cyclic vorticity $\Omega(t) = i\,\theta(v(t)) = i \sum_k [s_k, v_k(t)]$

$$\dot{\Omega}(t) = \mathcal{D}_{v(t)} \Omega(t)$$

cyclic vorticity equation ($\mathcal{B}_{\infty,1}$ context) additional equations for classical vorticity from div curl $= 0$ (in 3D) etc.

For cyclic vorticity, use results for cyclic gradients $C \circ \Theta = 0$, $\tau \circ \Theta = 0$.

Additional equations for $\Omega(t)$

$$\tilde{C} \circ \Omega(t) = 0, \quad \tau(\Omega(t)) = 0$$

($\mathcal{C}_{\infty,1}$ content).

Conserved quantities

some resemblance to classical where certain conserved quantities are related to vorticity

free Euler equations ($\mathcal{C}_{\infty,1}$ context)

$\tau(\Omega^m(t))$ $m>0$ constant in time

[so distribution of $\Omega(t)$ is constant hence $\tau(f(\Omega(t)))$ constant for any f bounded Borel] .