

# Subfactors from Penn on.

Vaughan Jones,  
Vanderbilt

March 30 2019



PROCEEDINGS OF THE SYMPOSIUM IN PURE MATHEMATICS OF  
THE AMERICAN MATHEMATICAL SOCIETY  
HELD AT QUEENS UNIVERSITY  
KINGSTON, ONTARIO  
JULY 14-AUGUST 2, 1980  
EDITED BY  
RICHARD V. KADISON  
Prepared by the American Mathematical Society  
with partial support from National Science Foundation grant MCS  
79-27061

Cohomological invariants for groups of outer automorphisms algebras

COLIN E. SUTHERLAND

Automorphism groups and invariant states

ERLING STØRMER

Compact ergodic groups of automorphisms

MAGNUS B. LANDSTAD

Actions of discrete groups on factors

V. F. R. JONES

Ergodic theory and von Neumann algebras

CALVIN C. MOORE

Topologies on measured groupoids

ARLAN RAMSAY

# The Seminar



Subfactors: definition.

Subfactors: definition.

For the purposes of this talk a  $\text{II}_1$  factor  $M$  should be thought of as a "field" (albeit noncommutative and with a ton of zero-divisors).



Subfactors: definition.

For the purposes of this talk a  $\text{II}_1$  factor  $M$  should be thought of as a "field" (albeit noncommutative and with a ton of zero-divisors).

This is because of von Neumann's result that a vector space  $H$  on which  $M$  acts is completely characterised as an  $M$ -module by its dimension  $\dim_M(H)$ .

Subfactors: definition.

For the purposes of this talk a  $\text{II}_1$  factor  $M$  should be thought of as a "field" (albeit noncommutative and with a ton of zero-divisors).

This is because of von Neumann's result that a vector space  $H$  on which  $M$  acts is completely characterised as an  $M$ -module by its dimension  $\dim_M(H)$ .

The novelty is that this  $\dim_M(H)$  is actually a *real number*  $\geq 0$  or  $\infty$ .

Subfactors: definition.

For the purposes of this talk a  $\text{II}_1$  factor  $M$  should be thought of as a "field" (albeit noncommutative and with a ton of zero-divisors).

This is because of von Neumann's result that a vector space  $H$  on which  $M$  acts is completely characterised as an  $M$ -module by its dimension  $\dim_M(H)$ .

The novelty is that this  $\dim_M(H)$  is actually a *real number*  $\geq 0$  or  $\infty$ .

Some intuition can be gained by considering the case of  $M = M_n(\mathbb{C})$  (which has all the attributes of a  $\text{II}_1$  factor except infinite dimensionality).

Subfactors: definition.

For the purposes of this talk a  $\text{II}_1$  factor  $M$  should be thought of as a "field" (albeit noncommutative and with a ton of zero-divisors).

This is because of von Neumann's result that a vector space  $H$  on which  $M$  acts is completely characterised as an  $M$ -module by its dimension  $\dim_M(H)$ .

The novelty is that this  $\dim_M(H)$  is actually a *real number*  $\geq 0$  or  $\infty$ .

Some intuition can be gained by considering the case of  $M = M_n(\mathbb{C})$  (which has all the attributes of a  $\text{II}_1$  factor except infinite dimensionality).

The smallest  $M = M_n(\mathbb{C})$  module has dimension  $n$  over  $\mathbb{C}$  and thus dimension  $\frac{1}{n}$  as measured by  $M$ .

Subfactors: definition.

For the purposes of this talk a  $\text{II}_1$  factor  $M$  should be thought of as a "field" (albeit noncommutative and with a ton of zero-divisors).

This is because of von Neumann's result that a vector space  $H$  on which  $M$  acts is completely characterised as an  $M$ -module by its dimension  $\dim_M(H)$ .

The novelty is that this  $\dim_M(H)$  is actually a *real number*  $\geq 0$  or  $\infty$ .

Some intuition can be gained by considering the case of  $M = M_n(\mathbb{C})$  (which has all the attributes of a  $\text{II}_1$  factor except infinite dimensionality).

The smallest  $M = M_n(\mathbb{C})$  module has dimension  $n$  over  $\mathbb{C}$  and thus dimension  $\frac{1}{n}$  as measured by  $M$ .

We see that all dimensions in this case are given by the numbers  $\frac{k}{n}$  where  $k$  is a non-negative integer (or infinity).

Subfactors: definition.

For the purposes of this talk a  $\text{II}_1$  factor  $M$  should be thought of as a "field" (albeit noncommutative and with a ton of zero-divisors).

This is because of von Neumann's result that a vector space  $H$  on which  $M$  acts is completely characterised as an  $M$ -module by its dimension  $\dim_M(H)$ .

The novelty is that this  $\dim_M(H)$  is actually a *real number*  $\geq 0$  or  $\infty$ .

Some intuition can be gained by considering the case of  $M = M_n(\mathbb{C})$  (which has all the attributes of a  $\text{II}_1$  factor except infinite dimensionality).

The smallest  $M = M_n(\mathbb{C})$  module has dimension  $n$  over  $\mathbb{C}$  and thus dimension  $\frac{1}{n}$  as measured by  $M$ .

We see that all dimensions in this case are given by the numbers  $\frac{k}{n}$  where  $k$  is a non-negative integer (or infinity).

Thus the  $\text{II}_1$  factors are infinite dimensional versions of  $M_n(\mathbb{C})$  the dimensions of whose modules "fill in the gaps" of those of  $M = M_n(\mathbb{C})$ .

That was something of a crash course in  $\text{II}_1$  factors and a bit impressionistic.

That was something of a crash course in  $\text{II}_1$  factors and a bit impressionistic.

If you want an example you can take the group  $PSL(2, \mathbb{Z}) < PSL_2(\mathbb{R})$ .

The "holomorphic discrete series" of  $PSL_2(\mathbb{R})$  is the natural unitary action of  $PSL_2(\mathbb{R})$  on  $L^2$  holomorphic functions on the upper half plane with measure  $\frac{dx dy}{y^{2-n}}$ .



That was something of a crash course in  $\text{II}_1$  factors and a bit impressionistic.

If you want an example you can take the group  $PSL(2, \mathbb{Z}) < PSL_2(\mathbb{R})$ .

The "holomorphic discrete series" of  $PSL_2(\mathbb{R})$  is the natural unitary action of  $PSL_2(\mathbb{R})$  on  $L^2$  holomorphic functions on the upper half plane with measure  $\frac{dx dy}{y^{2-n}}$ .

The "commutant"  $PSL_2(\mathbb{Z})'$  of the action of  $PSL_2(\mathbb{Z})$  is a  $\text{II}_1$  factor and the dimension of the Hilbert space is something like  $\frac{1}{n-1}$ .

That was something of a crash course in  $\text{II}_1$  factors and a bit impressionistic.

If you want an example you can take the group  $PSL(2, \mathbb{Z}) < PSL_2(\mathbb{R})$ .

The "holomorphic discrete series" of  $PSL_2(\mathbb{R})$  is the natural unitary action of  $PSL_2(\mathbb{R})$  on  $L^2$  holomorphic functions on the upper half plane with measure  $\frac{dx dy}{y^{2-n}}$ .

The "commutant"  $PSL_2(\mathbb{Z})'$  of the action of  $PSL_2(\mathbb{Z})$  is a  $\text{II}_1$  factor and the dimension of the Hilbert space is something like  $\frac{1}{n-1}$ .

Radulescu.

A  $\text{II}_1$  factor always has an identity. If  $M$  is a  $\text{II}_1$  factor a subfactor of  $N$  is a subfactor

A  $\text{II}_1$  factor always has an identity. If  $M$  is a  $\text{II}_1$  factor a subfactor of  $N$  is a subfactor containing the same identity as that of  $M$ .

A  $\text{II}_1$  factor always has an identity. If  $M$  is a  $\text{II}_1$  factor a subfactor of  $N$  is a subfactor containing the same identity as that of  $M$ .  
Thus one may define the *real* number  $[M : N] = \dim_N(M)$ .

A  $\text{II}_1$  factor always has an identity. If  $M$  is a  $\text{II}_1$  factor a subfactor of  $N$  is a subfactor containing the same identity as that of  $M$ .

Thus one may define the *real* number  $[M : N] = \dim_N(M)$ .

As an example one may take  $\Gamma < PSL_2(\mathbb{Z})$  a large subgroup so that the commutant  $\Gamma'$  of  $\Gamma$  is also  $\text{II}_1$  factor.

A  $\text{II}_1$  factor always has an identity. If  $M$  is a  $\text{II}_1$  factor a subfactor of  $N$  is a subfactor containing the same identity as that of  $M$ .

Thus one may define the *real* number  $[M : N] = \dim_N(M)$ .

As an example one may take  $\Gamma < PSL_2(\mathbb{Z})$  a large subgroup so that the commutant  $\Gamma'$  of  $\Gamma$  is also  $\text{II}_1$  factor. One then has

$$[\Gamma' : PSL_2(\mathbb{Z})'] = [PSL_2(\mathbb{Z}) : \Gamma]$$

A  $\text{II}_1$  factor always has an identity. If  $M$  is a  $\text{II}_1$  factor a subfactor of  $N$  is a subfactor containing the same identity as that of  $M$ .

Thus one may define the *real* number  $[M : N] = \dim_N(M)$ .

As an example one may take  $\Gamma < PSL_2(\mathbb{Z})$  a large subgroup so that the commutant  $\Gamma'$  of  $\Gamma$  is also  $\text{II}_1$  factor. One then has

$$[\Gamma' : PSL_2(\mathbb{Z})'] = [PSL_2(\mathbb{Z}) : \Gamma]$$

In this and similar ways it is easy to get all non-negative integers as indices of  $\text{II}_1$  factors.



A  $\text{II}_1$  factor always has an identity. If  $M$  is a  $\text{II}_1$  factor a subfactor of  $N$  is a subfactor containing the same identity as that of  $M$ .

Thus one may define the *real* number  $[M : N] = \dim_N(M)$ .

As an example one may take  $\Gamma < PSL_2(\mathbb{Z})$  a large subgroup so that the commutant  $\Gamma'$  of  $\Gamma$  is also  $\text{II}_1$  factor. One then has

$$[\Gamma' : PSL_2(\mathbb{Z})'] = [PSL_2(\mathbb{Z}) : \Gamma]$$

In this and similar ways it is easy to get all non-negative integers as indices of  $\text{II}_1$  factors. But it is supposed to be a real number.

The following result was proved when I was at Penn in 1981-1982:

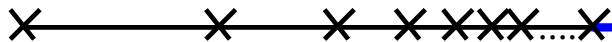
The following result was proved when I was at Penn in 1981-1982:

Theorem: If  $[M : N] < 4$  it is  $4 \cos^2 \pi/n$  for  $n = 3, 4, 5, 6, \dots$  and all these values occur. If  $> 4$  any real number possible.

The following result was proved when I was at Penn in 1981-1982:

Theorem: If  $[M : N] < 4$  it is  $4 \cos^2 \pi/n$  for  $n = 3, 4, 5, 6, \dots$  and all these values occur. If  $> 4$  any real number possible.

Picture:



1

2

3

4

$$2.6180339\dots = 4 \cos^2 \pi/5$$

About the proof I will only say that it involves a detailed understanding of what has become known as the Temperley-Lieb algebra  $TL(n+1, \tau)$

About the proof I will only say that it involves a detailed understanding of what has become known as the Temperley-Lieb algebra  $TL(n+1, \tau)$

This is the  $*$ -algebra generated by projections  $e_1, e_2, e_3, \dots, e_n$  with relations:

$$e_i^2 = e_i = e_i^*$$

About the proof I will only say that it involves a detailed understanding of what has become known as the Temperley-Lieb algebra  $TL(n+1, \tau)$

This is the  $*$ -algebra generated by projections  $e_1, e_2, e_3, \dots, e_n$  with relations:

$$e_i^2 = e_i = e_i^*$$

$$e_i e_j = e_j e_i \text{ for } |i - j| \geq 2$$

About the proof I will only say that it involves a detailed understanding of what has become known as the Temperley-Lieb algebra  $TL(n+1, \tau)$

This is the  $*$ -algebra generated by projections  $e_1, e_2, e_3, \dots, e_n$  with relations:

$$e_i^2 = e_i = e_i^*$$

$$e_i e_j = e_j e_i \text{ for } |i - j| \geq 2$$

$$e_i e_{i \pm 1} e_i = \tau e_i$$

Which possesses a trace functional  $tr$  completely defined by the formula

$$tr(xe_{n+1}) = \tau tr(x)$$

if  $x$  is a word on  $e_1, e_2, \dots, e_n$ .



About the proof I will only say that it involves a detailed understanding of what has become known as the Temperley-Lieb algebra  $TL(n+1, \tau)$

This is the  $*$ -algebra generated by projections  $e_1, e_2, e_3, \dots, e_n$  with relations:

$$e_i^2 = e_i = e_i^*$$

$$e_i e_j = e_j e_i \text{ for } |i - j| \geq 2$$

$$e_i e_{i \pm 1} e_i = \tau e_i$$

Which possesses a trace functional  $tr$  completely defined by the formula

$$tr(xe_{n+1}) = \tau tr(x)$$

if  $x$  is a word on  $e_1, e_2, \dots, e_n$ .

This algebra is loaded with structure and was the cause of a lot of discussion at the time at Penn.

About the proof I will only say that it involves a detailed understanding of what has become known as the Temperley-Lieb algebra  $TL(n+1, \tau)$

This is the  $*$ -algebra generated by projections  $e_1, e_2, e_3, \dots, e_n$  with relations:

$$e_i^2 = e_i = e_i^*$$

$$e_i e_j = e_j e_i \text{ for } |i - j| \geq 2$$

$$e_i e_{i \pm 1} e_i = \tau e_i$$

Which possesses a trace functional  $tr$  completely defined by the formula

$$tr(xe_{n+1}) = \tau tr(x)$$

if  $x$  is a word on  $e_1, e_2, \dots, e_n$ .

This algebra is loaded with structure and was the cause of a lot of discussion at the time at Penn. We talked about  $e_i e_i$  oh!

Today I would like to say something about what has happened since then.

Today I would like to say something about what has happened since then.  
I will split the talk up somewhat artificially into three parts:

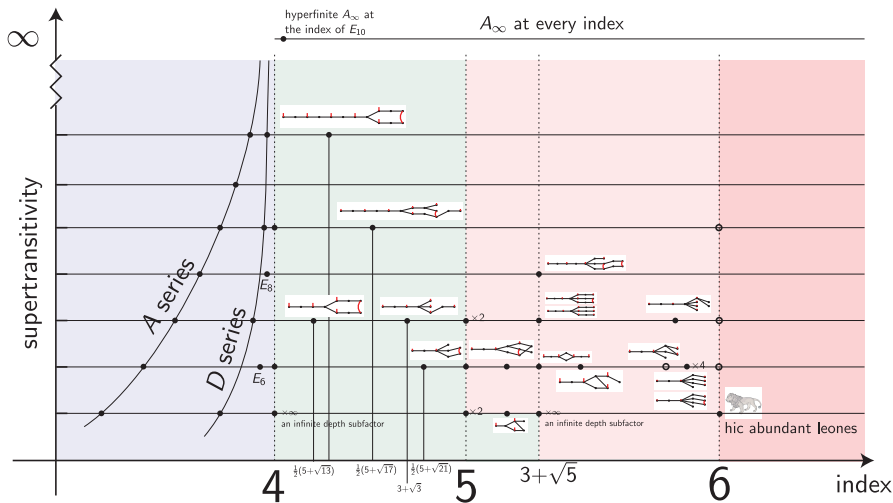
The internal theory of subfactors.

Interactions with other parts of mathematics.

Interactions with physics.

Internal theory:

One may sum up the internal theory developments with the following picture:



Ocneanu Popa Wenzl Goodman de la Harpe Haagerup Asaeda Izumi Bisch  
Graham Lehrer Morrison Peters Snyder Bigelow Xu Grossman Liu Penneys  
Tener Evans Gannon Etingof Nikschych Ostrik Yasuda Calegari Jobs  
Wolfram

What are all the stick insects?

What are all the stick insects?

It is important to note that subfactors are inextricably connected to bimodules (or correspondences in the sense of Connes).



What are all the stick insects?

It is important to note that subfactors are inextricably connected to bimodules (or correspondences in the sense of Connes).

A bimodule is a left and right (commuting) module over a pair of algebras.

What are all the stick insects?

It is important to note that subfactors are inextricably connected to bimodules (or correspondences in the sense of Connes).

A bimodule is a left and right (commuting) module over a pair of algebras.

$${}_M H_N$$

.

What are all the stick insects?

It is important to note that subfactors are inextricably connected to bimodules (or correspondences in the sense of Connes).

A bimodule is a left and right (commuting) module over a pair of algebras.

$${}_M H_N$$

.

It can be quite confusing because for a subfactor  $N \subseteq M$ ,  $M$  is an  $N - M$ ,  $M - M$ ,  $M - N$  and  $N - N$  bimodule whereas typically in bimodule theory one considers a single  ${}_M H_M$  and takes the tensor powers over  $M$ -  
 $\otimes_M^k H$ .

What are all the stick insects?

It is important to note that subfactors are inextricably connected to bimodules (or correspondences in the sense of Connes).

A bimodule is a left and right (commuting) module over a pair of algebras.

$${}_M H_N$$

.

It can be quite confusing because for a subfactor  $N \subseteq M$ ,  $M$  is an  $N - M$ ,  $M - M$ ,  $M - N$  and  $N - N$  bimodule whereas typically in bimodule theory one considers a single  ${}_M H_M$  and takes the tensor powers over  $M$ -  
 $\otimes_M^k H$ .

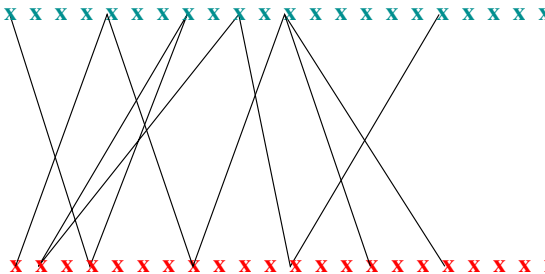
**N-M bimodules**    x

**N-N bimodules**    x

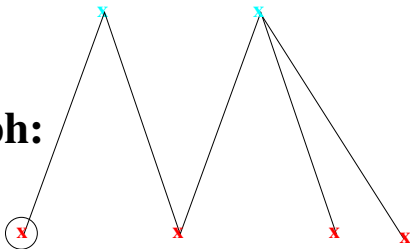
# N-M bimodules

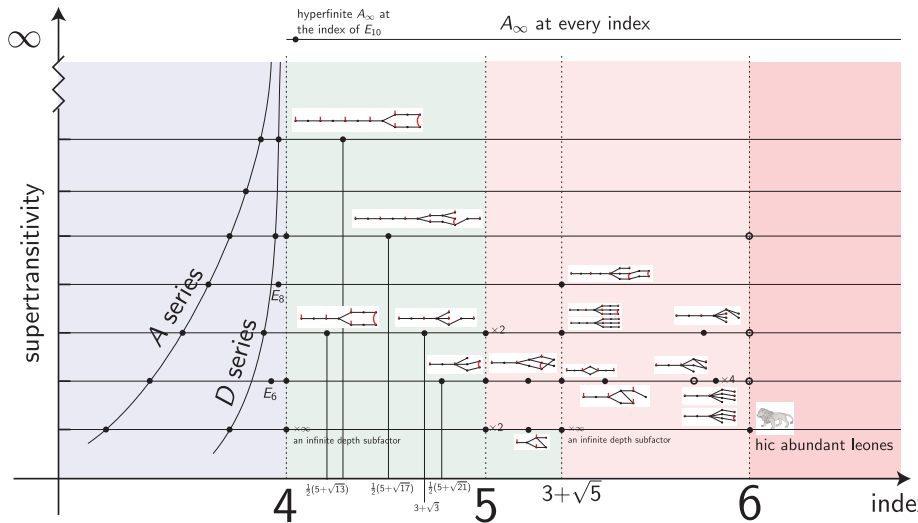
induction/restriction

# N-N bimodules



## The principal graph:





## Interactions with other parts of mathematics:

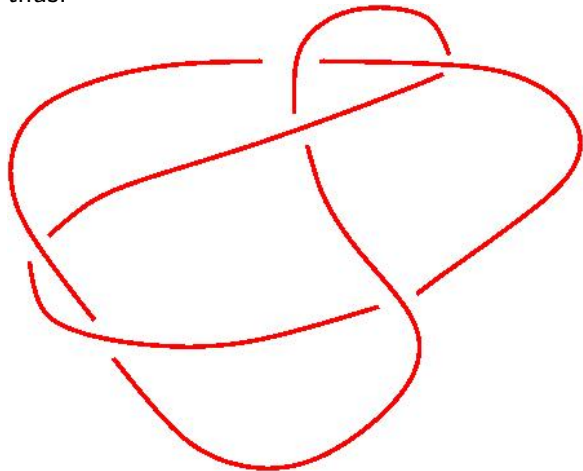


Interactions with other parts of mathematics:

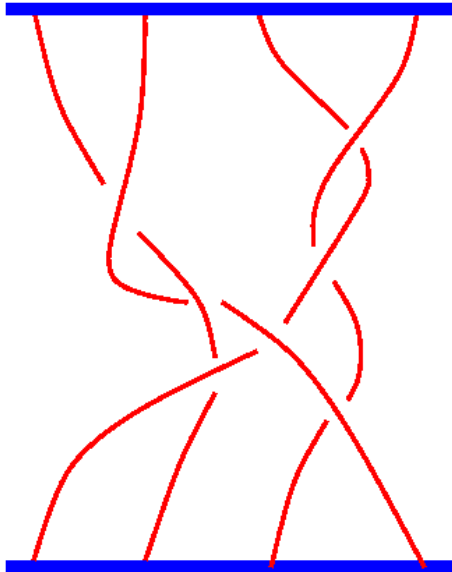
A knot is a smooth closed curve in 3 dimensional space. A link is a collection of knots.

Interactions with other parts of mathematics:

A knot is a smooth closed curve in 3 dimensional space. A link is a collection of knots. They can be represented by two dimensional pictures thus:

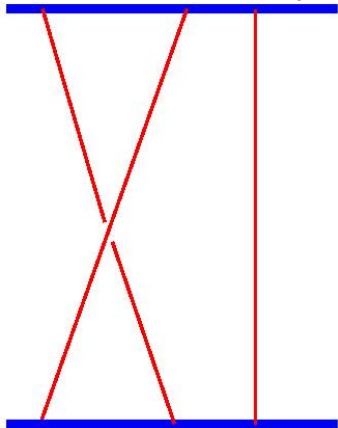


Braids are collections of curves in 3 dimensional space connecting two bars thus:

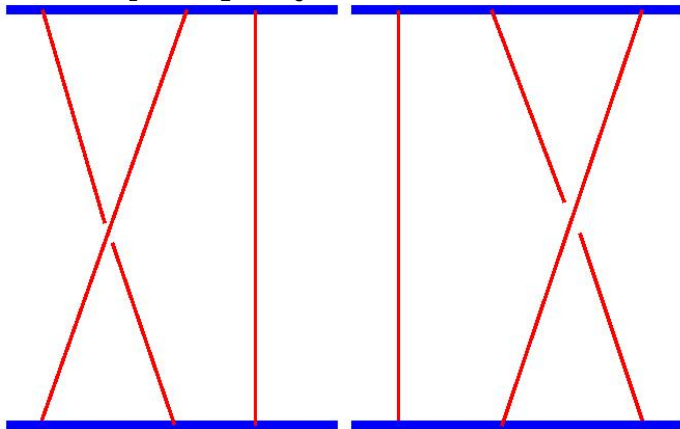


Braids with  $n$  strings form a *group*  $B_n$  under concatenation. This group has a simple algebraic presentation (Artin) on generators  $\sigma_i, i = 1, 2, \dots, n - 1$

Braids with  $n$  strings form a *group*  $B_n$  under concatenation. This group has a simple algebraic presentation (Artin) on generators  $\sigma_i, i = 1, 2, \dots, n - 1$ . Here are  $\sigma_1$  and  $\sigma_2$  in  $B_3$ :



Braids with  $n$  strings form a *group*  $B_n$  under concatenation. This group has a simple algebraic presentation (Artin) on generators  $\sigma_i, i = 1, 2, \dots, n - 1$ . Here are  $\sigma_1$  and  $\sigma_2$  in  $B_3$ :



The presenting relations are

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_j \sigma_i = \sigma_i \sigma_j \text{ if } |i - j| \geq 2$$

The presenting relations are

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_j \sigma_i = \sigma_i \sigma_j \text{ if } |i - j| \geq 2$$

Recall our projections  $e_i$ :

$$e_i = e_i^* = e_i$$

$$e_i e_{i \pm 1} e_i = \tau e_i \text{ where } \tau^{-1} = [M : N]$$

$$e_i e_j = e_j e_i \text{ if } |i - j| \geq 2$$



The presenting relations are

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_j \sigma_i = \sigma_i \sigma_j \text{ if } |i - j| \geq 2$$

Recall our projections  $e_i$ :

$$e_i = e_i^* = e_i$$

$$e_i e_{i \pm 1} e_i = \tau e_i \text{ where } \tau^{-1} = [M : N]$$

$$e_i e_j = e_j e_i \text{ if } |i - j| \geq 2$$

Represent  $B_n$  in  $\text{II}_1$  factor by

$$\sigma_i \rightarrow t e_1 - (1 - e_i)$$

with  $[M : N] = 2 + t + t^{-1}$ .

The connection between braids and links:

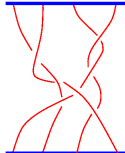
In 1984 while still at Penn, I visited Joan Birman at Columbia.

Recall our braid:

The connection between braids and links:

In 1984 while still at Penn, I visited Joan Birman at Columbia.

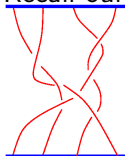
Recall our braid:



The connection between braids and links:

In 1984 while still at Penn, I visited Joan Birman at Columbia.

Recall our braid:

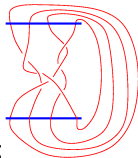
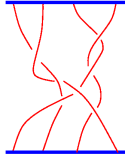


We can “close” it thus:

The connection between braids and links:

In 1984 while still at Penn, I visited Joan Birman at Columbia.

Recall our braid:

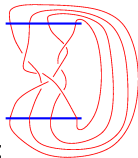
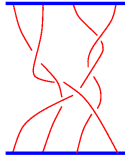


We can “close” it thus:

The connection between braids and links:

In 1984 while still at Penn, I visited Joan Birman at Columbia.

Recall our braid:



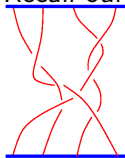
We can “close” it thus:

And remove the bars to obtain:

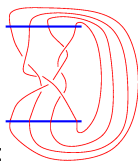
The connection between braids and links:

In 1984 while still at Penn, I visited Joan Birman at Columbia.

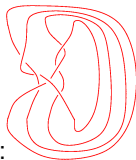
Recall our braid:



We can “close” it thus:



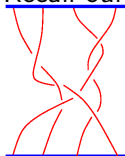
And remove the bars to obtain:



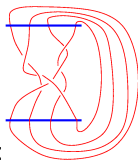
The connection between braids and links:

In 1984 while still at Penn, I visited Joan Birman at Columbia.

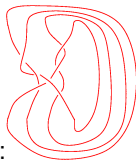
Recall our braid:



We can “close” it thus:



And remove the bars to obtain:



A LINK!



All links can be obtained in this way (Alexander) and Joan told me of a result of A.A. Markov which says exactly what different braids give the same link. Then a miracle occurred: Markov's theorem corresponds exactly (!! ) to our condition

All links can be obtained in this way (Alexander) and Joan told me of a result of A.A. Markov which says exactly what different braids give the same link. Then a miracle occurred: Markov's theorem corresponds exactly (!! ) to our condition

$$\text{trace}(xe_{n+1}) = \tau \text{trace}(x) \text{ if } x \text{ is a word on } 1, e_1, e_2, \dots, e_n$$

All links can be obtained in this way (Alexander) and Joan told me of a result of A.A. Markov which says exactly what different braids give the same link. Then a miracle occurred: Markov's theorem corresponds exactly (!! ) to our condition

$$\text{trace}(xe_{n+1}) = \tau \text{trace}(x) \text{ if } x \text{ is a word on } 1, e_1, e_2, \dots, e_n$$

(which in another miracle had already been called a Markov condition after A.A. Markov.....)

All links can be obtained in this way (Alexander) and Joan told me of a result of A.A. Markov which says exactly what different braids give the same link. Then a miracle occurred: Markov's theorem corresponds exactly (!! ) to our condition

$$\text{trace}(xe_{n+1}) = \tau \text{trace}(x) \text{ if } x \text{ is a word on } 1, e_1, e_2, \dots, e_n$$

(which in another miracle had already been called a Markov condition after A.A. Markov.....) Putting the ingredients together we get an invariant of knots/links, called  $V_L(t)$  by the following procedure:  
(i) Take  $L$  and represent it as the closure of a braid  $\alpha$ .

All links can be obtained in this way (Alexander) and Joan told me of a result of A.A. Markov which says exactly what different braids give the same link. Then a miracle occurred: Markov's theorem corresponds exactly (!! ) to our condition

$$\text{trace}(xe_{n+1}) = \tau \text{trace}(x) \text{ if } x \text{ is a word on } 1, e_1, e_2, \dots, e_n$$

(which in another miracle had already been called a Markov condition after A.A. Markov.....) Putting the ingredients together we get an invariant of knots/links, called  $V_L(t)$  by the following procedure:

- (i) Take  $L$  and represent it as the closure of a braid  $\alpha$ .
- (ii) Represent the braid  $\alpha$  in the algebra of the  $e_i$ 's via  $\sigma_i \mapsto te_i - (1 - e_i)$

All links can be obtained in this way (Alexander) and Joan told me of a result of A.A. Markov which says exactly what different braids give the same link. Then a miracle occurred: Markov's theorem corresponds exactly (!! ) to our condition

$$\text{trace}(xe_{n+1}) = \tau \text{trace}(x) \text{ if } x \text{ is a word on } 1, e_1, e_2, \dots, e_n$$

(which in another miracle had already been called a Markov condition after A.A. Markov.....) Putting the ingredients together we get an invariant of knots/links, called  $V_L(t)$  by the following procedure:

- (i) Take  $L$  and represent it as the closure of a braid  $\alpha$ .
- (ii) Represent the braid  $\alpha$  in the algebra of the  $e_i$ 's via  $\sigma_i \mapsto te_i - (1 - e_i)$
- (iii) Take the trace of the braid in the algebra, multiply by a simple fudge factor and voilà.

All links can be obtained in this way (Alexander) and Joan told me of a result of A.A. Markov which says exactly what different braids give the same link. Then a miracle occurred: Markov's theorem corresponds exactly (!! ) to our condition

$$\text{trace}(xe_{n+1}) = \tau \text{trace}(x) \text{ if } x \text{ is a word on } 1, e_1, e_2, \dots, e_n$$

(which in another miracle had already been called a Markov condition after A.A. Markov.....) Putting the ingredients together we get an invariant of knots/links, called  $V_L(t)$  by the following procedure:

- (i) Take  $L$  and represent it as the closure of a braid  $\alpha$ .
- (ii) Represent the braid  $\alpha$  in the algebra of the  $e_i$ 's via  $\sigma_i \mapsto te_i - (1 - e_i)$
- (iii) Take the trace of the braid in the algebra, multiply by a simple fudge factor and voilà.

e.g. for trefoil knot

$$V_K(t) = t + t^3 - t^4$$

All links can be obtained in this way (Alexander) and Joan told me of a result of A.A. Markov which says exactly what different braids give the same link. Then a miracle occurred: Markov's theorem corresponds exactly (!! ) to our condition

$$\text{trace}(xe_{n+1}) = \tau \text{trace}(x) \text{ if } x \text{ is a word on } 1, e_1, e_2, \dots, e_n$$

(which in another miracle had already been called a Markov condition after A.A. Markov.....) Putting the ingredients together we get an invariant of knots/links, called  $V_L(t)$  by the following procedure:

- (i) Take  $L$  and represent it as the closure of a braid  $\alpha$ .
- (ii) Represent the braid  $\alpha$  in the algebra of the  $e_i$ 's via  $\sigma_i \mapsto te_i - (1 - e_i)$
- (iii) Take the trace of the braid in the algebra, multiply by a simple fudge factor and voilà.

e.g. for trefoil knot

$$V_K(t) = t + t^3 - t^4$$

So 1984 was a bad year for Orwell characters, a great year for me (at Penn)!



Many questions arose from this work. One is somewhat notoriously still open: Is there a knot  $K$ , different from the unknot, for which  $V_K(t)$ ?

Many questions arose from this work. One is somewhat notoriously still open: Is there a knot  $K$ , different from the unknot, for which  $V_K(t)$ ?

The analogous question for links has been answered known thanks to Thistlethwaite and his computer.

Many questions arose from this work. One is somewhat notoriously still open: Is there a knot  $K$ , different from the unknot, for which  $V_K(t)$ ?

The analogous question for links has been answered known thanks to Thistlethwaite and his computer.

But it was Witten who in 1988 really opened the subject up with a functional integral interpretation of  $V_L(t)$  which led immediately to a generalisation of  $V_L(t)$ , at least for  $t$  a root of unity, to the case where  $L$  lives in an arbitrary closed 3 manifold.

Many questions arose from this work. One is somewhat notoriously still open: Is there a knot  $K$ , different from the unknot, for which  $V_K(t)$ ?

The analogous question for links has been answered known thanks to Thistlethwaite and his computer.

But it was Witten who in 1988 really opened the subject up with a functional integral interpretation of  $V_L(t)$  which led immediately to a generalisation of  $V_L(t)$ , at least for  $t$  a root of unity, to the case where  $L$  lives in an arbitrary closed 3 manifold.

First proved by Reshetikhin and Turaev. Then many others.

Thus began the subject of "quantum topology".

Many questions arose from this work. One is somewhat notoriously still open: Is there a knot  $K$ , different from the unknot, for which  $V_K(t)$ ?

The analogous question for links has been answered known thanks to Thistlethwaite and his computer.

But it was Witten who in 1988 really opened the subject up with a functional integral interpretation of  $V_L(t)$  which led immediately to a generalisation of  $V_L(t)$ , at least for  $t$  a root of unity, to the case where  $L$  lives in an arbitrary closed 3 manifold.

First proved by Reshetikhin and Turaev. Then many others.

Thus began the subject of "quantum topology".

The understanding of the structure and even the mathematical nature of these invariants is an ongoing body of research with strong interactions with physics, in particular conformal field theory.

(Kontsevich, Bar Natan, Lawrence, Le, Ohtsuki, Garoufalidis, Aganagic, Vafa, Gukov.....)

Interactions with other parts of mathematics:

Interactions with other parts of mathematics:  
Tensor categories, planar algebras.

Interactions with other parts of mathematics:

Tensor categories, planar algebras.

Bimodules over a  $\text{II}_1$  (or other) factor form a tensor category-whatever that means.



Interactions with other parts of mathematics:

Tensor categories, planar algebras.

Bimodules over a  $II_1$  (or other) factor form a tensor category-whatever that means.

If a subcategory contains only finitely many isomorphism classes of simple objects (=irreducible bimodules) it is called a "fusion category."

Interactions with other parts of mathematics:

Tensor categories, planar algebras.

Bimodules over a  $II_1$  (or other) factor form a tensor category-whatever that means.

If a subcategory contains only finitely many isomorphism classes of simple objects (=irreducible bimodules) it is called a "fusion category.

The main suspects in the study of fusion categories are Etingof Nikschykh and Ostrik.

Interactions with other parts of mathematics:

Tensor categories, planar algebras.

Bimodules over a  $\text{II}_1$  (or other) factor form a tensor category-whatever that means.

If a subcategory contains only finitely many isomorphism classes of simple objects (=irreducible bimodules) it is called a "fusion category.

The main suspects in the study of fusion categories are Etingof Nikschy and Ostrik.

There are many examples of fusion categories related to finite groups but it is fair to say that the most interesting ones have arisen as the  $N - N$  bimodules coming from a subfactor with finite principal graph.

Interactions with other parts of mathematics:

Tensor categories, planar algebras.

Bimodules over a  $\text{II}_1$  (or other) factor form a tensor category-whatever that means.

If a subcategory contains only finitely many isomorphism classes of simple objects (=irreducible bimodules) it is called a "fusion category.

The main suspects in the study of fusion categories are Etingof Nikschy and Ostrik.

There are many examples of fusion categories related to finite groups but it is fair to say that the most interesting ones have arisen as the  $N - N$  bimodules coming from a subfactor with finite principal graph.

In general these tensor categories admit a diagrammatic calculus which in the case of subfactors was formalised rigorously as "planar algebras".

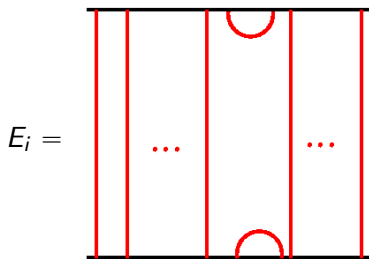
In general these tensor categories admit a diagrammatic calculus which in the case of subfactors was formalised rigorously as "planar algebras". They are a major tool in the construction and analysis of subfactors.

In general these tensor categories admit a diagrammatic calculus which in the case of subfactors was formalised rigorously as "planar algebras". They are a major tool in the construction and analysis of subfactors.

The original motivation for planar algebras included the desire to generalise a beautiful pictorial representation, due to L. Kauffman, of the TL algebra in which one uses the braid picture but with the following two-dimensional pictures replacing the crossings:

In general these tensor categories admit a diagrammatic calculus which in the case of subfactors was formalised rigorously as "planar algebras". They are a major tool in the construction and analysis of subfactors.

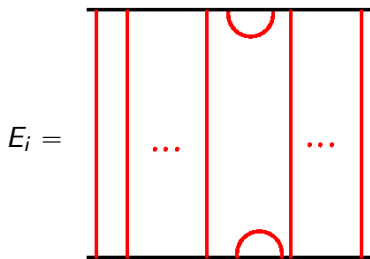
The original motivation for planar algebras included the desire to generalise a beautiful pictorial representation, due to L. Kauffman, of the TL algebra in which one uses the braid picture but with the following two-dimesional pictures replacing the crossings:





In general these tensor categories admit a diagrammatic calculus which in the case of subfactors was formalised rigorously as "planar algebras". They are a major tool in the construction and analysis of subfactors.

The original motivation for planar algebras included the desire to generalise a beautiful pictorial representation, due to L. Kauffman, of the TL algebra in which one uses the braid picture but with the following two-dimensional pictures replacing the crossings:



Checking the  $e_i$  relations is then a very pleasant exercise. (The  $\tau$  factor appears in a different place!)

Quantum groups.

## Quantum groups.

It was observed during a seminar at Penn (at which Kadison was present), by Mitch Baker, Bob Powers and myself, that the structure of the TL algebra is (for generic  $\tau$ ) the SAME as that of the  $SU(2)$ -invariant CAR algebra!

## Quantum groups.

It was observed during a seminar at Penn (at which Kadison was present), by Mitch Baker, Bob Powers and myself, that the structure of the TL algebra is (for generic  $\tau$ ) the SAME as that of the  $SU(2)$ -invariant CAR algebra!

This is now understood as a "quantum Schur-Weyl duality" in which  $SU(2)$  is deformed to the Jimbo-Drinfeld-Woronowicz quantum version  $SU_q(2)$ , together with its representations, and  $TL(n, 2 + q + q^{-1})$  is the commutant of the tensor product action of  $SU_q(2)$  on the tensor powers of the 2-dimensional representation.

## Quantum groups.

It was observed during a seminar at Penn (at which Kadison was present), by Mitch Baker, Bob Powers and myself, that the structure of the TL algebra is (for generic  $\tau$ ) the SAME as that of the  $SU(2)$ -invariant CAR algebra!

This is now understood as a "quantum Schur-Weyl duality" in which  $SU(2)$  is deformed to the Jimbo-Drinfeld-Woronowicz quantum version  $SU_q(2)$ , together with its representations, and  $TL(n, 2 + q + q^{-1})$  is the commutant of the tensor product action of  $SU_q(2)$  on the tensor powers of the 2-dimensional representation.

This has all been vastly generalised, but it should be stressed that the representation of the  $e'_i$ 's in the tensor power was in fact discovered in the subfactor context, by Pimsner and Popa!

Interactions with physics.

Interactions with physics.

The reason the  $e_i$  algebra is called the TL algebra is because two representations of it were used by Temperley and Lieb in proving a mathematical equivalence between two models in 2-D statistical mechanics.

Interactions with physics.

The reason the  $e_i$  algebra is called the TL algebra is because two representations of it were used by Temperley and Lieb in proving a mathematical equivalence between two models in 2-D statistical mechanics.

Lieb's "ice type" model and the Potts model.



Interactions with physics.

The reason the  $e_i$  algebra is called the TL algebra is because two representations of it were used by Temperley and Lieb in proving a mathematical equivalence between two models in 2-D statistical mechanics.

Lieb's "ice type" model and the Potts model. The models are on a square lattice and are analysed by a "transfer matrix" which corresponds to adding a row of spins to the lattice.

Interactions with physics.

The reason the  $e_i$  algebra is called the TL algebra is because two representations of it were used by Temperley and Lieb in proving a mathematical equivalence between two models in 2-D statistical mechanics.

Lieb's "ice type" model and the Potts model. The models are on a square lattice and are analysed by a "transfer matrix" which corresponds to adding a row of spins to the lattice.

Roughly speaking each  $e_i$  corresponds to adding a single spin to the transfer matrix which will thus have the form

$$(xe_1 + y)(xe_2 + y)(xe_3 + y).....(xe_n + y)$$

Interactions with physics.

The reason the  $e_i$  algebra is called the TL algebra is because two representations of it were used by Temperley and Lieb in proving a mathematical equivalence between two models in 2-D statistical mechanics.

Lieb's "ice type" model and the Potts model. The models are on a square lattice and are analysed by a "transfer matrix" which corresponds to adding a row of spins to the lattice.

Roughly speaking each  $e_i$  corresponds to adding a single spin to the transfer matrix which will thus have the form

$$(xe_1 + y)(xe_2 + y)(xe_3 + y).....(xe_n + y)$$

Taking the trace corresponds to periodic boundary conditions, although these were not the boundary conditions that T and L used-for their ones what they showed was that the calculation (of....) can be done ENTIRELY using the  $e_i$  relations.

Interactions with physics.

The reason the  $e_i$  algebra is called the TL algebra is because two representations of it were used by Temperley and Lieb in proving a mathematical equivalence between two models in 2-D statistical mechanics.

Lieb's "ice type" model and the Potts model. The models are on a square lattice and are analysed by a "transfer matrix" which corresponds to adding a row of spins to the lattice.

Roughly speaking each  $e_i$  corresponds to adding a single spin to the transfer matrix which will thus have the form

$$(xe_1 + y)(xe_2 + y)(xe_3 + y).....(xe_n + y)$$

Taking the trace corresponds to periodic boundary conditions, although these were not the boundary conditions that T and L used-for their ones what they showed was that the calculation (of....) can be done ENTIRELY using the  $e_i$  relations.

The  $e_i$  relations come from the nearest neighbour interaction between the spins.

Interactions with physics.

The reason the  $e_i$  algebra is called the TL algebra is because two representations of it were used by Temperley and Lieb in proving a mathematical equivalence between two models in 2-D statistical mechanics.

Lieb's "ice type" model and the Potts model. The models are on a square lattice and are analysed by a "transfer matrix" which corresponds to adding a row of spins to the lattice.

Roughly speaking each  $e_i$  corresponds to adding a single spin to the transfer matrix which will thus have the form

$$(xe_1 + y)(xe_2 + y)(xe_3 + y).....(xe_n + y)$$

Taking the trace corresponds to periodic boundary conditions, although these were not the boundary conditions that T and L used-for their ones what they showed was that the calculation (of....) can be done ENTIRELY using the  $e_i$  relations.

The  $e_i$  relations come from the nearest neighbour interaction between the spins.(David Evans.)

# Quantum spin chains, tensor networks.

Quantum spin chains, tensor networks.

If quantum spins are arranged in a one dimensional array with interactions between nearest neighbours, the Hilbert space for the spin chain is  $\otimes^n \mathbb{C}^2$ .

Quantum spin chains, tensor networks.

If quantum spins are arranged in a one dimensional array with interactions between nearest neighbours, the Hilbert space for the spin chain is  $\otimes^n \mathbb{C}^2$ .

On which the  $e_i'$ s act.



Quantum spin chains, tensor networks.

If quantum spins are arranged in a one dimensional array with interactions between nearest neighbours, the Hilbert space for the spin chain is  $\otimes^n \mathbb{C}^2$ .

On which the  $e'_j$ 's act. In a nearest neighbour fashion.

Quantum spin chains, tensor networks.

If quantum spins are arranged in a one dimensional array with interactions between nearest neighbours, the Hilbert space for the spin chain is  $\otimes^n \mathbb{C}^2$ .

On which the  $e'_j$ 's act. In a nearest neighbour fashion.

The operator

$$\sum_{i=1}^n e_i$$

has been used as a Hamiltonian governing the time evolution of a quantum spin chain of length  $n$ .

Quantum spin chains, tensor networks.

If quantum spins are arranged in a one dimensional array with interactions between nearest neighbours, the Hilbert space for the spin chain is  $\otimes^n \mathbb{C}^2$ .

On which the  $e'_j$ 's act. In a nearest neighbour fashion.

The operator

$$\sum_{i=1}^n e_i$$

has been used as a Hamiltonian governing the time evolution of a quantum spin chain of length  $n$ . Some think (e.g. Pasquier and Saleur) that the resulting physics can be rescaled to obtain a quantum field theory with conformal symmetry.

Quantum spin chains, tensor networks.

If quantum spins are arranged in a one dimensional array with interactions between nearest neighbours, the Hilbert space for the spin chain is  $\otimes^n \mathbb{C}^2$ .

On which the  $e_i$ 's act. In a nearest neighbour fashion.

The operator

$$\sum_{i=1}^n e_i$$

has been used as a Hamiltonian governing the time evolution of a quantum spin chain of length  $n$ . Some think (e.g. Pasquier and Saleur) that the resulting physics can be rescaled to obtain a quantum field theory with conformal symmetry.

Subfactors allow one to construct many other "anyonic" spin chains with other potential Hamiltonians.

# Conformal (and other) quantum field theory.

# Conformal (and other) quantum field theory.

No doubt the richest and most exciting interaction of subfactors with physics is the connection with conformal quantum field theory in 2 dimensions.

# Conformal (and other) quantum field theory.

No doubt the richest and most exciting interaction of subfactors with physics is the connection with conformal quantum field theory in 2 dimensions.

Need to go back to Haag-Kastler who proposed that one study QFT using *algebras of local observables*.

# Conformal (and other) quantum field theory.

No doubt the richest and most exciting interaction of subfactors with physics is the connection with conformal quantum field theory in 2 dimensions.

Need to go back to Haag-Kastler who proposed that one study QFT using *algebras of local observables*.

To each nice region  $\mathcal{O}$  of space time there is a von Neumann algebra  $\mathcal{A}(\mathcal{O})$  of observables localised in  $\mathcal{O}$ .



# Conformal (and other) quantum field theory.

No doubt the richest and most exciting interaction of subfactors with physics is the connection with conformal quantum field theory in 2 dimensions.

Need to go back to Haag-Kastler who proposed that one study QFT using *algebras of local observables*.

To each nice region  $\mathcal{O}$  of space time there is a von Neumann algebra  $\mathcal{A}(\mathcal{O})$  of observables localised in  $\mathcal{O}$ .

The association  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  satisfies several very general axioms, which prove in particular that the  $\mathcal{A}(\mathcal{O})$  are type  $\text{III}_1$  factors.

# Conformal (and other) quantum field theory.

No doubt the richest and most exciting interaction of subfactors with physics is the connection with conformal quantum field theory in 2 dimensions.

Need to go back to Haag-Kastler who proposed that one study QFT using *algebras of local observables*.

To each nice region  $\mathcal{O}$  of space time there is a von Neumann algebra  $\mathcal{A}(\mathcal{O})$  of observables localised in  $\mathcal{O}$ .

The association  $\mathcal{O} \rightarrow \mathcal{A}(\mathcal{O})$  satisfies several very general axioms, which prove in particular that the  $\mathcal{A}(\mathcal{O})$  are type  $\text{III}_1$  factors.

But the one that is of most interest to us is causality which asserts that, if  $\mathcal{O}$  and  $\mathcal{O}'$  are regions so that light cannot get from  $\mathcal{O}$  to  $\mathcal{O}'$  then

$$\mathcal{A}(\mathcal{O}) \text{ commutes with } \mathcal{A}(\mathcal{O}')$$

Or:

$$\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{O}')'$$

Or:

$$\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{O}')'$$

A Subfactor !

Or:

$$\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{O}')'$$

A Subfactor !

Doplicher, Haag and Roberts gave a much deeper analysis of this situation which involved bimodules(=endomorphisms).

Or:

$$\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{O}')'$$

A Subfactor !

Doplicher, Haag and Roberts gave a much deeper analysis of this situation which involved bimodules(=endomorphisms).

If spacetime is the circle  $S^1$  and  $\mathcal{A}(\mathcal{O})$  and  $\mathcal{A}(\mathcal{O}')$  are complementary intervals  $I$  and  $I^c$  we have the picture:

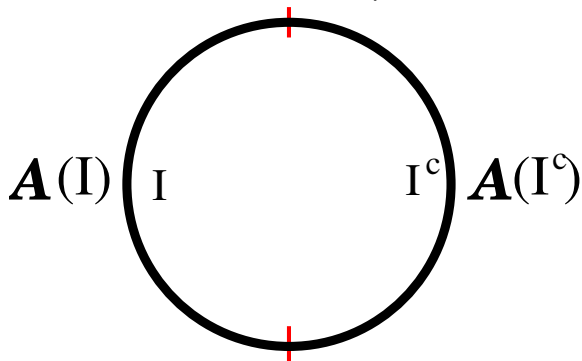
Or:

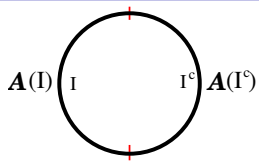
$$\mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}(\mathcal{O}')'$$

A Subfactor !

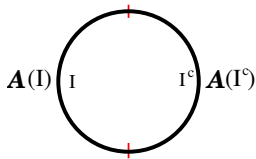
Doplicher, Haag and Roberts gave a much deeper analysis of this situation which involved bimodules(=endomorphisms).

If spacetime is the circle  $S^1$  and  $\mathcal{A}(\mathcal{O})$  and  $\mathcal{A}(\mathcal{O}')$  are complementary intervals  $I$  and  $I^c$  we have the picture:

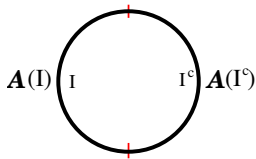






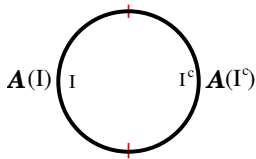


A. Wassermann and I proposed, and he proved, that the subfactor in this case can indeed, in the WZW models, be an interesting one, in particular obtaining subfactors of index  $4 \cos^2 \pi/n$  for all integers  $n \geq 3$ .



A. Wassermann and I proposed, and he proved, that the subfactor in this case can indeed, in the WZW models, be an interesting one, in particular obtaining subfactors of index  $4 \cos^2 \pi/n$  for all integers  $n \geq 3$ .

But we saw there are lots of interesting subfactors around, beginning with the Haagerup, which are unlikely to be obtainable from WZW in any way.

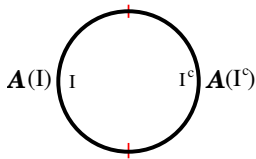


A. Wassermann and I proposed, and he proved, that the subfactor in this case can indeed, in the WZW models, be an interesting one, in particular obtaining subfactors of index  $4 \cos^2 \pi/n$  for all integers  $n \geq 3$ .

But we saw there are lots of interesting subfactors around, beginning with the Haagerup, which are unlikely to be obtainable from WZW in any way.

Over ten years ago Evans and Gannon saw structure in the Haagerup which suggested that there is actually a CFT which produces it!

Unfortunately this construction has not yet been achieved.



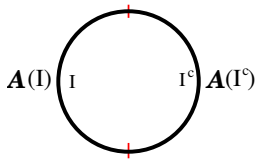
A. Wassermann and I proposed, and he proved, that the subfactor in this case can indeed, in the WZW models, be an interesting one, in particular obtaining subfactors of index  $4 \cos^2 \pi/n$  for all integers  $n \geq 3$ .

But we saw there are lots of interesting subfactors around, beginning with the Haagerup, which are unlikely to be obtainable from WZW in any way.

Over ten years ago Evans and Gannon saw structure in the Haagerup which suggested that there is actually a CFT which produces it!

Unfortunately this construction has not yet been achieved. We are left with the intriguing question, which is no doubt the main question in the external theory of subfactors:

Does EVERY (finite depth) subfactor arise in Conformal Field Theory?



A. Wassermann and I proposed, and he proved, that the subfactor in this case can indeed, in the WZW models, be an interesting one, in particular obtaining subfactors of index  $4 \cos^2 \pi/n$  for all integers  $n \geq 3$ .

But we saw there are lots of interesting subfactors around, beginning with the Haagerup, which are unlikely to be obtainable from WZW in any way.

Over ten years ago Evans and Gannon saw structure in the Haagerup which suggested that there is actually a CFT which produces it!

Unfortunately this construction has not yet been achieved. We are left with the intriguing question, which is no doubt the main question in the external theory of subfactors:

Does EVERY (finite depth) subfactor arise in Conformal Field Theory?

This may take some time-perhaps even a good chunk of the 21st century.