Embeddings for 3-dimensional CR-manifolds

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Abstract. We consider the problem of projectively embedding strictly pseudoconcave surfaces, X_- containing a positive divisor, Z and affinely embedding its 3-dimensional, strictly pseudoconvex boundary, $M = -bX_-$. We show that embeddability of M in affine space is equivalent to the embeddability of X_- or of appropriate neighborhoods of Z inside X_- in projective space. Under the cohomological hypotheses: $H^2_{comp}(X_-, \Theta) = 0$ and $H^1(Z, N_Z) = 0$ these embedding properties are shown to be preserved under convergence of the complex structures in the C^∞-topology.

§1. Conditions for embeddability (geometric approach).

Let M denote a smooth compact, strictly pseudoconvex 3-dimensional CR-manifold. Such a structure is induced on a strictly pseudoconvex, real compact hypersurface in a complex space. The CR-manifold, M is called fillable if M can be realized as the boundary of a 2-dimensional (Stein) space. From the results of H.Grauert, 1958, J.J.Kohn, 1963, H.Rossi, 1965, it follows that M is fillable iff M is embeddable by a CR-mapping into affine complex space. It is now well known that the generic CR-structure on M is not fillable or embeddable (H.Rossi,1965, A.Andreotti-Yum Tong Siu,1970, L.Nirenberg, 1974, D.Burns,1979, H.Jacobovitz, F.Treves,1982, D.Burns, C.Epstein,1990.)

On the other hand L.Lempert,1995, has proved that any embeddable strictly pseudoconvex CR-manifold M can be realized as a separating hypersurface in a projective variety X. This means, that if a strongly pseudoconvex, compact 3-dimensional CR-manifold M bounds a strongly pseudoconvex surface, X_+ then $-M$ also bounds a strongly pseudoconcave complex surface, X_- containing a smooth holomorphic curve, Z with positive normal bundle, N_Z.

Suppose that the CR-manifold, $-M$ is the boundary of a two-dimensional strictly pseudoconcave manifold, X_- which contains a smooth curve, Z with positive normal bundle, N_Z. It is quite possible that this assumption is valid for any strictly pseudoconvex compact CR-manifold, M.

Definition. X_- will be called weakly embeddable in $\mathbb{C}P^N$ if there exists a holomorphic map $\varphi : X_- \to \mathbb{C}P^N$ injective in some neighborhood of Z in X_-. A

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weakly embeddable X_- will be called almost embeddable in $\mathbb{C}P^N$ if there exists an almost injective holomorphic map $\varphi : X_- \to \mathbb{C}P^N$, i.e. a map which embeds the complement of a proper analytic subset of X_-.

The next result (Epstein-Henkin,1998) shows that the embeddability of M in affine space depends on the embeddability of X_- in projective space.

Theorem 1. *The following properties are equivalent:*
i)M is embeddable in complex affine space
ii)X_- is embeddable in complex projective space.
iii)X_- is almost embeddable in complex projective space.

Sketch of the proof.

The implication *i)* $\Rightarrow$ *ii)* follows from the classical results. Indeed, if M is embeddable in affine space then by H.Rossi,1965, we have $M = bX_+$, where X_+ is strictly pseudoconvex Stein space. Hence, $X = X_+ \sqcup_M X_-$ is compact complex space, satisfying condition of projectivity of K.Kodaira, 1960 and H.Grauert,1962.

The implication *ii)* $\Rightarrow$ *i)* is proved by the following scheme. If X_- is embeddable in projective space $\mathbb{C}P^N$ (by the holomorphic mapping, $\varphi : X_- \to \mathbb{C}P^N$) then by A.Andreotti, 1963, there exists an irreducible compact projective 2-dimensional variety $W \subset \mathbb{C}P^N$ such that $W_- = \varphi(X_-) \subset W$.

Let p_0 be a non-negative defining function for bX_- which is strictly plurisubharmonic in a neighborhood of bX_-. For sufficiently small $\varepsilon > 0$ the sets

$$X_{\varepsilon,-} = \{x \in X_- \quad : \quad p_0(x) > \varepsilon\}$$

are domains in X_- with smooth strictly pseudoconcave boundaries. We let $W_{\varepsilon,-} = \varphi(X_{\varepsilon,-})$ and $W_{\varepsilon,+} = W \backslash \bar{W}_{\varepsilon,-}$. By H.Grauert,1958, for any small $\varepsilon > 0$ the CR-manifold $bW_{\varepsilon,+}$ is embeddable in complex affine space. Hence the CR-manifold, $bX_{\varepsilon,-}$ is also embeddable in complex affine space for any small $\varepsilon > 0$. The embeddability of $bX_- = \lim_{\varepsilon \to 0} bX_{\varepsilon,-}$ follows from here, by applying the relative index theory for CR-structures from C.Epstein,1998.

The proof of implication *iii)* $\Rightarrow$ *ii)* is based on Andreotti,1963, theorem. We show that image, $\varphi(X_-)$ is an open subset of an irreducible algebraic variety $W \subset \mathbb{C}P^N$ and that the image $\varphi(G)$ of the analytic subset, $G \subset X_-$, where φ fails to be an embedding, is a discrete set in $\varphi(X_-)$. Using Hironaka's 1964 theorem as well as a germ version of the Castelnuovo criterion proved in Epstein, Henkin 1998 we show that there exists a desingularization $\tilde{\pi} \quad : \tilde{W} \to W$ and a meromorphic map $\tilde{\varphi} : X_- \to \tilde{W}$, which is a biholomorphism on the complement of a discrete set and such that $\varphi = \tilde{\pi} \circ \tilde{\varphi}$. From this and from Andreotti-Tomassini,1970, theorem follows the embeddability of X_- in $\mathbb{C}P^N$.

Let $H^0(X_-, [d \cdot Z])$ denotes the space of holomorphic sections of the line bundle $[d \cdot Z]$, defined by the divisor Z. The following result (Epstein-Henkin,1998) shows that weak embeddability of X_- in complex projective space depends on the asymptotic behavior of dim $H^0(X_-, [d \cdot Z])$ as $d \to \infty$.

Theorem 2. *Let X_- be strictly pseudoconcave surface, which contains a smooth curve, Z with positive normal bundle, N_Z. Let*

$$M(d) = (deg\, N_Z)\frac{d(d+1)}{2} + (1-g)(d+1),$$

where g is a genus of Z. Then the following properties are equivalent
i) X_- is weakly embeddable in complex projective space,
ii) $M(d) \leq \dim H^0(X_-, [d \cdot Z]) \leq M(d) + C$,
where C is a constant, depending only on Z and N_Z.

Proof. The proof of implication $i) \Rightarrow ii)$ is based on the L^2- Kodaira vanishing theorem of Pardon-Stern,1991, for a projective variety with singularities and on the Riemann-Roch theorem for line bundles, N_Z over smooth curves, Z. The proof of implication $ii) \Rightarrow i)$ uses a refinement of the Nakai- Moishezon theorem (see Hartshorne, 1977). We show that ii) implies that there is a map, $\varphi : X_- \to \mathbb{C}P^N$, which is an embedding of a neighborhood, U of Z and such that $\varphi(Z) = \varphi(U) \cap H$, where H is a hyperplane in $\mathbb{C}P^N$.

Problem. Is a weakly embeddable strictly pseudoconcave surface $X_- \supset Z$ embeddable?

§2. Conditions for embeddability (analytic approach).

Weighted L_2-estimates for the $\bar{\partial}$-equation on appropriate singular domains give a different approach to the results above and also lead to more general conditions of embeddability of strictly pseudoconvex CR-varieties.

Definition. A compact subset M of the almost complex manifold X is called a strictly pseudoconvex compact CR-variety if M is a compact level set of a C^∞ strictly plurisubharmonic function (with possible critical points). Such a CR-variety, M is called embeddable in $\mathbb{C}^N$ if there exists a real embedding $\Phi : X \to \mathbb{C}^N$ with the property $\bar{\partial}\Phi\big|_M = 0$. A smooth CR-variety is called CR-manifold.

Definition. Two compact CR-varieties M_0 and M_1 are called strictly CR-cobordant if there exists a complex manifold X, embedded as an open subset in some almost complex manifold $\tilde{X}$, and a C^∞ strictly plurisubharmonic function ρ on $\tilde{X}$ such that $bX = M_1 - M_0$, $0 < \rho(x) < 1$, $x \in X$, $\rho\big|_{M_0} = 0$ and $\rho\big|_{M_1} = 1$.

Theorem 3. *Let M_0 be a compact, 3-dimensional, strictly pseudoconvex CR-manifold. Then M_0 is embeddable iff there exists a complex space X with $bX = M_1 - M_0$, where M_1 is strictly pseudoconvex CR-manifold, embeddable by a CR-mapping $\varphi : M_1 \to \mathbb{C}^N$ which admits a holomorphic almost injective extension to X.*

Sketch of the proof.

The proof of necessity is obtained by the following arguments. If M_0 is embeddable in affine space, then by Rossi, 1965, and Harvey, Lawson, 1975, results there exists a Stein space with isolated singularities, X^0 such that $bX^0 = M_0$. Applying the results of Ohsawa, 1984, and Heuneman, 1986, and (or) the technique of §3 below we show that the space X^0 admits a proper embedding into a bigger Stein space, X^1 with a strictly pseudoconvex smooth boundary $bX^1 = M_1$. From

Kohn, 1964, result it follows that the strictly pseudoconvex CR-manifold, M_1 is embeddable in affine space by a mapping holomorphic on X and smooth on $\bar{X}$.

The proof of sufficiency contains several steps. Step 1. The proof of embeddability of X in projective space and the reduction of the sufficiency to a special case.

Let $\varphi : X \to \mathbb{C}^N$ be the holomorphic almost injective mapping which admits a smooth extension to M_1 and realizes an embedding of M_1 into $\mathbb{C}^N$. From the concavity of X near M_0 and from the Cauchy formula on analytic discs embedded in X it follows that φ has also a smooth extension to M_0. One can show further that for set, $G \subset \bar{X}$, where φ fails to be an embedding, the subset, $G \cap X$ is at most a 1-dimensional analytic set in X with boundary $b(G \cap X) \subset M_0$. Besides, $\varphi(G \cap X)$ is a discrete subset in $\varphi(X)$. By results of Rossi, 1965, or Harvey, Lawson, 1975, there exists a Stein space with isolated singularities, W embedded in $\mathbb{C}^N$ and such that $bW = \varphi(M_1)$. We have $W_- = \varphi(X) \subset W$. Let $W_+ = W \backslash \bar{W}_-$. Applying Andreotti, Narasimhan, 1964, result, we obtain the Steinness property of W_+. Applying as in the proof of Theorem 1 the Hironaka, 1964, theorem, we obtain the embeddability of X in some projective space, $\mathbb{C}P^N$ by a mapping ψ. By the Hartogs-Levi extension theorem the mapping $\psi \circ \varphi^{(-1)} : \varphi(M_1) \to \mathbb{C}P^N$ has meromorphic extension, g on W.

Let p_0 be non-negative defining function for $M_0 \subset \bar{X}$ which is smooth on X and strictly plurisubharmonic in a neighborhood of M_0. Because $g(W) \supset \psi(X)$ and from the Steinness property of $g(W)$ it follows that for almost all small $\varepsilon > 0$ the CR-manifold $M_\varepsilon = \{p_0^{(-1)}(\varepsilon)\}$ is fillable and hence embeddable. Hence, to prove Theorem 3 it suffices to prove it in the special case when there exists a strictly plurisubharmonic function, p defined on X such that $M_0 = p^{(-1)}(0)$ and $M_1 = p^{(-1)}(1)$. In such a case the analytic set $G \cap X$ must be discrete and $\varphi(G) \subset$ *Sing* W.

Step 2. Weighted L_2-estimates for $\bar{\partial}$-equation in singular domains $W_\pm$.

Let $\rho(z) = \sum_{z^* \in Sing\, W} \ln|z - z^*|$ and $r = e^\rho$. Let $L^2(W, e^{-\rho})$ be the space of functions φ on W with the norm $\int_W |\varphi|^2 e^{-2\rho} dv$, where dv is the induced volume form on W. Let $\Lambda^1_{0,q}(\bar{W}_\pm)$ be the spaces of $(0,q)$-forms on $\bar{W}_\pm$ with coefficients in the space of Lipschitz functions. For real numbers $\nu_\pm$ we define the spaces $\Lambda^{1,\nu_\pm}_{0,q}(\bar{W}_\pm) = r^{-\nu_\pm}\Lambda^1_{0,q}(\bar{W}_\pm)$. Let $C^{\perp s}_{0,1}(\bar{X})$ be the space of s-times differentiable(0,1)-forms on $\bar{X}$, which are $\bar{\partial}$-closed on $\bar{X}$ and $\bar{\partial}_b$-exact on M_0.

Using a result in Epstein, Henkin, 1998 (2) it follows that for the given mapping $\varphi : X \to W_-$ there exists a $\nu_- \geq 0$ such that the operator $\varphi_* : C^{\perp 1}_{0,1}(\bar{X}) \to \Lambda^{1,\nu_-}_{0,1}(\bar{W}_-)$ is continuous. Using the Lipschitz extension theorem from Danzer, Grünbaum, Klee, 1963, it follows that for the given $W_\pm$ and $\nu_- \geq 0$ there exists a $\nu_+ \geq 0$ and a continuous linear extension operator,

$$\mathcal{E}_+ : \Lambda^{1,\nu_-}_{0,1}(\bar{W}_-) \to \Lambda^{1,\nu_+}_{0,1}(\bar{W}_+).$$

There exists also a $\mu_+ \geq 0$ such that the operator $\bar{\partial} : \Lambda^{1,\nu_+}_{0,1}(\bar{W}_+) \to L^2_{0,2}(W_+, e^{\mu_+\rho})$ is continuous.

One can check further that for any $f \in C^{\perp 1}_{0,1}(\bar{X})$ the form, $b_+ = \bar{\partial}\mathcal{E}_+\varphi_* f$ belongs to $L^{\perp 2}_{0,2}(W_+, e^{\mu_+\rho})$, i.e $b_+ \in L^2_{0,2}(W_+, e^{\mu_+\rho})$ and satisfies the orthogonality property $\int_{W_+} b_+ \wedge h = 0$ for any $h \in L^2_{n,n-2}(W_+, e^{-\mu_+\rho}) : \bar{\partial}h = 0$. From results of Andreotti, Vesentini, 1965, and Demailly, 1982, follows the existence of a continuous linear operator, $T_+ \ : \ L^{\perp 2}_{0,2}(W_+, e^{\mu+\rho}) \to L^2_{0,1}(W_+, e^{\mu+\rho})$ such that $T_+ b_+\big|_{b \, Reg \, W_+} = 0$ in the L_2-distribution sense and $\bar\partial T_+ b_+ = b_+$ on $Reg \, W_-, \ \ \forall \ b_+ \in L^{\perp 2}_{0,2}(W_+, e^{\mu+\rho})$.

Step 3. Embeddability of M_0 from estimates for $\bar\partial_b$.

Let $\tilde C^{\nu}_{0,1}(M_0)$ be the subspace of $C^{\nu}_{0,1}(M_0)$ consisting of $\bar\partial_b$-exact forms. Following Epstein, Henkin, 1997, we can construct a continuous extension operator,

$$E_- \ : \ \tilde C^{3/2}_{0,1}(M_0) \to C^{\perp 1}_{0,1}(\bar X).$$

For any $f \in \tilde C^{3/2}_{0,1}(M_0)$ we have $\tilde f = E_- f \in C^{\perp 1}_{0,1}(\bar X)$ and $g_- = \varphi_* \tilde f \in \Lambda^{1,\nu_-}_{0,1}(\bar W_-)$. For $g_- \in \Lambda^{1,\nu_-}_{0,1}(\bar W_-)$ we have an extension operator,

$$g_- \mapsto E_+ g_- = \mathcal{E}_+ g_- - T_+(\bar\partial \mathcal{E}_+ g_-)$$

with the properties $E_+ g_- \in L^2_{0,1}(W_+, e^{\mu+\rho})$, $E_+ g_-\big|_{bW_-} = g_-\big|_{bW_-}$ and $\bar\partial E_+ g_- = 0$ on W_+. For

$$g = \begin{cases} E_+ g_- \ \text{for} \ z \in W_+ \\ g_- \ \text{for} \ z \in W_- \end{cases}$$

we have $g \in L^2_{0,1}(W, e^{\mu\rho})$, where $\mu = \max \, \mu_\pm$, and $\bar\partial g = 0$ on $Reg \, W$. Applying Kohn, Rossi, 1965, and Henkin, 1977, results we conclude that for any g from a finite-codimensional subspace, $B^2_{0,1}(W, e^{\mu\rho})$ of the space

$$\{g \in L^2_{0,1}(W, e^{\mu\rho}) \ : \ \bar\partial g = 0\}$$

we have $g = \bar\partial T g$, where

$$T \ : \ L^2_{0,1}(W, e^{\mu\rho}) \to L^2(W, e^{\mu\rho})$$

is a continuous linear operator. Hence, for $\tilde f$ from the finite-codimensional subspace

$$\varphi^* B^2_{0,1}(W, e^{\mu\rho}) \cap C^{\perp 1}_{0,1}(\bar X) \subset C^{\perp 1}_{0,1}(\bar X)$$

we have

$$\tilde f = \varphi^* g = \bar\partial T(g(\varphi(x))) = \bar\partial R \tilde f,$$

where the function $R\tilde f = Tg(\varphi(x))$ on X has at most L^2- polynomial growth near the inverse image of $Sing \, W$.

From the concavity of the variety, X near $M_0 \cup (G \cap X)$, the continuity of $\tilde f$ on $\bar X$ and Cauchy type estimates for $\bar\partial$ on analytic discs, embedded in $X \backslash G$, we obtain the regularity of $R\tilde f$ on M_0 of class at least $C^{1/2}(M_0)$. So, we have constructed a continuous linear operator $R \ : \ \tilde C^{3/2}_{0,1}(M_0) \to C^{1/2}(M_0)$ such that $\bar\partial_b R f = f$ for a finite-codimensional subspace of $\tilde C^{3/2}_{0,1}(M_0)$. From here follows the embeddability of M_0 in affine space, applying some modifications of results from Boutet de Monvel, 1975, Henkin, 1977.

Remark 1. In a future paper we will obtain a version of theorem 3, (useful in applications), for non-smooth strictly pseudoconvex CR varieties. For this some generalizations of the regularity results for $\bar{\partial}_b$ on strictly pseudoconvex CR-variety from Henkin, Leiterer, 1984, will be applied.

Theorem 4. *Let M_0 be a compact, 3-dimensional, strictly pseudoconvex CR-manifold. Then M_0 is embeddable in affine space iff there exists an embeddable compact strictly pseudoconvex CR-manifold, M_1 which is strictly CR-cobordant to M_0, and an exhaustion strictly plurisubharmonic function on the complex cobordism which has no critical points.*

Sketch of the proof.

The necessity follows from the same arguments as the necessity in Theorem 3. To prove sufficiency let us suppose that M_0 is strictly CR-cobordant to an embeddable CR-manifold M_1 : there exist a complex space X and a strictly plurisubharmonic function on X such that $bX = M_1 - M_0$. This function ρ has no critical points and $0 < \rho(x) < 1$, $\rho\big|_{M_0} = 0$ and $\rho\big|_{M_1} = 1$. Besides there exists a CR-embedding $\varphi_1 \ : \ M_1 \to \mathbb{C}^N$. Let $X_\theta = \{x \in X \ : \ \rho(x) > \theta\}$ and $M_\theta = \{x \in X \ : \ \rho(x) = \theta\}$.

By the H.Lewy extension theorem the mapping φ_1 admits an holomorphic extension as a holomorphic embedding $\psi_\theta \ : \ X_\theta \to \mathbb{C}^N$ for some $\theta < 1$. Let θ_1 be the infimum of numbers θ such that there exists an embedding $\psi_\theta \ : \ X_\theta \to \mathbb{C}^N$. Applying H.Rossi, 1965, result we obtain the existence of an embedding $\psi_{\theta_1} \ : \ X_{\theta_1} \to \mathbb{C}^N$. From Hartogs extension theorem (see, Henkin,Leiterer, 1988) it follows that holomorphic mapping ψ_{θ_1} admits holomorphic extension on X. From Theorem 3 follows the existence of an embedding $\varphi_{\theta_1} \ : \ M_{\theta_1} \to \mathbb{C}^N$. To finish the proof it is sufficient to show that $\theta_1 = 0$. Suppose that $\theta_1 > 0$. From H.Lewy's extension theorem it follows that the mapping φ_{θ_1} admits an holomorphic extension as a holomorphic embedding $\tilde{\psi}_{\theta_2} \ : \ (X_{\theta_2} \backslash X_{\theta_1}) \to C^N$ for some $\theta_2 < \theta_1$. From Hartogs extension theorem and Oka-Weil approximation theorem it follows that the holomorphic embedding $\tilde{\psi}_{\theta_2}$ can be chosen to be holomorphic on X. Hence holomorphic functions separate all points of X_{θ_2} and we can apply H.Rossi,1965, result to obtain the existence of an embedding $\psi_{\theta_2} \ : \ X_{\theta_2} \to \mathbb{C}^N$ with $\theta_2 < \theta_1$. This contradicts the minimality of θ_1.

Remark 2. Basing on Remark 1 and an appropriate generalization of H.Lewy's extension theorem, we plan in a future paper to remove the hypothesis that the strictly plurisubharmonic exhaustion function has no critical points.

The following statement answers to question of Falbel, 1992.

Corollary . *Let X be an analytic space of pure dimension 2 with boundaries M_1 and M_0, where M_1 is strictly pseudoconvex and M_0 is strictly pseudoconcave. Assume that holomorphic functions separate points in X. Then the pseudoconcave holes of X can be filled, i.e. X is biholomorphic to an open subset of a complex space $\tilde{X}$ such that $\tilde{X} \backslash X$ is compact in $\tilde{X}$.*

Proof. Let ρ be a smooth function on $\bar{X}$ such that $0 < \rho < 1$ on X, $\rho = 0$ and $d\rho \neq 0$ on M_0 and ρ is strictly plurisubharmonic in the neighborhood of M_0. Let $X_\theta = \{x \in X \ : \ \rho(x) > \theta\}$, $M_\theta = \{x \in X \ : \ \rho(x) = \theta\}$. Let the constant $0 < \theta_1 < 1$ be such that $X \backslash \bar{X}_{\theta_1}$ is manifold, $d\rho \neq 0$ on $X \backslash X_{\theta_1}$ and ρ is strictly plurisubharmonic on the $X \backslash X_{\theta_1}$. From the hypotheses it follows that for any $\theta_0 \ : \ 0 < \theta_0 < \theta_1$ we can find an injective holomorphic mapping

$$\varphi \ : \ \bar{X}_{\theta_0} \backslash X_{\theta_1} \to \mathbb{C}^N.$$

Let us prove that for any $\theta \in (\theta_0, \theta_1)$, the CR-manifold M_θ is embeddable.

Let $W_- = \varphi(X_{\theta_0} \backslash \bar{X}_{\theta_1})$. From the Harvey-Lawson, 1975, result it follows that $\varphi(M_{\theta_0})$ is the boundary (in the sense of currents) of a complex space W_+ with finite volume in $\mathbb{C}^N$. Denote by W the set $\bar{W}_+ \cup W_-$. This set W defines a locally closed rectifiable (2,2)-current of integration $[W]$:

$< [W], \chi >= \int_{W_+} \chi + \int_{W_-} \chi$ for any $\chi \in C^\infty_{2,2}(\mathbb{C}^N)$.

From Harvey-Shiffman, 1974, theorem it follows that W is a complex space with boundary $bW = \varphi(M_{\theta_1})$. Let us consider the normalizations $\hat{W}_\pm$ of $W_\pm$. By definition $\hat{W}_\pm$ are the (unique) normal complex spaces, admitting holomorphic, finite and bimeromorphic mappings $\varphi_\pm : \hat{W}_\pm \to W_\pm$.

Because $\varphi : X_{\theta_0} \backslash \bar{X}_{\theta_1} \to W_-$ is such mapping we have the equality $\hat{W}_- = X_{\theta_0} \backslash \bar{X}_{\theta_1}$. Besides we have $\hat{W} = \bar{\hat{W}}_+ \cup \hat{W}_-$. Hence, the strictly pseudoconvex CR-manifold M_θ, $\theta \in (\theta_0, \theta_1)$,bounds a strictly pseudoconvex complex space $\bar{\hat{W}}_+ \cup (X_{\theta_0} \backslash \bar{X}_\theta)$ with at most isolated singularities. So, the manifold M_θ is embeddable. By Theorem 4 the manifold M_0 is also embeddable. By Harvey, Lawson, 1975 result M_0 is fillable. Hence, the pseudoconcave holes of X can be filled also.

Problem. The Theorem 4 suggests the question: in order for compact, strictly pseudoconvex CR-manifold M_0 to be embeddable does it suffice that there exists a complex manifold X with $bX = M_1 - M_0$ where M_1 is an embeddable, strictly pseudoconvex CR-manifold? The answer is known to be positive under some additional assumptions, for example if X is embeddable in projective space.

§3. **Extension of CR-structures.**

The CR-structure on M can be described as a subbundle, $T^{0,1}M$ of the complexified tangent bundle with fiber dimension 1. For each $p \in M$ we require that $T^{0,1}_p M \cap T^{1,0}_p M = \{0\}$, where $T^{1,0}_p M = \overline{T^{0,1}_p} M$. There is a real two-plane field, $H \subset TM$ such that $H \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}M$. The plane-field H is a contact field iff the CR-structure defining it is strictly pseudoconvex. All of the CR-structures with a given underlying plane field are, up to orientation, deformations of one another.

Suppose that $-M$ is the boundary of a strictly pseudoconcave surface, X_- containing a smooth curve, Z with positive normal bundle, N_Z. Let σ_0 be a holomorphic section of the line bundle, $[Z]$. A section $\omega \in C^\infty(M; \mathrm{Hom}\,(T^{0,1}M, T^{1,0}M))$ with $\|\omega\|_{L^\infty} < 1$ defines a deformation of the CR-structure with the same underlying contact field. For each $p \in M$ we set

$$^{\omega}T^{0,1}_p M = \{\bar{Z} + \omega(\bar{Z}) \quad : \quad \bar{Z} \in T^{0,1}_p M\}.$$

Note that ω does not have to satisfy an integrability condition. If

$$\Omega \in C^\infty(\bar{X}_-; \mathrm{Hom}\,(T^{0,1}X_-, T^{1,0}X_-))$$

satisfies $\|\Omega\|_{L^\infty} < 1$ and the integrability condition

$$\bar{\partial}\Omega = \frac{1}{2}[\Omega, \Omega]$$

then

$$^{\Omega}T^{0,1}X = \{\bar{Z} + \Omega(\bar{Z}) \quad : \quad \bar{Z} \in T^{0,1}X\}, \quad x \in X$$

defines an integrable almost complex structure on X_-. If $\Omega_b = \Omega\big|_{T^{0,1}M} = \omega$ then we say that Ω is an extension of ω to X_-. If ω is a deformation of the CR-structure on M which has an extension, Ω to X_- which satisfies $\bar\partial\Omega = \frac{1}{2}[\Omega,\Omega]$ and $\sigma_0^{-j}\Omega$ has a smooth extension across Z then we say that ω has an integrable extension to X_-, vanishing to order j along Z.

The following result shows that deformations of the CR-structure on M can often be extended to deformations of the complex structure on the pseudoconcave manifold, X_-.

Theorem 5. *For any $j \geq 0$ the set of small deformations ω of the CR-structure on bX_-, which extend to complex structures on X_- with deformation tensors, Ω, vanishing to order j along Z contains an analytic submanifold of finite codimension $= \dim H^2_{comp}(X_-, \Theta \otimes [-j \cdot Z])$ in the set of all small deformations of CR-structure on M and does not contain an analytic submanifold of smaller codimension.*

Here Θ is the complex tangent bundle and $H^2_{comp}(X_-, \Theta \otimes [-j \cdot Z])$ denotes the 2-dimensional cohomology with compact support in X_- and coefficients in the bundle $\Theta \otimes [-j \cdot Z])$.

Kiremidjian,1979, proved a version of Theorem 5 under the assumptions $j = 0$ and $H^2_{comp}(X_-, \Theta) = 0$. He used appropriate versions of Kohn-Nirenberg,1965, a priori estimates for $\bar\partial$-Neumann problem and the Nash-Moser implicit function theorem. Our simpler and more precise arguments are based on the sharp estimates for the $\bar\partial$-Neumann problem on X_- from Beals, Greiner, Stanton,1987, and on the standard implicit function theorem in a Banach space. We have shown in addition that the extended structure, Ω depends analytically on the data, ω. Theorem 5 is a small modification of a statement proved in Epstein,Henkin,1997. As in our previous paper the deformations we obtain have, *a priori* only finitely many derivatives.

Namely, we use the following homotopy formulas: for each integer j there exist linear operators

$$R_j \; : \; C^\infty_{0,2}(\bar X_-, \Theta \otimes [-jZ]) \to C^\infty_{0,1}(\bar X_-, \Theta \otimes [-jZ])$$

$$H_j \; : \; C^\infty_{0,2}(\bar X_-, \Theta \otimes [-jZ]) \to C^\infty_{0,2}(\bar X_-, \Theta \otimes [-jZ]),$$

which satisfy the conditions

$$R_j\Phi\big|_{bX_-} = 0 \;\; \forall \;\; \Phi \in C^\infty_{0,2}(\bar X_-, \Theta \otimes [-jZ]),$$

$$\text{range} \;\; H_j \simeq H^{0,2}_{comp}(\bar X_-, \Theta \otimes [-jZ]),$$

$$\begin{aligned} &\Phi = \bar\partial R_j\Phi + H_j\Phi \;\; \text{and} \\ &\|R_j\Phi\|_{L^k_b(X_-)} \leq C\,\|\Phi\|_{L^{k-1}(X_-)}, \quad k = 1, 2, \ldots, \\ &\forall \;\; \Phi \in C^\infty_{0,2}(\bar X_-, \Theta \otimes [-jZ]), \end{aligned}$$

where C denotes a constant depending only on X_-, j, k.

H_j is of finite rank and has a Schwartz kernel in $C^\infty(\bar X_- \times \bar X_-)$. $L^k(X_-)$ is the standard L^2-Sobolev space; $L^k_b(X_-)$ is an anisotropic Sobolev space, such that $L^k(X_-) \subset L^k_b(X_-) \subset L^{k-1/2}(X_-)$ and

$$\|[U,V]\| \leq C\,\|U\|\,\|V\|$$

for any U, V belonging to $L_b^k(X_-, \Theta \otimes \Lambda^{0,1} \otimes [-jZ])$.

We also use an extension operator,

$$\mathcal{E} \ : \ C^\infty(M, \mathrm{Hom}\,(T^{0,1}M, T^{1,0}M) \otimes [-jZ]) \to C^\infty_{0,1}(\bar{X}_-, \Theta \otimes [-jZ])$$

with the properties:

$$\begin{aligned} &\mathcal{E}(\sigma_0^{-j}\omega)\big|_{bX_-} = \sigma_0^{-j}\omega, \\ &\|\mathcal{E}(\sigma_0^{-j}\omega)\|_{L_b^k(X_-)} \leq C\,\|(\sigma_0^{-j}\omega)\|_{L^k(X_-)} \\ &\|\bar\partial\mathcal{E}(\sigma_0^{-j}\omega)\|_{L^{k-1}(X_-)} \leq C\,\|\mathcal{E}(\sigma_0^{-j}\omega)\|_{L_b^k(X_-)}. \end{aligned}$$

Finally for any small deformation, ω of the CR-structure on M we replace the sufficient conditions for the existence of an extension, Ω by the following integral equations for the form, ψ :

$$\begin{aligned} &\Omega = \sigma_0^j[\mathcal{E}(\sigma_0^{-j}\omega) - \psi], \\ &\psi + R_j([\mathcal{E}(\sigma_0^{-j}\omega) - \psi, \mathcal{E}(\sigma_0^{-j}\omega) - \psi]) = R_j(\bar\partial\mathcal{E}(\sigma_0^{-j}\omega)), \\ &H_j([\mathcal{E}(\sigma_0^{-j}\omega) - \psi, \mathcal{E}(\sigma_0^{-j}\omega) - \psi]) = H_j(\bar\partial\mathcal{E}(\sigma_0^{-j}\omega)). \end{aligned}$$

The mapping in the second equation is bounded and invertible in an appropriate Banach space. The third equation is a finite rank analytic equation.

Problem. Theorem 5 suggests the following natural conjecture. The set of small deformations, ω of the CR-structure on bX_-, which extend to complex structures on X_- is precisely analytic submanifold of codimension $=$ dim $H^2_{comp}(X_-, \Theta)$ in the set of all small deformations of CR-structure on M.

§4. **Closedness of embeddable deformations of CR-structure.**

For the construction of a moduli space for embeddable CR-manifolds it would be useful to have answers to the following questions:

1. When is the set of small embeddable deformations, ω of $T^{0,1}M$ stable, i.e. when does the entire algebra of CR-functions deforms continuously with ω (Burns, Epstein, 1990)?

2. Is the set of small embeddable deformations of $(M, T^{0,1}M)$ closed in the C^∞-topology (Burns,1979) and even more precisely an infinite codimensional, closed, locally connected, analytic subset in the space of all deformations (Lempert,1994)?

For strictly pseudoconvex compact hypersurfaces M in $\mathbb{C}^2$ the results of Burns, Epstein, 1990, Lempert,1992,94, Epstein,1992,98, Bland,1994, Bland, Duchamp, 1997, Epstein, Henkin,1997 provide affirmative answers to these questions.

Using the Theorems 1,2,5 we have proved (Epstein,Henkin,1998) that the set of small embeddable deformations is closed in the C^∞-topology for many new classes of CR-manifolds. Suppose that M is a compact 3-dimensional, strictly pseudoconvex, embeddable CR-manifold, such that $M = bX_+$ and $-M$ is also the boundary of pseudoconcave manifold X_-, containing a smooth, compact holomorphic curve Z with positive normal bundle N_Z. Let $X = X_+ \sqcup_M X_-$.

Theorem 6. *Suppose that either $H^2_{comp}(X_-, \Theta\otimes[-Z]) = 0$ or $H^2_{comp}(X_-, \Theta) = 0$ and $H^1(Z, N_Z) = 0$. Then the set of sufficiently small embeddable perturbations of the CR-structure on M is closed in the C^∞-topology. If, in addition, $Z \simeq \mathbb{C}P^1$*

then every sufficiently small embeddable deformation of the CR-structure on M is stable.

We have also obtained several results on the stability of embeddable deformations of the CR-structure on M which extend to deformations of X_- with various orders of vanishing on Z. One of these results is the following.

Theorem 7. *Assume that for any $d > 0$ the restriction mapping,*

$$Sym^d H^0(X, [Z]) \to H^0(Z, N_Z^d)$$

is surjective. Let ω define an embeddable deformation of the CR-structure on M which has an extension to an integrable almost complex structure, Ω on X_-, vanishing to order 3 along Z. We denote X_- with this complex structure by X'_- and the line bundle on X'_- defined by the divisor Z by $[Z']$. Then ω is stable in the sense that for any $d > 0$ we have:

$$\dim\, H^0(X'_-, [d \cdot Z']) = \dim\, H^0(X_-, [d \cdot Z]).$$

Example 1. (Neighborhoods of curves in $\mathbb{C}P^2$.) Let $Z \subset \mathbb{C}P^2$ be a smoothly embedded curve of degree d. Let X_- be a neighborhood of Z with strictly pseudoconcave boundary. Calculations show that $H^1(Z, N_Z) = 0$ and $H^2_{comp}(X_-, \Theta) = 0$. From Theorem 6 we obtain that, for any $d \geq 1$ the set of sufficiently small, embeddable perturbations of the CR-structure on bX_- is closed in the C^∞-topology and at least for $d = 1, 2$ is stable. The stability part of this statement is L. Lempert's,1994, result.
Problem. Is the embedding of bX_- in $\mathbb{C}P^2$ stable for any $d > 2$?

Example 2. (Quadric hypersurfaces). Let $X \subset \mathbb{C}P^3$ be a quadric hypersurface, not necessarily smooth. Let $Z = X \cap \mathbb{C}P^2$ be a smooth hyperplane section and X_- a smoothly bounded strictly pseudoconcave neighborhood of Z. From Theorem 6 we obtain Hua-Lun Li,1995, result that the set of sufficiently small, embeddable perturbations of the CR-structure on bX_- is stable.

Example 3. (Cubic hypersurfaces). Let $X \subset \mathbb{C}P^3$ be a cubic surface not necessarily smooth. Let $Z = X \cap \mathbb{C}P^2$ be a smooth hyperplane section and X_- a smoothly bounded strictly pseudoconcave neighborhood of Z. A computation shows that both $H^2_{comp}(X_-, \Theta \otimes [-j \cdot Z]) = 0$, $j = 0, 1, 2$ and $H^1(Z, N_Z) = 0$. Applying Theorem 6 we conclude that the set of sufficiently small, embeddable perturbations of the CR-structure on bX_- is closed in the C^∞-topology. It is very plausible that this set of deformations is also stable.

Example 4. (quartic hypersurfaces). Let $X \subset \mathbb{C}P^3$ be a quartic surface, not necessarily smooth. Let $Z_d = X \cap Y_d$ be a smooth intersection in $\mathbb{C}P^3$ of the X with a hypersurface Y_d of degree d. Let X_- be a smoothly bounded, strictly pseudoconcave neighborhood of Z_d. Computations show in this case that, for all $d \geq 1$ dim $H^1(Z_d, N_{Z_d}) = 1$ On the other hand dim $H^2_{comp}(X_-, \Theta \otimes [-Z_d])$ is equal to the codimension of the set of deformations of Z_d extendible to deformations of X in the space of all deformations of the complex structure on Z_d. If $d = 1$ then $H^0(X_-, \Theta \otimes [-Z_1]) = 0$ and we can apply Theorem 5 to conclude that the set of small embeddable perturbations of the CR-structure on bX_- is closed in the C^∞-topology.

If $d \geq 2$ then we can not apply Theorem 6 directly. Nevertheless with help of the information above and Theorem 5 we obtain that for any $d \geq 1$ the set of small embeddable perturbations of the CR-structure on bX_- is closed in the C^∞-topology. By modifying the construction of Catlin-Lempert,1992, one can obtain an example of a singular quartic hypersurface, X such that the embedding of bX_- into $\mathbb{C}P^3$ is not stable. In this example we have a case where the algebra of CR-functions is not stable under embeddable deformations, but the set of such deformations is closed in the C^∞-topology.

Problem. Let $X \subset \mathbb{C}P^3$ be a quintic surface and X_- be a smooth strictly pseudoconcave neighborhood of a hyperplane section Z in X. Is the set of all sufficiently small, embeddable deformations of the CR-structure on bX_- closed in the C^∞-topology?

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