C*-Algebras and Tempered Representations

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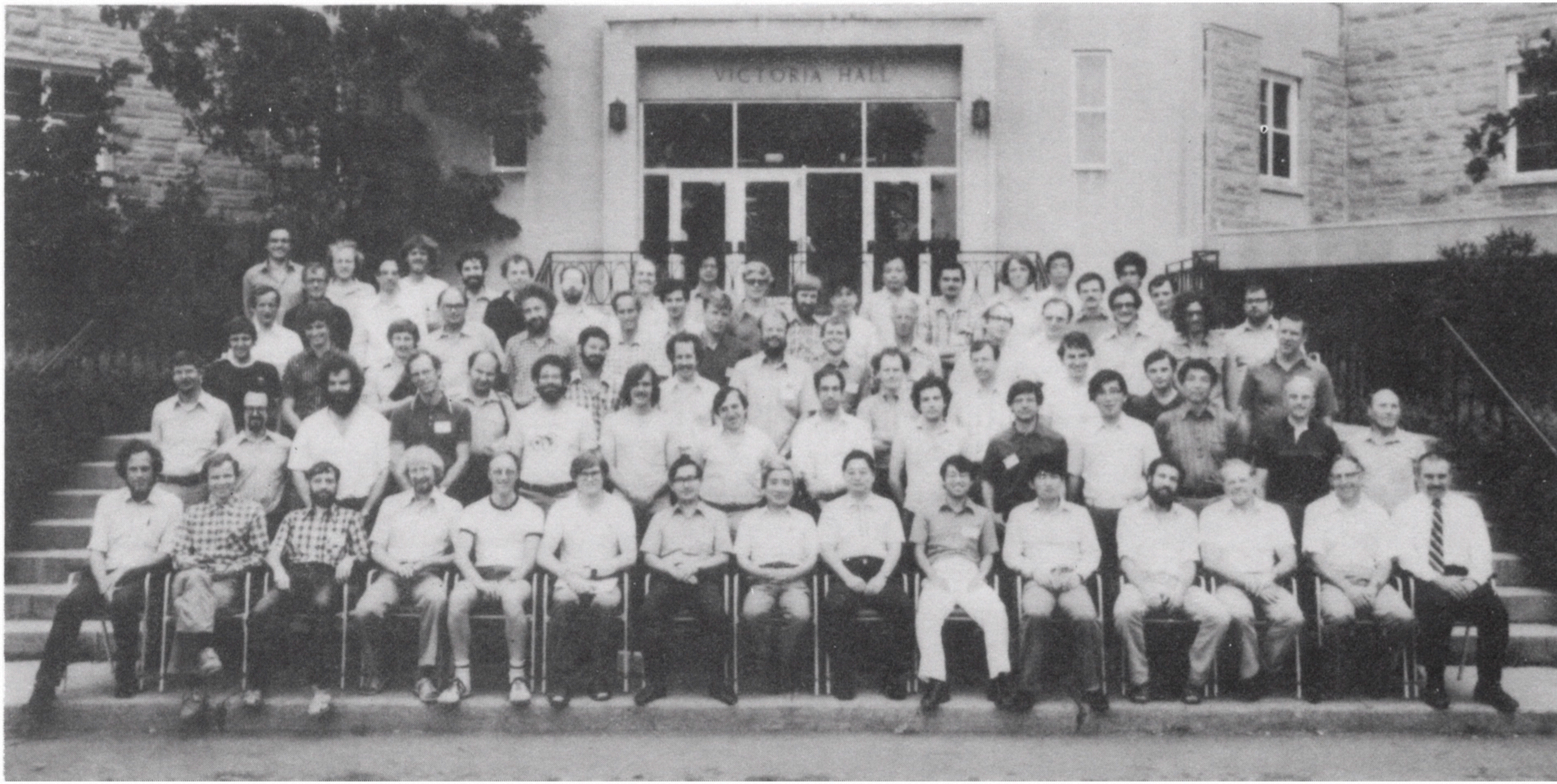


Speakers and Organizers

Introduction

I'm going to be talking about the tempered representation theory of real reductive groups, more or less as in Harish-Chandra's work. This was never far from the attention of Dick and the operator algebras group at Penn.

I want to show how C^*-algebra techniques can help clarify some basic principles in the theory, especially those related to the dichotomy between discrete series and continuous series of representations.

I also want to indicate one feature of tempered representation theory (not the only one!) that is a bit of a puzzle from the C^*-point of view.

Harish-Chandra and Unitary Representation Theory

Recall Plancherel's formula: if f is a test function on the line, and if

$$\widehat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx,$$

then

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(s)\, ds.$$

In the 1950's, Segal proved a version of this for (suitable) locally compact groups:

$$f(e) = \int_{\widehat{G}} \mathrm{Trace}\big(\pi(f)\big)\, d\mu(\pi)$$

Here $\widehat{G}$ is the unitary dual of G, and μ is the Plancherel measure for G.

SOME REMARKS ON REPRESENTATIONS OF CONNECTED GROUPS

BY RICHARD V. KADISON* AND I. M. SINGER

INSTITUTE FOR ADVANCED STUDY AND MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated by John von Neumann, March 19, 1952

1. *Introduction.*—The purpose of this note is to bring to light a fact which has escaped notice, viz., in the direct integral reduction of the regular representation of a connected separable[1] locally compact group, factors of Type II_1 occur almost nowhere[2] (cf. Corollary 3). This proof is carried out by the following scheme of argument. We show first that a connected locally compact group which has sufficiently many unitary representations which generate rings of finite type is the group direct product of a compact group and an abelian group[3] (cf. Theorem 1). From this it follows quite easily that a unitary representation of a connected locally compact group generates a ring of operators which has no summand of Type II_1 (cf. Theorem 1) and, in particular, is not itself a factor of Type II_1. Employing a theorem of Mautner,[4] to the effect that, for almost every factor in the direct integral reduction of the regular representation of a group, there exists a strongly continuous representation of the group which generates the factor, we obtain the final result.

A THEORY OF SPHERICAL FUNCTIONS. I

ROGER GODEMENT

THEOREM 2. *Let G be a semi-simple connected Lie group with a faithful representation and let K be a maximal compact subgroup of G; then every irreducible representation $\mathfrak{d}$ of K is contained at most* $\dim(\mathfrak{d})$ *times in every completely irreducible representation of G.*

Harish-Chandra and Unitary Representation Theory

Harish-Chandra made the Plancherel formula completely explicit for real reductive groups.

For instance, when $G = SL(2,\mathbb{R})$, Harish-Chandra's formula is

$$f(e) = \sum_{n\neq 0} \mathrm{Trace}\big(\pi_n(f)\big)\cdot |n| + \frac{1}{2}\int_0^\infty \mathrm{Trace}\big(\pi_s^{even}(f)\big)\cdot s\tanh(\pi s/2)\,ds + \frac{1}{2}\int_0^\infty \mathrm{Trace}\big(\pi_s^{odd}(f)\big)\cdot s\coth(\pi s/2)\,ds.$$

(Actually this special case was obtained earlier, by Bargmann.)

Primer on Reductive Groups

Definition

A *real reductive group* is a closed, (almost) connected subgroup of some $GL(n, \mathbb{R})$ with the property that

$$g \in G \quad \Leftrightarrow \quad g^{\text{transpose}} \in G.$$

Examples

$SL(n, \mathbb{R})$, $SO(p, q)$, $Sp(2n, \mathbb{R})$, etc.

Notation

- $K = O(n) \cap G =$ max. compact subgroup
- $A =$ max. commuting group of positive-definite elts in G

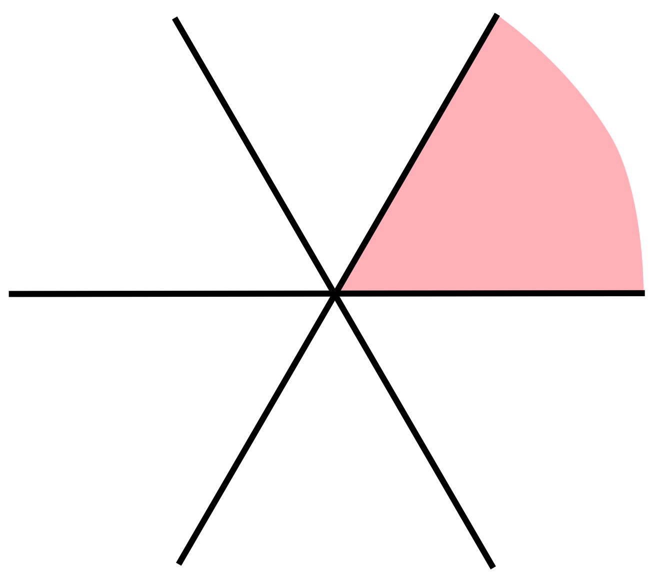
Theorem

$G = KAK$.

Some Examples

If $G = SL(n, \mathbb{R})$, then

$$K = SO(n)$$
$$A = \text{positive diagonal matrices}$$



It is often important to divide up A into smaller parts—chambers and the walls between them. When $G = SL(3, \mathbb{R})$ there are 6 of them. For instance the shaded area is

$$A_+ = \{ \operatorname{diag}(a, b, c) : a > b > c \}$$

One has

$$G = K \cdot \overline{A_+} \cdot K.$$

C*-Algebras and Representations

Definition

The *group C^*-algebra* $C^*(G)$ is the (universal) completion of the convolution algebra of test functions on G into a C^*-algebra.

Theorem

Each unitary representation π of G integrates to a C^*-algebra representation:

$$\pi(f) = \int_G f(g)\pi(g)\, dg$$

Moreover all representations of $C^*(G)$ (as bounded operators on Hilbert spaces) come this way.

Tempered Representations

For the most part Harish-Chandra studied only tempered representations, and so shall we:

Definition

The *reduced group C^*-algebra* $C_r^*(G)$ is the image of $C^*(G)$ in the representation of G by left translation on $L^2(G)$.

So the reduced C^*-algebra is a quotient of $C^*(G)$.

Definition

A unitary representation of G is *tempered* if

As a representation of $C^*(G)$, it factors through $C_r^*(G)$.

$\Leftrightarrow$ Its matrix coefficients (functions on G) decay sufficiently rapidly at infinity.

$\Leftrightarrow$ It decomposes into irreducible unitary representations in the support of the Plancherel measure.

Discrete Series and Continuous Series

Let's consider $G = SL(2,\mathbb{R})$ for a moment. As we saw in the Plancherel formula, there are:

$$\text{discrete series} \quad \pi_n$$

and

$$\text{continuous series} \quad \pi_s^{even/odd}$$

in the tempered dual.

The continuous series are built from homogeneous functions

$$h(av) = |a|^{is-1}\operatorname{sign}(a)^{\varepsilon} h(v)$$

on the plane.

The discrete series are perhaps a bit more complicated ...

Discrete Series from the C*-Algebra Perspective

...but the discrete series are easily *defined* abstractly, using C^*-algebra theory.

Definition (Standard)

A tempered irreducible representation is a *discrete series representation* if it is isolated in the tempered dual.

Definition (Interesting)

A tempered irreducible representation is a *discrete series representation* if it is associated to the ideal in $C_r^*(G)$ consisting of elements that act as compact operators on $L^2(G)^\sigma$ for every $\sigma \in \widehat{K}$.

Here $L^2(G)^\sigma$ is the σ-isotypical part of $L^2(G)$ for the right action of K on $L^2(G)$.

Remedial Lesson in C*-Algebra Theory

Each ideal in a C^*-algebra, $J \triangleleft A$, determines a open subset of the dual of A (the irreducible representations of A up to equivalence):

$$\{\, [\pi] : \pi|_J \neq 0 \,\}$$

These are all the open subsets of the dual.

Theorem

The discrete series ideal (according to the standard definition) consists precisely of elements in $C_r^*(G)$ that act as compact operators on $L^2(G)^\sigma$ for every $\sigma \in \widehat{K}$.

Discrete Series and Hilbert's Integral Operators

For a manifold M, the compact operators on $L^2(M)$ more or less correspond to the integral operators

$$(Th)(x) = \int_M k(x,y)h(y)\,dy$$

with $k(x,y)$ smooth and compactly supported, as studied by Hilbert and Schmidt.

For a group G, the convolution operator on $L^2(G)$ associated to a test function f is

$$(f \star h)(x) = \int_G k(x,y)h(y)\,dy$$

where $k(x,y) = f(xy^{-1})$ which is not compactly supported.

So the existence of discrete series is a bit of a miracle!

Parabolic Induction, or Transfer

On to the continuous series . . . and back to the subgroup $A \subseteq G$ (of positive diagonal matrices, let's say) ...

Fix a one-parameter subgroup in A, $a(t) = \exp(tH)$, and write

$$L = \{\, g \in G : a(t) \cdot g \cdot a(-t) = g \quad \forall t \,\}$$
$$N = \{\, g \in G : a(t) \cdot g \cdot a(-t) \to e, \quad t \to +\infty \,\}$$

Example

Let $G = SL(2, \mathbb{R})$. If $a(t) = \begin{bmatrix} e^t & \\ & e^{-t} \end{bmatrix}$, then

$$L = \left\{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \right\} \quad \text{and} \quad N = \left\{ \begin{bmatrix} 1 & \\ b & 1 \end{bmatrix} \right\}$$

In general, L normalizes N, and so L acts on the right on G/N.

Parabolic Induction, or Transfer

The test functions on the homogeneous space G/N may be completed to a C^*-bimodule (or correspondence) $C^*_r(G/N)$.

The *transfer* or *parabolic induction* functor

$$H \longmapsto C^*_r(G/N) \otimes_{C^*_r(L)} H$$

takes tempered representations of L to tempered representations of G.

Definition

For H irreducible, these are the *continuous series* representations of G.

Why continuous? Since $\{a(t)\}$ is a central subgroup of L, any such representation can be rescaled by a character of $\{a(t)\}$.

Harish-Chandra/Langlands Principle

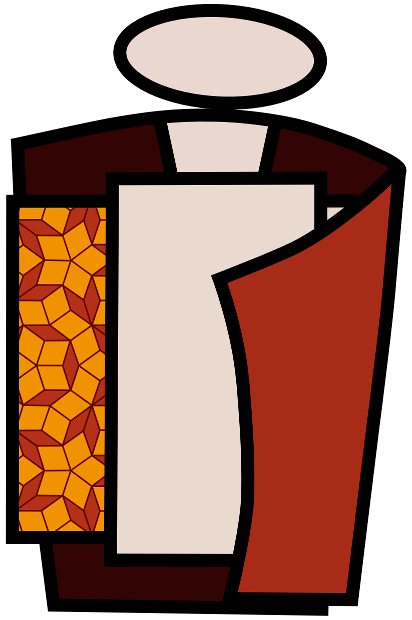
Theorem

Every tempered irreducible representation of G is either a discrete series representation (mod center) or a summand of the transfer of a discrete series representation (mod center).

I want to explain why the theorem is true—the C^*-algebra viewpoint is very helpful here.

Geometry and Noncommutative Geometry

The idea is to translate the geometric or dynamic information encoded in the definition of N ...



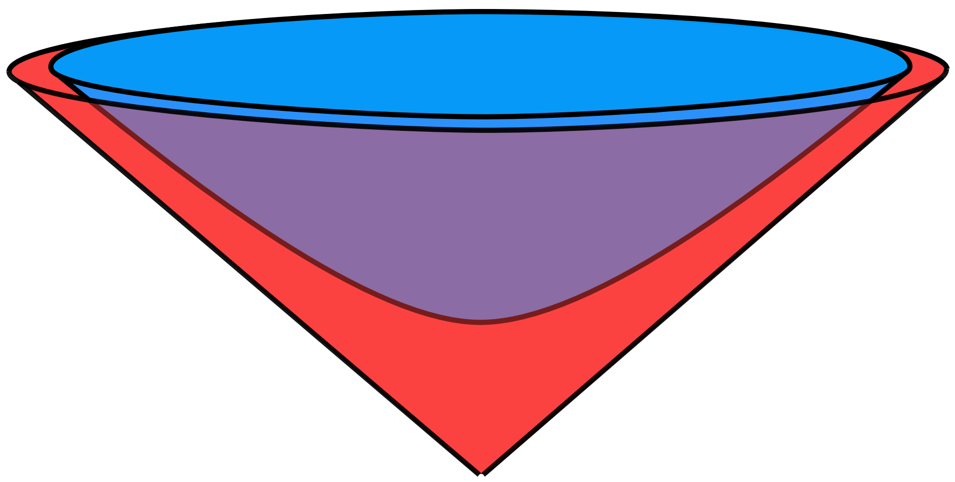
... into a Hilbert-space theoretic statement ... which will be equivalent to the theorem.

The Geometry

Fundamental fact:

$$a(t) \cdot K \cdot a(-t) \longrightarrow (K \cap L) \cdot N \qquad (t \to +\infty)$$

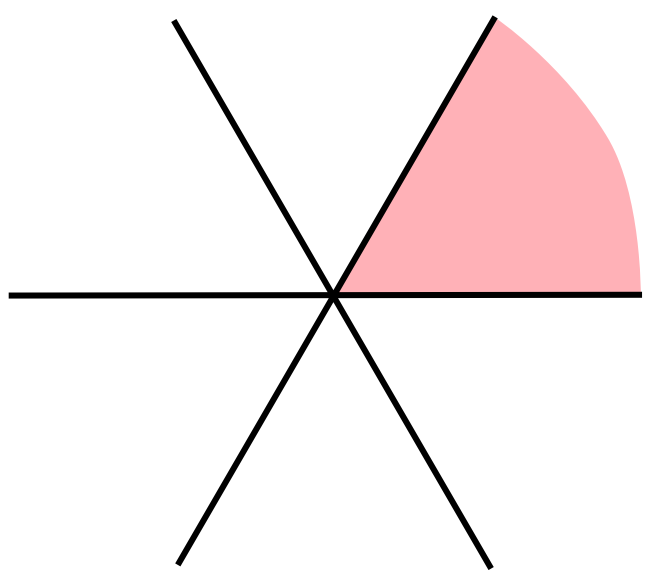
A reinterpretation: Keeping in mind that $\{a(t)\}$ normalizes L and N, $G/(L \cap K)N$ and G/K are asymptotically equivalent to one another as G-spaces.



The picture shows G/K (blue) and $G/(K \cap L)N$ (red) embeded in $\mathfrak{g}^*$ as coadjoint orbits in the example where $G = SL(2, \mathbb{R})$.

Remark

To approximate G/K by some $G/(K \cap L)N$ towards infinity in G/K in different directions, one needs to use different $\{a(t)\}$ and different L, N.



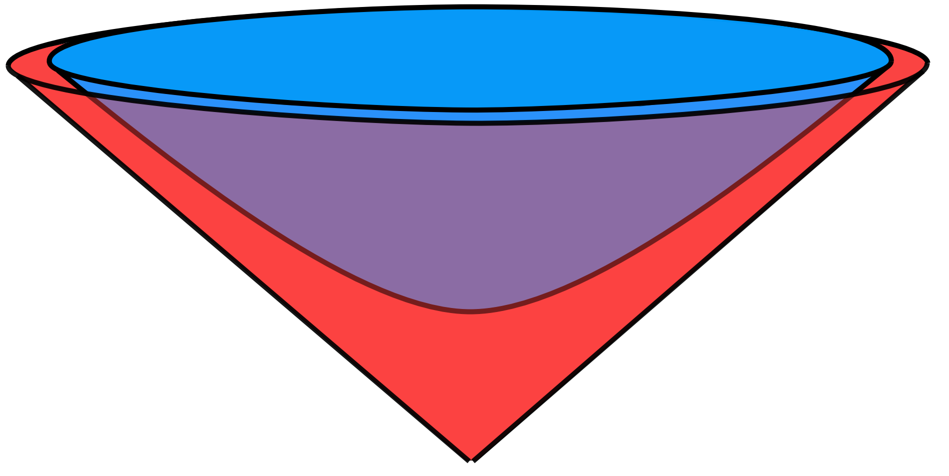
The case where $G = SL(3, \mathbb{R})$:

$G = K \cdot \overline{A_+} \cdot K$.

Hilbert Space Theory

Theorem

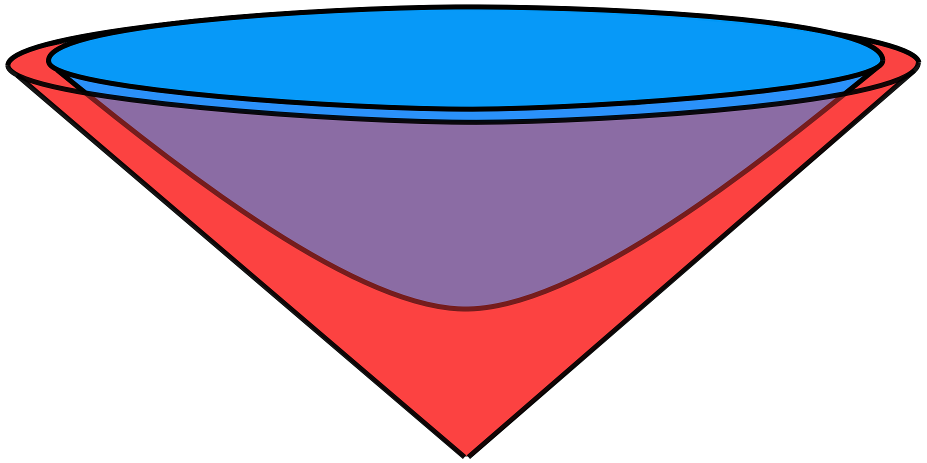
If an element of $C_r^*(G)$ acts trivially on each $L^2(G/N)$, then it acts as a compact operator on $L^2(G)^\sigma$ for every $\sigma \in \widehat{K}$.



Theorem

The mutual kernel of the representations of $C_r^*(G)$ on the spaces $L^2(G/N)$ is precisely the discrete series ideal in $C_r^*(G)$.

An Asymptotic Inclusion of Representations



Conjugation by $\{a(t)\}$ gives rise to a one-parameter group action

$$U_t \colon L^2(G/(K \cap L)N) \longrightarrow L^2(G/(K \cap L)N)$$

and an *asymptotic inclusion* of representations

$$V_t \colon L^2(G/(K \cap L)N) \longrightarrow L^2(G/K)$$

It is possible to extract a formula for the Plancherel meansure from this (work with Qijun Tan, following Weyl's Plancherel formula Sturm-Liouville operators) . . .

An Adjoint C*-Bimodule

Theorem (Work with Pierre Clare and Tyrone Crisp)

- *There is an adjoint Hilbert $C_r^*(L)$-$C_r^*(G)$-bimodule $C_r^*(N\backslash G)$.*
- *There is a natural isomorphism*

$$\mathrm{Hom}_G\big(H, C_r^*(G/N) \otimes_{C_r^*(L)} K\big) \cong \mathrm{Hom}_L\big(C_r^*(N\backslash G) \otimes_{C_r^*(G)} H, K\big).$$

The parabolic restriction functor

$$H \longmapsto C_r^*(N\backslash G) \otimes_{C_r^*(G)} H$$

is a two-sided adjoint to parabolic induction (*c.f.* Frobenius reciprocity and Bernstein's second adjunction).

The Problem of Uniform Admissibility

Theorem

Fix a irreducible representation σ of K. There are at most finitely many discrete series representations of G that include σ.

This is a consequence of Harish-Chandra's classification work on the discrete series. It plays an essential role in desribing the tempered dual as a (noncommutative) topological space.

Is there a geometric/noncommutative geometric proof?

There is an analogous theorem in the context of p-adic groups, due to Bernstein. It uses similar geometry to what was used earlier . . . plus the fact that the Hecke convolution algebras that arise in p-adic groups are Noetherian.

Uniform Admissibility

An equivalent formulation:

Theorem

Fix a irreducible representation σ of K. The representation $L^2(G)^\sigma$ includes at most finitely many discrete series representations.

This is plausible enough: the representation $L^2(G)^\sigma$ “mostly looks like” a combination of the spaces $L^2(G/N)^{\sigma|_{K\cap L}}$ ... and there is only a compact part that does not.

But I don’t know how to devise a proper argument along these lines.

Thank you