Topological Quantum Matter

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Topology in Condensed Matter

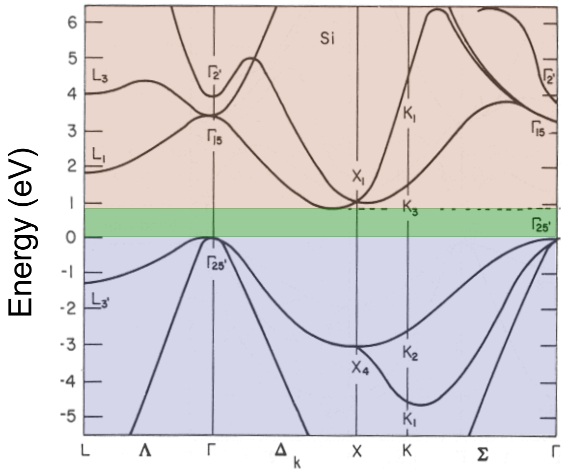
I. Topological Band Theory

Band Theory of Solids

- A theory of materials based on single particle quantum mechanics
- Theoretical foundation for modern electronics technology

Topological Band Structures

- Quantum Hall Effect
- Topological Insulators
- Topological Crystalline Insulators, Topological superconductors, etc.



Band structure of Silicon

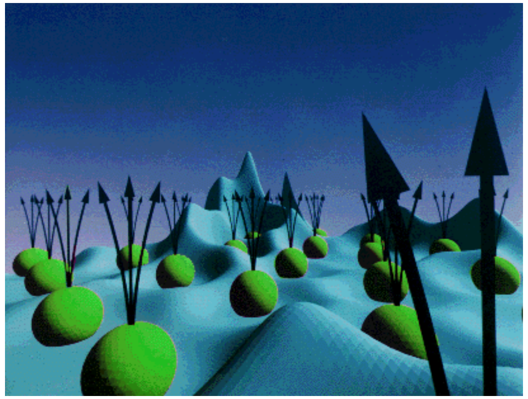
II. Topological Field Theory

Effective Field Theory

- A theory of Phases of Matter
- Characterizes emergent properties of interacting many body systems

Topological Phases of Matter

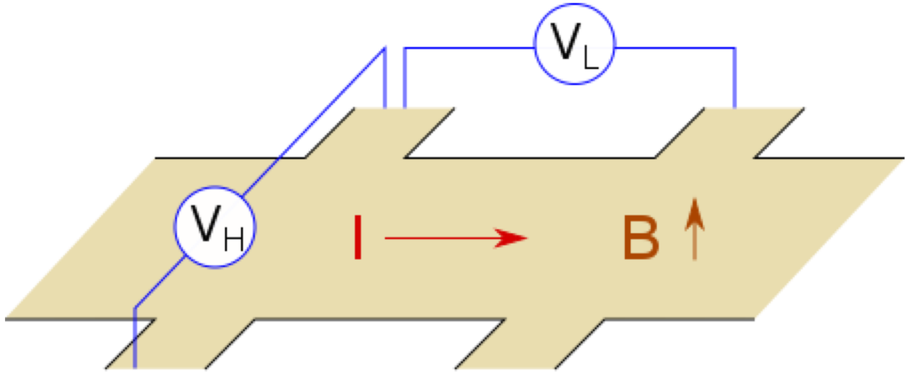
- Fractional Quantum Hall Effect
- Fractional Charge and Statistics
- Topological ground state degeneracies and quantum information

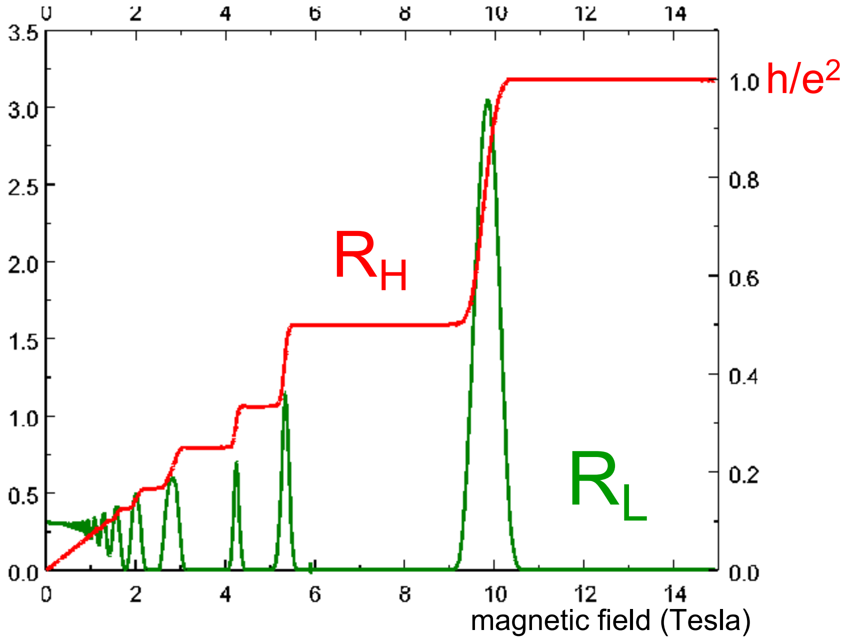


Binding of charge and flux in
the fractional quantum Hall effect

Integer Quantized Hall Effect

von Klitzing, Dorda, Pepper 1980



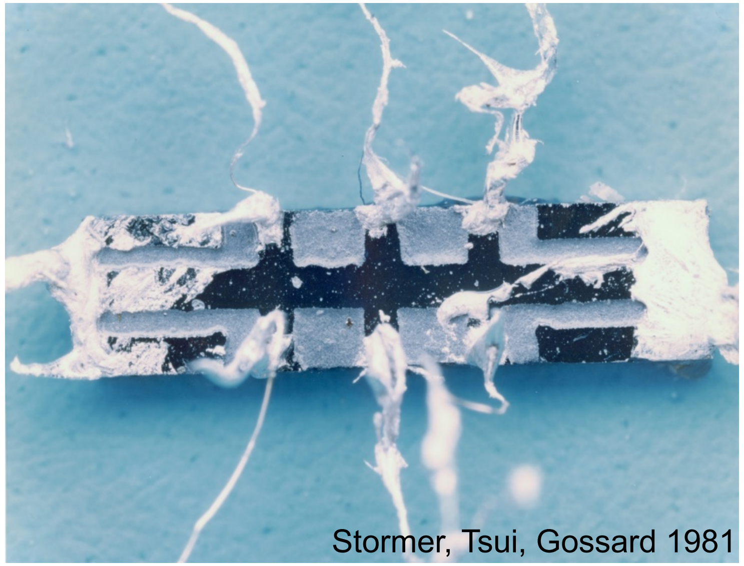


2D electrons in perpendicular magnetic field B

Hall Resistance: $R_H = \frac{V_H}{I} = \frac{1}{N}\frac{h}{e^2}$

h/e^2 = 25, 812.807 557 Ohms

N = integer (measured to 1 part in a billion!)



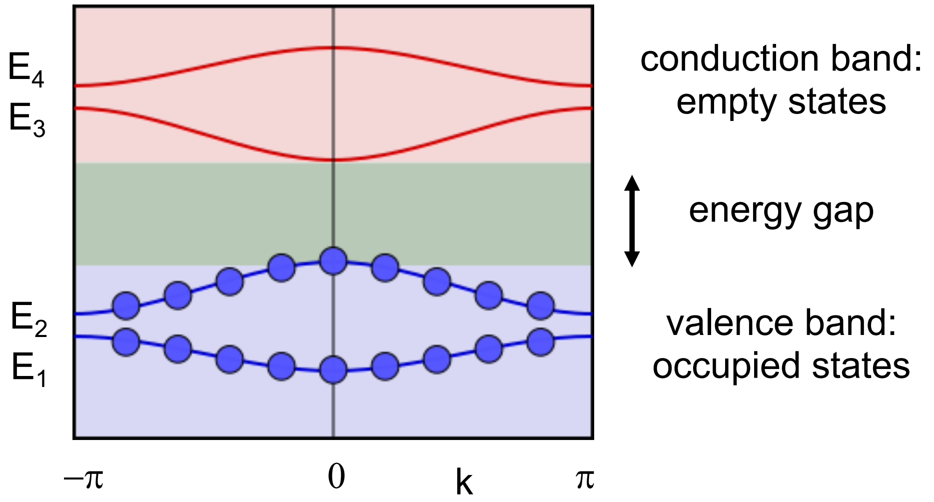
Stormer, Tsui, Gossard 1981

Topological Band Theory

Band structure :

Collection of energy eigenvalues $E_i(\mathbf{k})$ and (complex) eigenvectors $|u_i(\mathbf{k})\rangle$ of a Hermitian Hamiltonian *H(**k**)* parametrized by momentum $\mathbf{k} \in T^d$.

If occupied and empty states are separated by a gap, then the *N* occupied bands define a *U(N)* vector bundle over the torus T^d.



d=2: Band structures with a gap are classified by the first Chern class

(aka TKNN invariant: Thouless, Kohomoto, Nightengale and den Nijs, 1984)

$\mathbf{A}_{ij} = -i\langle u_i \| du_j \rangle$	Berry connection
$\mathbf{F} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$	Berry curvature
$n = \frac{1}{2\pi}\int_{T^2} \mathrm{Tr}[\mathbf{F}] \in$	1st Chern class

Hall Conductivity : $\sigma_H = n\frac{e^2}{h}$

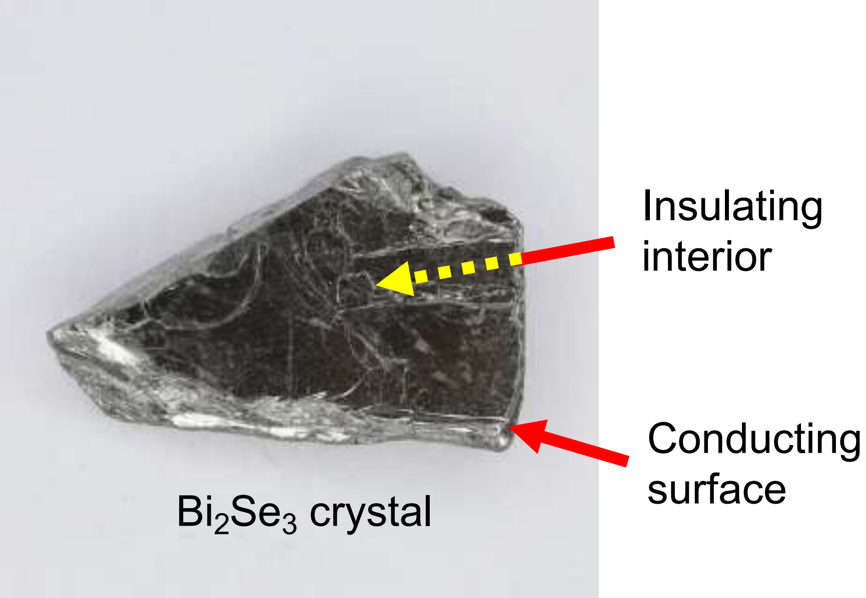
Trivial Insulator: n = 0

Integer Quantum Hall state: n ≠ 0

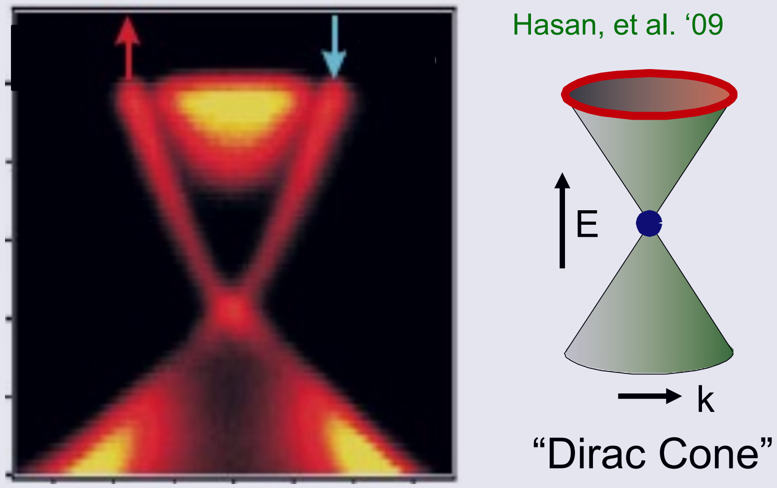
Time Reversal Invariant Topological Insulator

Symmetry introduces new topological structures. (Kane, Mele 2005)

- Time reversal symmetry (ie magnetic field **B**=0): $H(\mathbf{k}) = \Theta^{-1} H(-\mathbf{k}) \Theta$
- Chern number: $n = 0$
- T- invariant 2D or 3D band structures classified by Z_2 invariant: $\nu = 0,\ 1$
- Non trivial class: Topologically protected conducting surface states



Surface states of Bi_2Se_3 probed by Angle Resolved Photoemission Spectroscopy



Chern-Simons form for Z_2 invariant

Dimensional Reduction:

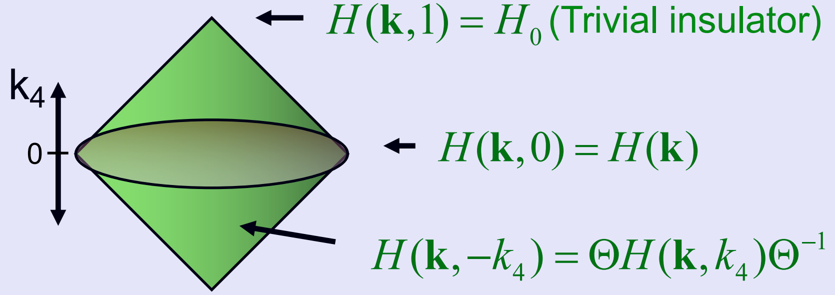
Add an extra parameter, k_4, that smoothly connects the topological insulator to a trivial insulator (while breaking time reversal symmetry) (Qi, Hughes, Zhang 2008).

$\tilde{H}(\mathbf{k},k_4)$ characterized by second Chern class

$$n_2 = \frac{1}{8\pi^2}\int d^4k \,\mathrm{Tr}[\mathbf{F}\wedge\mathbf{F}]$$

n_2 depends on how $H(\mathbf{k})$ is connected to H_0, but due to time reversal, the difference must be even.

$$\nu = n_2 \ \mathrm{mod}\ 2$$



Express in terms of Chern-Simons 3-form

$$\mathrm{Tr}[\mathbf{F}\wedge\mathbf{F}] = dQ_3$$

$$Q_3(\mathbf{k}) = \mathrm{Tr}[\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}]$$

$$\nu = \frac{1}{4\pi^2}\int \mathrm{d}^3k\, Q_3(\mathbf{k}) \ \mathrm{mod}\ 2$$

Gauge invariant up to an even integer.

Periodic Table of Topological Insulators and Superconductors

Kitaev, 2008; Schneider, Ryu, Furusaki, Ludwig 2008

Anti-Unitary Symmetries :

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})$; $\Theta^2 = \pm 1$
- Particle - Hole : $\Xi H(\mathbf{k})\Xi^{-1} = -H(-\mathbf{k})$; $\Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k})\Pi^{-1} = -H(\mathbf{k})$; $\Pi = \Theta\Xi$

10 = 8 + 2 symmetry classes

Mathematical framework:

Real and complex K theory

symmetry class | dimension

		Θ^2	Ξ^2	Π^2	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=0$
insulator (no symmetry)	A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
	AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
	AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
	BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
superconductor (no symmetry)	D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
T-invariant superconductor	DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
T-invariant insulator	AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
	CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
	C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
	CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

$d \to d+2$ (A, AIII)

Bott periodicity

$d \to d+8$ (AI – CI)

Richer (but more fragile) topological structures when, in addition, spatial (space group) symmetries are taken into account.

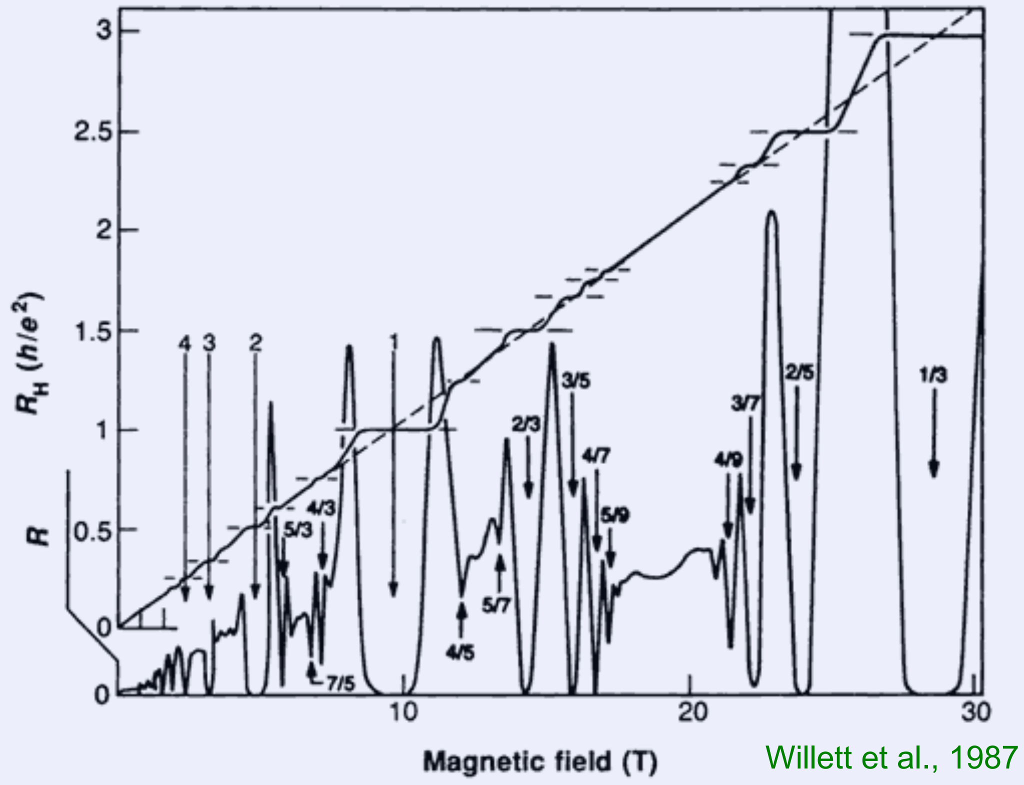
Fractional Quantized Hall Effect

(Tsui, Stormer, Gossard, 1982)

- Quantized Hall conductance observed at rational fractions :

$$R_H = \frac{1}{\nu}\frac{h}{e^2} \qquad \nu = \frac{p}{q}$$

- Can not be explained by band theory (single particle quantum mechanics).
- Initiated a revolution in our understanding of many particle phases of matter.



Willett et al., 1987

Laughlin State: $\nu = 1/m$ (Laughlin, 1983)

- A strongly correlated quantum fluid that supports emergent quasiparticle excitations with

fractional charge: $e^* = e/m$ fractional exchange statistics: $\Theta = e^{i\pi/m}$

- Low energy behavior is characterized by a 2+1D Chern-Simons effective field theory.

Topological Field Theory

Effective action for ν=1/m Laughlin state :

Abelian $U(1)_m$ Chern-Simons Theory

$$S=\int d^2x\,dt\left(\underbrace{\frac{m}{4\pi}\varepsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda}_{\text{Chern-Simons term}}+\frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda\right)$$

Electromagnetic Field:

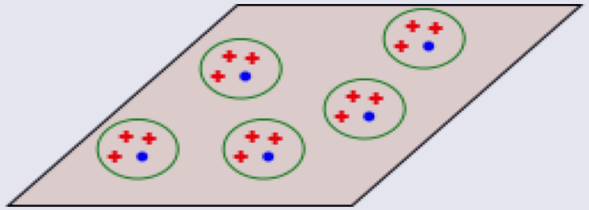
$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

Electric Current:

$$J_\mu=\frac{\delta S}{\delta A_\mu}=\frac{1}{2\pi}\varepsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$$

$$=\frac{1}{4\pi m}\varepsilon_{\mu\nu\lambda}F_{\nu\lambda}$$

The Chern-Simons term describes the binding of electric charge to magnetic flux.



composite boson:

(e + m ϕ_0)

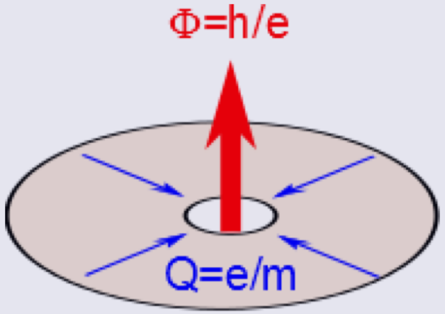
ϕ_0 = h/e : flux quantum

charge density

$$\rho=J_0=\frac{1}{m}\frac{e}{h}B$$

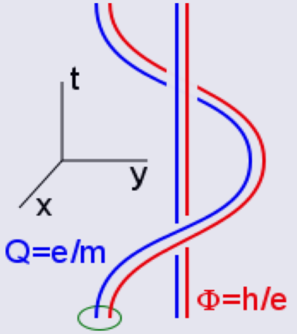
electric current

$$\mathbf{J}=\frac{1}{m}\frac{e^2}{h}\hat{z}\times\mathbf{E}$$



fractional charge

$$e^*=e/m$$



fractional statistics

phase $e^{2\pi i/m}$

Non-Abelian Topological Phases

Non-Abelian quasiparticle statistics: Moore and Read 1991

- Topological ground state degeneracies
- Method to store and manipulate quantum information (Kitaev 2001)
- Foundation for topological quantum computation

Ising topological order:

e.g. $SU(2)_2$ Chern-Simons theory → Ising TQFT

- Fractional quantum Hall state at $\nu = 5/2$
- Topological superconductivity in proximity effect devices

Promising experimental evidence

Fibonacci topological order:

e.g. $SU(2)_3$ Chern-Simons theory → Z_3 parafermion TQFT

- Candidate for fractional quantum Hall state at $\nu = 12/5$

e.g. $(G_2)_1$ Chern-Simons theory → Fibonacci TQFT

- Proposed in strongly correlated topological superconductor

Challenge for the future

Evidence for non-Abelian phase at ν=5/2

e.g. Heiblum et al. (Weizmann)

- 2008: Shot noise experiments measure fractional quasiparticle charge.

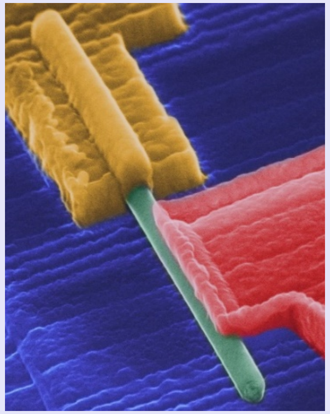
  $e^* \sim e/4$

- 2018: Thermal conductance experiments measure fractional quantized chiral central charge.

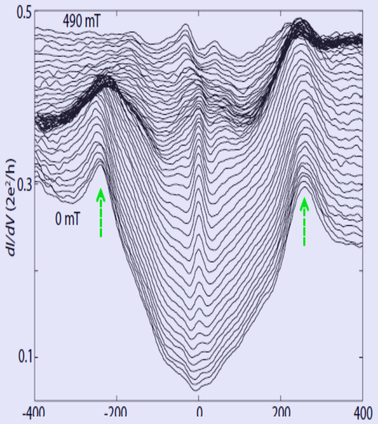
  $c \sim 5/2$

Evidence for topological superconductivity

e.g. Kouwenhoven et al. 2012 (Delft)



semiconductor quantum wire on superconductor

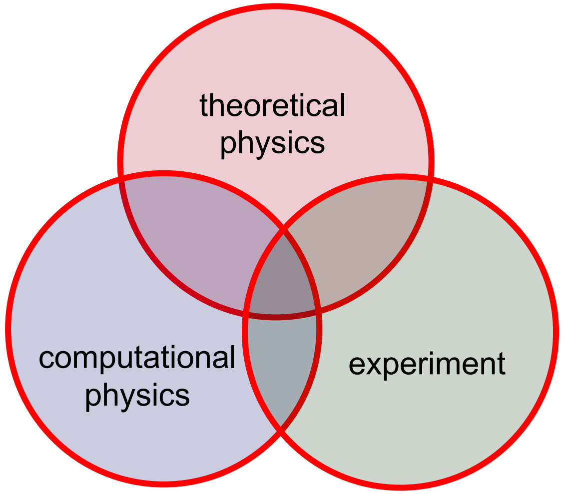


zero bias anomaly observed in tunneling

Outlook

There are many more examples of topological phenomena in band theory:

- Topological superconductors
- Topological crystalline insulators
- Topological semimetals
- Topological photonics
- Topological mechanics ……..



Frontier of strongly interacting topological phases:

- Can we do for the fractional quantum Hall effect what topological insulators did for the integer quantum Hall effect?
- Can we control non-Abelian topological phases?
- Can we construct a topological quantum computer?