GEOMETRIC BOUNDS ON THE RELATIVE INDEX

CHARLES L. EPSTEIN

Abstract. Let X be a compact surface such that $Y \hookrightarrow X$ as a separating, strictly pseudoconvex, real hypersurface;

$$X \setminus Y = X_+ \sqcup X_-,$$

where X_+ (X_-) is the strictly pseudoconvex (pseudoconcave) component of the complement. Suppose further that X_- contains a positively embedded, compact curve Z. Under cohomological hypotheses on (X_-, Z) we show that if $\bar{\partial}_b'$ is a sufficiently small, embeddable deformation of the CR-structure on Y, then

$$\text{R-Ind}(\bar{\partial}_b, \bar{\partial}_b') \geq -[\dim H^{0,2}(X_-) + \dim H^0(Z, \mathcal{O}_Z)].$$

This implies that the set of small, embeddable deformations of the CR-structure on Y is closed, in the $\mathcal{C}^\infty$-topology on the set of all deformations.

Contents

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1. Introduction

Let Y be a manifold of dimension $2n-1$. A CR-structure on Y is defined as a subbundle $T^{0,1}Y \subset TY \otimes \mathbb{C}$ which satisfies the following conditions

[dimension] fiber-$\dim_{\mathbb{C}} T^{0,1}Y = n-1$.

[non-degeneracy] $T^{0,1}Y \cap \overline{T^{0,1}Y} =$ zero section of TY.

[integrability] If $\bar{W}, \bar{Z} \in \mathcal{C}^\infty(Y; T^{0,1}Y)$ then their Lie bracket, $[\bar{W}, \bar{Z}]$ is as well.

If we let $T^{1,0}Y = \overline{T^{0,1}Y}$ then there is a real hyperplane bundle $H \subset TY$ such that

$$T^{0,1}Y \oplus T^{1,0}Y = H \otimes \mathbb{C}. \tag{1}$$

If $T^{0,1}Y$ is a CR-structure on Y for which (1) holds then we say that $T^{0,1}Y$ is a CR-structure *supported* by H.

The CR-structure defines a differential operator on functions by the rule

$$\bar{\partial}_b f = df\big|_{T^{0,1}Y}.$$

A function satisfying

$$\bar{\partial}_b f = 0$$

is called a CR-function. For θ a non-vanishing one form such that $H = \ker\theta$ we define the "Levi form" to be the Hermitian pairing defined on $T^{1,0}Y$ by

$$(Z, W) \longrightarrow id\theta(Z, \bar{W}).$$

If θ' is another 1-form defining H then there is non-vanishing function f so that $\theta' = f\theta$ and therefore

$$d\theta'\big|_{T^{1,0}Y \oplus T^{0,1}Y} = f d\theta\big|_{T^{1,0}Y \oplus T^{0,1}Y}.$$

From this it is clear that, up to an overall sign, the signature of the Levi form is determined by the CR-structure. If the Levi form is definite then the CR-structure on Y is strictly pseudoconvex, if it is positive or strictly pseudoconcave, if it is negative. For an abstract CR-manifold whether one wishes to regard the Levi form as positive or negative is simply a matter of convention. As it is fixed by choosing a non-vanishing vector field transverse to H, the choice of a sign for the Levi form is called a *transverse orientation.* The Levi-form is everywhere non-degenerate if and only if the underlying hyperplane field defines a contact structure.

1.1. Deformations of the CR-structure.

Due to a theorem of Gray on the rigidity of contact structures, see [12, §5], every deformation of a strictly pseudoconvex CR-structure is equivalent, under the action of the diffeomorphism group, to one supported by H. A smooth section ω of the homomorphism bundle, $\mathrm{Hom}(T^{0,1}Y, T^{1,0}Y)$, defines an "almost CR-structure" with fiber at $y \in Y$ given by

$${}^{\omega}T^{0,1}_y Y = \{\bar{Z} + \omega_y(\bar{Z}) \,:\, \bar{Z} \in T^{0,1}_y Y\}.$$

As $T^{1,0}Y \cap T^{0,1}Y$ is the zero section, the dimension condition is immediate for structures defined in this way. The almost CR-structure is non-degenerate if

${}^{\omega}T^{0,1}Y \cap {}^{\omega}T^{1,0}Y$ is the zero section in $TY \otimes \mathbb{C}$. The non-degeneracy condition is equivalent to the statement

$$+1 \text{ is not an eigenvalue of } \bar{\omega}_y \circ \omega_y \text{ for any } y \in Y.$$

Such deformations are said to at *finite distance* from the reference structure. We study deformations of this type which can be connected to the zero section through sections satisfying these non-degeneracy conditions. If $\dim Y = 3$ this simply means that $|\omega_y| < 1$ for all $y \in Y$. If $\dim Y \geq 5$ then, in order to define a CR-structure, ω must also satisfy an integrability condition which can be expressed as a partial differential equation, see [1, pg. 619]. The $\bar{\partial}_b$-operator defined by the deformed structure is denoted ${}^{\omega}\bar{\partial}_b$. We often use the notation ${}^{\omega}\bar{\partial}_b$ to refer to the CR-structure itself. In this connection the *reference CR-structure*, $T^{0,1}Y$ is denoted by $\bar{\partial}_b$.

Let $\operatorname{Def}(Y, \bar{\partial}_b)$ denote the connected neighborhood of the zero section in $\mathcal{C}^{\infty}(Y; \operatorname{Hom}(T^{0,1}Y, T^{1,0}Y))$ consisting of integrable deformations of the reference CR-structure. If $\dim Y = 3$ then the integrability condition is vacuous and

$$\operatorname{Def}(Y, \bar{\partial}_b) = \{\omega \in \mathcal{C}^{\infty}(Y; \operatorname{Hom}(T^{0,1}Y, T^{1,0}Y)) \ : \ \|\omega\|_{L^{\infty}} < 1\}.$$

The group of contact diffeomorphisms of (Y, H) acts on $\operatorname{Def}(Y, \bar{\partial}_b)$ by push forward. Two structures, ω_1 and ω_2 are equivalent if there exists an orientation preserving, contact diffeomorphism ψ with

$$\psi_* {}^{\omega_1}T^{0,1}_y Y = {}^{\omega_2}T^{0,1}_{\psi(y)} Y \text{ for all } y \in Y.$$

In this case

$$\ker {}^{\omega_2}\bar{\partial}_b = \psi^*(\ker {}^{\omega_1}\bar{\partial}_b).$$

We sometimes say that ${}^{\omega_1}\bar{\partial}_b$ and ${}^{\omega_2}\bar{\partial}_b$ define the same *geometric CR-structure*.

In this paper we are concerned with the behavior of $\ker {}^{\omega}\bar{\partial}_b$ under deformations of the CR-structure. If $\dim Y \geq 5$ then theorems of Boutet de Monvel and Kohn and Rossi imply that, for any strictly pseudoconvex CR-structure, $\ker \bar{\partial}_b$ is quite large. Indeed it contains enough functions to define an embedding $\varphi : Y \to \mathbb{C}^N$ for some N. If $\dim Y = 3$ then this is usually not the case; for "most" choices of CR-structure, $\ker \bar{\partial}_b$ contains only the constant functions. We assume that the reference CR-structure is embeddable, that is $\ker \bar{\partial}_b$ contains enough functions to embed Y into $\mathbb{C}^N$ for some N. In three dimensions this property is very unstable under deformations. Starting with [4], several authors have worked, over the last decade, to describe the set of embeddable deformations of the CR-structure on a 3-manifold, see [20, 21, 22], [6], [8], [3]. Though a comprehensive theory has yet to emerge, quite a few cases are now understood.

1.2. **CR-manifolds as boundaries.** Let X denote a complex manifold of dimension at least 2. A CR-structure is induced on a real hypersurface $Y \subset X$ by the rule

$$T^{0,1}Y = T^{0,1}X\big|_Y \cap TY \otimes \mathbb{C}.$$

If X is a complex manifold with boundary Y, then the same construction induces a CR-structure on the boundary. Suppose that Y is a level set of the smooth function ρ and that $d\rho$ does not vanish along Y. The non-vanishing 1-form $\theta = -i\bar{\partial}\rho\big|_Y$ defines H and the Levi form is represented by the $(1,1)$-form

$$\mathcal{L}_\rho = \partial\bar{\partial}\rho,$$

restricted to Y. The boundary components of a complex manifold have induced transverse orientations. Suppose that Y is a connected component of the boundary of a complex manifold X. Let ρ be a smooth, *non-positive* function which vanishes on Y such that $d\rho \neq 0$ along Y. If $\mathcal{L}_\rho > 0$ on $T^{1,0}Y$ then Y is a strictly pseudoconvex boundary component of X, if $\mathcal{L}_\rho < 0$ then Y is a strictly pseudoconcave boundary component of X. The sign of the Levi form is well defined under local biholomorphisms. Let J denote the almost complex structure on X. A direction T, transverse to $H \subset TY$ is determined by the condition that the JT is an *outward* pointing vector field along $Y = bX$. A choice of sign for the Levi form is often called a "co-orientation." From work of Harvey and Lawson and Kohn, it is well understood that a strictly pseudoconvex CR-structure is embeddable if and only if it can be realized as the boundary of a normal Stein space, see [13] and [18].

A very important innovation in the study of embeddability of CR-manifolds was introduced in [20] by Lempert. Lempert's idea was to think of a CR-manifold as the boundary of both a strictly pseudoconvex manifold, X_+ and a strictly pseudoconcave manifold, X_-. Indeed Lempert showed that an embeddable strictly pseudoconvex, CR-manifold is also the boundary of a strictly pseudoconcave space, see [22]. Lempert's result does not preclude the possibility that X_- has singularities, away from its boundary, though we always assume that X_- is a smooth, strictly pseudoconcave manifold. Forming $X = X_+ \sqcup_Y X_-$ leads to a compactification of the problem. Technically this is very useful, because the problem of extending a deformation of the CR-structure to the pseudoconcave side is well posed. Let Θ denote the tangent sheaf of a complex space. The linear obstruction to extending an integrable deformation of the CR-structure on bX_- to an integrable deformation of the complex structure on X_- is the cohomology group $H^2_c(X_-;\Theta)$. Kiremidjian showed that if $H^2_c(X_-;\Theta) = 0$ then any sufficiently small, integrable deformation of the CR-structure on bX_- extends to an integrable deformation of the complex structure on X_-, see [16]. If $\dim X_- = 2$ then this cohomology group is finite dimensional. The case where $H^2_c(X_-;\Theta) \neq 0$ is treated in [7] where it is shown that an extension exists for data belonging to a finite co-dimensional subvariety.

1.3. **The relative index.** In three dimensions the algebra of CR-functions is very unstable under deformations of the CR-structure. In [6] the *relative index* is introduced, it is an invariant which measures the change in this algebra under deformations. The relative index is vastly generalized in [9] and

[10]. We recall its definition. Let $(Y, \bar{\partial}_b)$ and $(Y, \bar{\partial}_b')$ be strictly pseudoconvex CR-structures with the same underlying contact field. Choosing a volume form fixes orthogonal projections onto the null-spaces of the $\bar{\partial}_b$-operators. We denote these by S and S' respectively. If both structures are embeddable (which is automatic if $\dim Y \geq 5$) then the restriction

$$S : \ker \bar{\partial}_b' \longrightarrow \ker \bar{\partial}_b$$

is a Fredholm map. The relative index, $\text{R-Ind}(\bar{\partial}_b, \bar{\partial}_b')$ is defined to be the Fredholm index of this map.

Many choices are made to define this index, but the results in [6] and [9] show that it only depends on the underlying geometric CR-structures. In [6] a filtration of the space of embeddable structures is defined, the union of strata are defined by

$$\mathfrak{S}_n = \{\omega \ : \ \text{R-Ind}(\bar{\partial}_b, {}^{\omega}\bar{\partial}_b) \geq -n\}.$$

A principal result in [6] is that, in three dimensions, the sets $\mathfrak{S}_n$ are locally closed in the $\mathcal{C}^\infty$-topology on $\text{Def}(Y, \bar{\partial}_b)$. This in turn led to the following conjecture:

CONJECTURE: Given an embeddable, compact, strictly pseudoconvex, 3-manifold $(Y, \bar{\partial}_b)$ there is a non-negative integer N such that, if $\omega \in \text{Def}(Y, \bar{\partial}_b)$ is a sufficiently small, embeddable deformation of the reference CR-structure then

$$\text{R-Ind}(\bar{\partial}_b, {}^{\omega}\bar{\partial}_b) > -N.$$

This in turn, would imply that the set of small, embeddable deformations is closed in the $\mathcal{C}^\infty$-topology on $\text{Def}(Y, \bar{\partial}_b)$. As we show in Proposition 1, the analogous statement in higher dimensions is quite easy to prove.

In [6] the conjecture is verified for the case of a domain in $\mathbb{C}^2$. Using a deep result of Eliashberg it is also shown that if $Y = S^3$, with the reference structure induced from its embedding as the unit sphere in $\mathbb{C}^2$ then $\text{R-Ind}(\bar{\partial}_b, {}^{\omega}\bar{\partial}_b) = 0$ for *any* embeddable deformation. In [8, pg. 225] the conjecture is verified for strictly pseudoconvex domains in the total space of a line bundle over $\mathbb{P}^1$. The relative index is again always zero for small, embeddable perturbations. In [8] the closedness of the set of small, embeddable perturbations is proved for many classes of 3-dimensional CR-manifolds without however verifying the relative index conjecture. The proof of the closedness is a rather intricate, geometric argument. In this paper we prove the relative index conjecture for these cases.

Theorem 1. *Let Y be a compact, embeddable, strictly pseudoconvex, 3-dimensional CR-manifold. Suppose that there is a strictly pseudoconcave manifold X_- with boundary Y and suppose further that X_- contains a smooth, compact holomorphic curve Z. If either of the following hypotheses hold*

$$\begin{gathered} H^2_c(X_-; \Theta \otimes [-Z]) = 0 \text{ or} \\ H^2_c(X_-; \Theta) = 0 \text{ and } H^1(Z; N_Z) = 0 \end{gathered} \tag{2}$$

then for small, embeddable deformations ω of the CR-structure on Y the estimate

$$\text{R-Ind}(\bar{\partial}_b, {}^{\omega}\bar{\partial}_b) \geq -[\dim H^1(Z, \mathcal{O}_Z) + \dim H^{2,0}(X_-)]$$

is valid.

Remark 1. The cohomological hypotheses are identical to those under which the closedness of the set of small embeddable perturbations is established in [8, pg. 188]. This latter result is a simple corollary of the theorem and the fact, proven in [6, pg. 51], that the sets $\mathfrak{S}_n$ are locally closed. The proof of Theorem 1 is analytic and much less delicate than the geometric proof of the weaker result in [8].

Remark 2. In [8] many examples are presented which satisfy the cohomological hypotheses. Among them are compact, strictly pseudoconvex hypersurfaces in line bundles over Riemann surfaces, Σ where the degree of the bundle exceeds $4g(\Sigma) - 3$. These examples have deformations for which the relative index is not zero. These therefore provide the first examples where the relative index conjecture is proved *and* the relative index assumes values besides 0 and $-\infty$. Another class of examples is given by neighborhoods of compact, holomorphic curves $Z \subset \mathbb{P}^2$. If the degree of Z is greater than 2 and U is a small neighborhood with a smooth strictly pseudoconcave boundary then $H^2_c(U; \Theta) = 0$ and $H^1(Z; N_Z) = 0$.

Remark 3. The ultimate goal of this subject is to give a "nice" description of the set of small, embeddable deformations of the CR-structure on a compact 3-manifold. Theorem 1 gives the first indications of such a structure in non-trivial examples. In particular it gives support for the hope that there is a finite codimension subspace of the algebra of CR-functions, for the reference structure, which is stable under all sufficiently small embeddable deformations.

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2. The $\Box_b$-operator

In addition to the $\bar{\partial}_b$-operator, it is often convenient to work with the associated Laplacian. This is called the $\Box_b$-operator, it is defined by

$$\Box_b = \bar{\partial}_b^* \bar{\partial}_b.$$

Here $\bar{\partial}_b^*$ is the formal adjoint of $\bar{\partial}_b$ defined by choosing a metric on the bundle $\Lambda_b^{0,1} Y = (T^{0,1} Y)'$. The $\Box_b$-operator has a natural, self adjoint extension as an unbounded operator on $L^2(Y)$. The null-space of $\Box_b$ equals that of $\bar{\partial}_b$. If $\bar{\partial}_b$ is embeddable then $\Box_b$ has an infinite dimensional null-space, its non-zero spectrum is a sequence $\{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ of positive numbers tending to infinity. We let ${}^{\omega}\Box_b$ denote the $\Box_b$-operator defined by ${}^{\omega}\bar{\partial}_b$. If ${}^{\omega}\bar{\partial}_b$ is **not** embeddable then the spectrum of ${}^{\omega}\Box_b$ accumulates at zero.

2.1. **Small eigenvalues.** Choose an $\epsilon << 1$ and suppose that ω is a sufficiently small, embeddable perturbation. Let λ_1 denote the smallest positive eigenvalue of $\Box_b$ and $\{0 < \mu_1 \leq \mu_2 \leq \cdots\}$ denote the non-zero eigenvalues of ${}^\omega\Box_b$. There is a $k \geq 1$ so that

$$0 < \mu_i < \epsilon\lambda_1 \text{ for } i < k,$$

while μ_k is comparable to λ_1. The first condition is vacuous if $k = 1$. We call $\{\mu_1, \ldots, \mu_{k-1}\}$ the *small eigenvalues* of ${}^\omega\Box_b$; they are small relative to the smallest, positive eigenvalue of the reference structure. In [6, pg. 44] it is shown that if ω is sufficiently small then

$$\text{R-Ind}(\bar{\partial}_b, {}^\omega\bar{\partial}_b) = 1 - k.$$

In other words, for small embeddable deformations, the relative index is minus the number of small eigenvalues. This implies that a lower bound for the k^{th}-eigenvalue of ${}^\omega\Box_b$ gives a lower bound for R-Ind$(\bar{\partial}_b, {}^\omega\bar{\partial}_b)$.

The proof of Theorem 1 is effected by obtaining such a bound on a particular eigenvalue of ${}^\omega\Box_b$. In [6, pg. 38] this idea was used to prove bounds on the relative index for a family of CR-structures bounding a family of strictly pseudo*convex* manifolds over which we could exercise considerable control. For example a family of hypersurfaces which arise from wiggling a hypersurface in its ambient space. The basic idea, going back to Kohn in [18], is to solve the $\bar{\partial}_b$ equation for $(0,1)$-forms on Y, with estimates, by using the estimates for the $\bar{\partial}$-Neumann problem on X_+. Using a similar idea we solve the $\bar{\partial}_b$-equation for $(2,1)$-forms on Y, with estimates, by using estimates for the $\bar{\partial}$-Neumann problem on X_-.

The cohomological hypotheses in Theorem 1 ensure that small deformations of the CR-structure on Y can be realized as boundaries of pseudoconcave manifolds over which we again exercise considerable control. The argument has three ingredients: 1. An identity for the "relative Euler characteristic" proved in [10]. In the case of small perturbations of the CR-structure on a 3-dimensional CR-manifold, this reduces to the observation that $\bar{\partial}_b^*\bar{\partial}_b$ and $\bar{\partial}_b\bar{\partial}_b^*$ are isospectral away from the zero eigenvalue. Using this observation, we can replace an analysis of $\bar{\partial}_b$ on $(0,0)$-forms with an analysis of $\bar{\partial}_b^*$ on $(0,1)$-forms. Using duality this is equivalent to analyzing $\bar{\partial}_b$ on $(2,0)$-forms. 2. Estimates for the $\bar{\partial}$-Neumann problem on X_- are deduced from Lempert's estimates for the $\bar{\partial}$-operator acting on sections of a holomorphic line bundle over a pseudoconcave manifold. 3. Using 2. we obtain a $(2,0)$-form, u which solves $\bar{\partial}_b u = \bar{\partial}_b\eta$, satisfying estimates, for $\bar{\partial}_b\eta$ belonging to a finite codimension subspace of the range of $\bar{\partial}_b$. The codimension of this subspace is bounded by using an exact sequence in cohomology proved by Andreotti and Hill, [2, pg. 352]. This, in turn shows that a particular eigenvalue of ${}^\omega\Box_b$ satsifies a lower bound, which therefore proves the theorem.

We close this section with a definition of the Kohn-Rossi complex and a discussion of step 1.

2.2. **The Kohn-Rossi complex.** In addition to the $\Lambda_b^{0,1}Y$, Kohn and Rossi defined a $\bar{\partial}_b$-complex on a CR-manifold, see [19, pg. 465]. The bundles $\Lambda_b^{0,q}Y$ are defined as $(\Lambda^q T^{0,1}Y)'$. If α is a section of $\Lambda_b^{0,q}Y$ then

$$\bar{\partial}_b\alpha \overset{d}{=} d\alpha\big|_{\underbrace{(T^{0,1}Y)'\times\cdots\times(T^{0,1}Y)'}_{(q+1)-\text{times}}}.$$

It is a consequence of integrability that $\bar{\partial}_b^2 = 0$. To define $\Lambda_b^{p,q}Y$ we follow Tanaka, see [26]. The bundle $\mathcal{T}Y = TY\otimes\mathbb{C}/T^{0,1}Y$ naturally carries the structure of a holomorphic (or CR) bundle. If $Y\hookrightarrow X$ is a real hypersurface in a complex manifold with the induced CR-structure then it is simply $T^{1,0}X\big|_Y$. We let

$$\Lambda_b^{p,0}Y = \Lambda^p(\mathcal{T}Y)' \text{ and } \Lambda_b^{p,q}Y = \Lambda_b^{p,0}\otimes\Lambda_b^{0,q}Y.$$

It is then immediate that the action of $\bar{\partial}_b$ extends to define a map

$$\bar{\partial}_b : \mathcal{C}^\infty(Y;\Lambda_b^{p,q}Y) \longrightarrow \mathcal{C}^\infty(Y;\Lambda_b^{p,q+1}Y)$$

satisfying $\bar{\partial}_b^2 = 0$. For clarity we sometimes denote this operator by $\bar{\partial}_b^{p,q}$.

The Kohn-Rossi cohomology groups are defined as

$$H_b^{p,q}(Y) = \frac{\ker\bar{\partial}_b : \mathcal{C}^\infty(Y;\Lambda_b^{p,q}Y)\to\mathcal{C}^\infty(Y;\Lambda_b^{p,q+1}Y)}{\bar{\partial}_b\mathcal{C}^\infty(Y;\Lambda_b^{p,q-1}Y)}.$$

If $q=0$ then $H_b^{p,0}(Y)$ consists of the CR-sections of $\Lambda_b^{p,0}Y$. If the structure is embeddable then these groups are infinite dimensional. If $\dim Y = 2n-1$ then the groups $H_b^{p,n-1}(Y)$ are isomorphic to the null-space of $\bar{\partial}_b^*$ acting on $\mathcal{C}^\infty(Y;\Lambda_b^{p,n-1}Y)$ and are also infinite dimensional. If Y is strictly pseudoconvex and $2n-1=\dim Y\geq 5$ then Kohn and Rossi showed that

$$\dim H_b^{p,q}(Y) < \infty \text{ for } 1\leq q\leq n-2.$$

2.3. **The relative Euler characteristic.** As above let $\dim Y = 2n-1$ be at least 5 and define the finite part of the CR-Euler characteristic to be

$$\chi_b(Y) = \sum_{j=1}^{n-2}(-1)^j\dim H_b^{0,q}(Y).$$

If $\dim Y = 3$ then $\chi_b(Y)=0$. In [10] it is shown that there is an analogous theory of relative indices for the operators $(\bar{\partial}_b^{p,n-1})^*$. If $\bar{S}$ is an orthogonal projection on $\ker(\bar{\partial}_b^{0,n-1})^*$ and ω is an embeddable deformation then $\text{R-Ind}(\bar{\partial}_b^*, {}^\omega\bar{\partial}_b^*)$ is the Fredholm index of the map

$$\bar{S} : \ker({}^\omega\bar{\partial}_b^{0,n-1})^* \longrightarrow \ker(\bar{\partial}_b^{0,n-1})^*.$$

In [10] the following relationship between these relative indices and the finite parts of Euler characteristic is established for two embeddable CR-structures with the same underlying contact field.

Theorem 2 ([10])**.** *Let Y be a compact $2n-1$ dimensional manifold with two strictly pseudoconvex CR-structures $\bar{\partial}_b$ and $\bar{\partial}'_b$ with the same underlying contact field. If both structures are embeddable then*

$$\text{R-Ind}(\bar{\partial}_b, \bar{\partial}'_b) - \chi_b(Y; \bar{\partial}_b) + \chi_b(Y; \bar{\partial}'_b) + (-1)^{n-1}\,\text{R-Ind}(\bar{\partial}^*_b, \bar{\partial}'^*_b) = 0. \tag{3}$$

There is no smallness hypothesis in the statement of this theorem. For the case of 3-manifolds and small perturbations it is easy to establish that

$$\text{R-Ind}(\bar{\partial}_b, {}^{\omega}\bar{\partial}_b) - \text{R-Ind}(\bar{\partial}^*_b, {}^{\omega}\bar{\partial}'^*_b) = 0.$$

The fact that, for small perturbations, the relative index is minus the number of small eigenvalues holds *mutatis mutandis* for the $\bar{\partial}^*_b$-operator with $\Box^{0,1}_b = \bar{\partial}_b\bar{\partial}^*_b$. The observation that

$$\bar{\partial}_b\Box^{0,0}_b = \Box^{0,1}_b\bar{\partial}_b$$

shows that the unbounded self adjoint operators $\Box^{0,0}_b$ and $\Box^{0,1}_b$ are isospectral away from the zero eigenvalue. In other words the operators ${}^{\omega}\Box^{0,0}_b$ and ${}^{\omega}\Box^{0,1}_b$ have the same number of small eigenvalues and therefore

$$\text{R-Ind}(\bar{\partial}_b, {}^{\omega}\bar{\partial}_b) = \text{R-Ind}(\bar{\partial}^*_b, {}^{\omega}\bar{\partial}^*_b).$$

From this discussion and Theorem D in [6] the following corollary is evident.

Corollary 1. *Let $(Y, \bar{\partial}_b)$ be a compact, embeddable, strictly pseudoconvex CR-manifold and $\mathcal{F}$ a family of deformations of the CR-structure. Suppose there exists a positive constant C and an integer N so that, for every $\omega \in \mathcal{F}$ there exist N L^2-bounded linear functionals, $\{l^{\omega}_1, \dots, l^{\omega}_N\}$, so that for every $\alpha \in \mathcal{C}^{\infty}(Y; {}^{\omega}\Lambda^{0,1}_b Y)$, satisfying*

$$l^{\omega}_j(\bar{\partial}^*_b\alpha) = 0 \text{ for } j = 1, \dots, N,$$

there exists $\beta \in \mathcal{C}^1(Y; {}^{\omega}\Lambda^{0,1}_b Y)$ with

$${}^{\omega}\bar{\partial}^*_b\beta = {}^{\omega}\bar{\partial}^*_b\alpha \text{ and } \|\beta\|_{L^2} \leq C\|\bar{\partial}^*_b\alpha\|_{L^2}.$$

Then for sufficiently small $\omega \in \mathcal{F}$

$$\text{R-Ind}(\bar{\partial}_b, {}^{\omega}\bar{\partial}_b) \geq -N.$$

Proof. Theorem D in [6] states that, for sufficiently small embeddable deformations, the relative index is minus the number of small eigenvalues of ${}^{\omega}\Box^{0,0}_b$. The hypotheses of the corollary imply that the $(N+1)^{\text{st}}$ positive eigenvalue of ${}^{\omega}\Box^{0,1}_b$ is at least C^{-2}. Let Q denote the partial inverse of ${}^{\omega}\Box^{0,0}_b$,

$$Q\,{}^{\omega}\Box^{0,0}_b = \text{Id} - \mathcal{S} = {}^{\omega}\Box^{0,0}_b Q,$$

here $\mathcal{S}$ is the orthogonal projection onto the $\ker\bar{\partial}_b$. Note that $\bar{\partial}_b Q\bar{\partial}^*_b$ is the orthogonal projection onto the range of $\bar{\partial}^*_b$. Let S denote the L^2-closure of

$$\{\bar{\partial}^*_b\alpha \,:\, \alpha \in \mathcal{C}^{\infty}(Y; {}^{\omega}\Lambda^{0,1}_b Y),\, l^{\omega}_j(\bar{\partial}^*_b\alpha) = 0,\, j = 1, \dots, N\}.$$

This is a closed subspace of the range of $[\ker\bar{\partial}_b]^{\perp}$ of codimension N.

For $u \in S \cap \bar{\partial}_b^* \mathcal{C}^\infty(Y; {}^\omega\Lambda_b^{0,1} Y)$ with β, as above, satisfying $\bar{\partial}_b^*\beta = u$, we have

$$\begin{aligned} \frac{< Qu, u >}{\|u\|^2} &= \frac{< Q\bar{\partial}_b^*\beta, \bar{\partial}_b^*\beta >}{\|u\|^2} \\ &= \frac{< \bar{\partial}_b Q\bar{\partial}_b^*\beta, \beta >}{\|u\|^2} \\ &\leq \frac{\|\beta\|^2}{\|\bar{\partial}_b^*\beta\|^2} \leq C^2 \end{aligned} \tag{4}$$

In the last line we use the fact that $\bar{\partial}_b Q \bar{\partial}_b^*$ is an orthogonal projection and the estimate satisfied by β. As Q is a bounded operator this estimate holds for all $u \in S$. The min-max characterization of eigenvalues shows that the $(N+1)^{\text{st}}$ eigenvalue of Q is at most C^2 and therefore the $(N+1)^{\text{st}}$ eigenvalue of ${}^\omega\Box_b^{0,0}$ is at least C^{-2}. The corollary therefore follows from Theorem D in [6]. □

We close the introduction with the argument showing that the relative index conjecture holds if $\dim Y \geq 5$.

Proposition 1. *Let Y be a compact, strictly pseudoconvex CR-manifold of dimension at least 5. For sufficiently small, integrable deformations of the CR-structure the estimate*

$$\text{R-Ind}(\bar{\partial}_b, {}^\omega\bar{\partial}_b) \geq -\dim H_b^{0,1}(Y; \bar{\partial}_b)$$

is valid.

Proof. The integrability of the CR-structures implies the identity

$$\bar{\partial}_b \Box_b^{0,0} = \Box_b^{0,1} \bar{\partial}_b.$$

This shows that the non-zero spectrum of $\Box_b^{0,0}$ is a subset of the spectrum of $\Box_b^{0,1}$. Kohn and Rossi proved that, if $\dim Y \geq 5$, then $\dim H_b^{0,1}(Y) < \infty$. Using the Heisenberg calculus it is not difficult to show that the spectrum of $\Box_b^{0,1}$ behaves continuously under small, integrable deformations of the CR-structure, see [10]. If ω is such a deformation then the number of small eigenvalues of ${}^\omega\Box_b^{0,1}$ is therefore bounded by $\dim H_b^{0,1}(Y; \bar{\partial}_b)$. As the non-zero spectrum of ${}^\omega\Box_b^{0,0}$ is a subset of the spectrum of ${}^\omega\Box_b^{0,1}$ the number of small eigenvalues of ${}^\omega\Box_b^{0,0}$ also cannot exceed $\dim H_b^{0,1}(Y; \bar{\partial}_b)$. This completes the proof of the proposition. □

3. The Andreotti-Hill exact sequences

Let X be a compact manifold and $Y \hookrightarrow X$ a smooth, separating hypersurface. The complement of Y has two connected components

$$X \setminus Y = X_+ \sqcup X_-.$$

In our applications $\bar{X}_+$ (resp. $\bar{X}_-$) denotes the closure of the pseudoconvex (resp. pseudoconcave) component of $X \setminus Y$. Andreotti and Hill proved a variety of long exact sequences relating the *smooth* Dolbeault cohomology on $\bar{X}_+, \bar{X}_-$ and the Kohn-Rossi cohomology on Y. In addition to the ordinary Dolbeault groups and the Kohn-Rossi cohomology, Andreotti and Hill also work with cohomology groups defined by a differential ideal $\mathcal{J} \subset \mathcal{C}^\infty(X; \Lambda^{*,*})$. To describe this ideal we let ρ denote a smooth defining function for Y. A (p,q)-form $\eta \in \mathcal{J}^{p,q}$ if there are smooth forms $\alpha \in \mathcal{C}^\infty(X; \Lambda^{p,q})$ and $\beta \in \mathcal{C}^\infty(X; \Lambda^{p,q-1})$ so that

$$\eta = \rho\alpha + \bar{\partial}\rho \wedge \beta.$$

From its definition it is apparent that $\bar{\partial} : \mathcal{J}^{p,q} \to \mathcal{J}^{p,q+1}$; we let $H^{p,q}(X; \mathcal{J})$ denote the (p,q)-cohomology group of this sub-complex of the Dolbeault complex.

We now restrict to the case of X a complex surface. The crucial point for our analysis is to control the kernel of the map

$$r_1 : H^{2,1}(\bar{X}_-) \longrightarrow H_b^{2,1}(Y).$$

The map, r_1, defined on page 352 of [2] is that induced by restriction of forms to the boundary. Suppose that α is a smooth representative of a class in $H_b^{p,q}(Y)$ and let $\widetilde{\alpha}$ denote a smooth extension of α to $\bar{X}_\pm$. The operators $\bar{\partial}'_\pm$ are defined by

$$\bar{\partial}'_\pm \alpha = \bar{\partial}\widetilde{\alpha}\big|_{\bar{X}_\pm}.$$

In [2, pg 353] it is shown that the classes of $\bar{\partial}'_\pm \alpha \in H^{p,q+1}(\bar{X}_\pm; \mathcal{J})$ are well defined. In the sequel we drop the $\pm$ subscript from $\bar{\partial}'_\pm$ as we only use the $-$ case.

With these preliminaries we can state the basic exact sequences we need. The first is

$$0 \longrightarrow H^{2,0}(\bar{X}_-) \xrightarrow{r_0} H_b^{2,0}(Y) \xrightarrow{\bar{\partial}'} H^{2,1}(\bar{X}_-; \mathcal{J}) \tag{5}$$

$$\xrightarrow{i_1} H^{2,1}(\bar{X}_-) \xrightarrow{r_1} H_b^{2,1}(Y) \longrightarrow \cdots$$

see Proposition 4.3 in [2]. From this exact sequence it follows that

$$\ker r_1 = \operatorname{Im} i_1 \simeq \frac{H^{2,1}(\bar{X}_-; \mathcal{J})}{\bar{\partial}' H_b^{2,0}(Y)} \tag{6}$$

Because $H^{2,0}(X_+)$ consists of holomorphic sections the inclusion,

$$H^{2,0}(\bar{X}_+) \hookrightarrow H_b^{2,0}(Y)$$

is injective and $\bar{\partial}' : H^{2,0}(\bar{X}_+) \to H^{2,1}(\bar{X}_-; \mathcal{J})$ factors through it; therefore

$$\dim \left[\frac{H^{2,1}(\bar{X}_-; \mathcal{J})}{\bar{\partial}' H_b^{2,0}(Y)} \right] \leq \dim \left[\frac{H^{2,1}(\bar{X}_-; \mathcal{J})}{\bar{\partial}' H^{2,0}(\bar{X}_+)} \right].$$

On the other hand, Proposition 4.3 in [2] implies that

$$0 \longrightarrow \frac{H^{2,1}(\bar{X}_-;\mathcal{J})}{\bar{\partial}' H^{2,0}(\bar{X}_+)} \xrightarrow{i_1} H^{2,1}(X) \xrightarrow{r_1'} H^{2,1}(\bar{X}_+) \longrightarrow \cdots, \tag{7}$$

is also exact.

Lemma 1. *If X_+ is a smooth complex surface with a strictly pseudoconvex boundary then*

$$H^{2,1}(\bar{X}_+) = 0.$$

Proof. Because the boundary is smooth and strictly pseudoconvex we can apply the results of Hormänder and Ohsawa and Takegoshi to conclude that

$$H^{2,1}(\bar{X}_+) \simeq H^{2,1}(X_+) \simeq H^{1,2}(X_+),$$

see, [15], [23]. Let Ω^1 be the sheaf of germs of holomorphic 1-forms, then the Dolbeault isomorphism implies that

$$H^{1,2}(X_+) \simeq H^2(X_+;\Omega^1).$$

Finally we let $A \subset\subset X_+$ be the maximal, compact analytic subset of X_+. Theorem V in [24] implies that

$$H^2(X_+;\Omega^1) \simeq H^2(A;\Omega^1\big|_A).$$

As A is a one dimensional, analytic set the group on the right vanishes. This completes the proof of the lemma. □

Combining the lemma with (6) and (7) completes the proof of the following proposition.

Proposition 2. *Suppose that Y is a separating, strictly pseudoconvex hypersurface in a smooth, compact, complex surface, X then*

$$\dim\ker r_1 \leq \dim H^{2,1}(X) = \dim H^{0,1}(X).$$

Remark 4. As X is compact the group $H^{2,1}(X)$ is automatically finite dimensional.

Proof. Everything but the last equality has already been proved. Because X is a smooth and compact this follows from Serre duality. □

4. Lempert's estimates and bounded geometry

In [21] estimates are proved for sections of holomorphic line bundles over pseudoconcave manifolds. The main point of our exposition is to recast Lempert's estimates in a form more familiar from the strictly pseudoconvex case and to discuss the dependence of the constants in these estimates on the underlying geometry.

Proposition 3 (Lempert)**.** *Let X_- be an n-dimensional complex manifold with boundary and suppose that the Levi form of the boundary has least one negative eigenvalue at each point. Let $E \to X_-$ be a holomorphic line bundle and let $\bar{\partial}_E$ be the $\bar{\partial}$-operator on sections of E. Finally let g be an hermitian metric on X_- and h an Hermitian metric on E, there is a constant C*

which depends on finite geometric bounds on (X_-, g, E, h) *so that for any* $\mathcal{C}^1$*-section,* s *of* E *we have the estimate*

$$\int_{bX_-} \|s\|^2 \,\mathrm{dV} \leq C \left[\|\bar{\partial}_E s\|^2_{L^2} + \|s\|^2_{L^2}\right]. \tag{8}$$

Remark 5. Note the similarity between (8) and the Morrey-$\frac{1}{2}$ estimate, valid for $(0,1)$-forms on a strictly pseudoconvex domain.

Remark 6. In the proposition it is stated that the constant depends on **finite geometric bounds** on (X_-, g, E, h). We need to explain what this means. Suppose that X_- is covered by open coordinate neighborhoods $\{(U_1, \psi_1), \ldots, (U_N, \psi_N)\}$ so that

$$\Phi_j : E\big|_{U_j} \longrightarrow \psi_j(U_j) \times \mathbb{C}, \quad j = 1, \ldots, N$$

are smooth trivializations. We suppose that the coordinate charts $\{\psi_j(U_j)\}$ have $\mathcal{C}^k$-bounded geometry in $\mathbb{C}^n$, i.e. diameters bounded above and below, $\mathcal{C}^k$-estimates on the regularity of the boundary, etc. On each open neighborhood we suppose that the metrics $\psi_j^*(g), \Phi_j^*(h)$ are within a given $\epsilon_1 > 0$, in the $\mathcal{C}^1$-norm, of the flat metrics on $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}$ respectively. The curvatures of g and h are assumed to be bounded above and below by $\pm K$ and to have $\mathcal{C}^k$ variation over each coordinate neighborhood bounded by $\epsilon_2 > 0$. Using deformation tensors, the complex structures on $\psi_j(U_j)$ and $\Phi_j(E\big|_{U_j})$ can also be compared to the flat complex structures on $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}$ respectively. We finally suppose that these deformation tensors are of $\mathcal{C}^k$-norm less than ϵ_3.

If N is fixed and k is sufficiently large then for any (X_-, g, E, h) satisfying these conditions with fixed constants K, $0 < \epsilon_1, \epsilon_2$ and $0 < \epsilon_3 < 1$ there exists a constant C making the inequality (8) true which is otherwise independent of (X_-, g, E, h). This is an immediate consequence of the argument used in section 3 of [21]. Throughout the paper this is what is meant when it is said that a constant "*depends on finite geometric bounds*."

In the pseudoconvex case the Morrey estimate is used to derive the so called "$\frac{1}{2}$-estimate". As this argument only involves the symbolic properties of the Laplacian $\bar{\partial}_E^* \bar{\partial}_E$, and has nothing to do with the convexity of the boundary, it applies to this case as well. For latter applications we state this result in terms of the (k,s)-norms, which are better adapted to the analysis of boundary value problems. Let (W, g) be a Riemannian manifold with boundary and choose a diffeomorphism of a neighborhood of bW onto $bW \times [0,2]_r$. Let C denote this "collar neighborhood" of bW. The metric on W induces a metric on bW. Let ∂_r denote a vector field transverse to the level sets of r. For $k \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{R}$ define the boundary part of the (k,s)-norm by

$$\|u\|^2_{(k,s)b} = \sum_{j=0}^{k} \int_0^1 \|\partial_r^j u(r, \cdot)\|^2_{H^{k-j+s}(bW)} dr.$$

Here $H^t(bW)$ are the standard L^2-Sobolev spaces on bW. Choose a function $\psi \in \mathcal{C}^\infty(W)$ with $\psi = 0$ in $bW \times [0, \frac{1}{2}]$ and $\psi = 1$ the complement of $bW \times [0,1]$. The (k,s)-norm is then defined by

$$\|u\|^2_{(k,s)} = \|u\|^2_{(k,s)b} + \|\psi u\|^2_{H^{k+s}(W)}.$$

The space $H_{(k,s)}(W)$ is the set of distributions on W with finite (k,s)-norm. These norms depend on the choice of tubular neighborhood, transverse vector field and cut-off ψ; though different choices lead to equivalent norms. Using a partition of unity and local trivializations these norms are easily extended to sections of vector bundles over W.

The space $H_{(1,-\frac{1}{2})}(W) \subset H^{\frac{1}{2}}(W)$. It has the very useful property that the restriction map $H_{(1,-\frac{1}{2})}(W) \to L^2(bW)$ is bounded, i.e. there is a constant C' so that if u is a smooth section of a vector bundle over W then

$$\|u\big|_{bW}\|_{L^2(bW)} \leq C'\|u\|_{(1,-\frac{1}{2})}. \tag{9}$$

It is easy to see a tubular neighborhood and cut-off ψ can be chosen so that the constant C' also depends only on finite geometric bounds on W and the vector bundle.

Using a standard argument, see [27, pg. 402] and Proposition 3 we obtain the "$\frac{1}{2}$-estimate" for a pseudoconcave manifold.

Proposition 4. *Let (X_-, g, E, h) be as in Proposition 3. There is a constant C_1, depending only on finite geometric bounds on (X_-, g, E, h),* vide *remark 6, so that for any $\mathcal{C}^1$ section s of E the following estimate holds*

$$\|s\|^2_{(1,-\frac{1}{2})} \leq C_1\left[\|\bar{\partial}_E s\|^2_{L^2} + \|s\|^2_{L^2}\right]. \tag{10}$$

Note that no boundary condition is needed for this estimate to hold.

Remark 7. If $V \to X_-$ is a holomorphic vector bundle then there is a canonical extension of the $\bar{\partial}$-operator to sections of $E \otimes V$. Let l denote a hermitian metric on V. Our notion of bounded geometry extends in an obvious way to (V, l). The estimate (10) extends to $E \otimes V$ with the constant again depending on finite geometric bounds on $(X_-, g, E \otimes V, h \otimes l)$.

Higher norm estimates can be derived precisely as in the pseudoconvex case. We state these in terms of the standard L^2-Sobolev norms and the associated Kohn-Laplacian

$$\Box_{E\otimes V} = \bar{\partial}^*_{E\otimes V}\bar{\partial}_{E\otimes V}.$$

These estimates require that $\bar{\partial}_{E\otimes V}u$ lie in the domain of the Hilbert space adjoint of $\bar{\partial}_{E\otimes V}$. The precise condition depends on the choice of metric. Briefly there exists a $(0,1)$-vector field ν defined along bX_- such that $\operatorname{Re}\nu$ is everywhere transverse to the boundary. An $E \otimes V$-valued $(0,1)$-form ω belongs to $\operatorname{Dom}(\bar{\partial}^*_{E\otimes V})$ if the weak derivative $\bar{\partial}^*_{E\otimes V}\omega$ is in L^2 and

$$i_\nu\omega\big|_{bX_-} = 0. \tag{11}$$

The latter condition is the $\bar{\partial}$-Neumann boundary condition, see [11, pg. 16].

Proposition 5. *With $(X_-, g, E \otimes V, h \otimes l)$ as in Proposition 4, for each $k \in [0, \infty)$ there is a constant C_k, depending only on finite geometric bounds on $(X_-, g, E \otimes V, h \otimes l)$, so that any $\mathcal{C}^{k+1}$-section ω of $E \otimes V$ for which $\bar\partial_{E\otimes V}\omega$ satisfies (11) satisfies the estimate*

$$\|\omega\|_{H^{k+1}} \leq C_k \left[\|\Box_{E\otimes V}\omega\|^2_{H^k} + \|\omega\|^2_{L^2}\right]. \tag{12}$$

The Kohn-Laplacian $\Box_{E\otimes V}$ is an unbounded self adjoint operator with a purely discrete spectrum lying in $[0, \infty)$. The following observation is a corollary of these estimates.

Corollary 2. *Let $(X_-, g, E \otimes V, h \otimes l)$ be as in Proposition 5. For any given $\lambda \geq 0$ there is a constant N_λ depending only on finite geometric bounds on $(X_-, g, E \otimes V, h \otimes l)$ such that the number of eigenvalues of $\Box_{E\otimes V}$, counted with multiplicity, less than or equal to λ is bounded by N_λ.*

Proof. If $\{(s_j, \lambda_j)\}$ is an orthonormal eigenbasis for $\Box_{E\otimes V}$ with $\lambda_j \leq \lambda_{j+1}$, then we can apply (12) with $k = 0$ to conclude that

$$\|s_j\|_{H^1} \leq C_1(\lambda_j + 1)\|s_j\|_{L^2}.$$

Using the Courant-Fischer min-max principle this estimate implies that the n^{th}-eigenvalue of standard Neumann Laplacian acting on sections of $E \otimes V$ is less than or equal to $C_1(\lambda_n + 1)$. Using the well known bounded geometry, lower bounds for these eigenvalues the conclusion of the corollary follows, see [5, pg. 333]. □

5. The $\bar\partial$-Neumann problem on a pseudoconcave surface

Let dV denote the volume form on X_-, the metric on TX_- induces metrics on the bundles $\Lambda^{p,q}X_-$. To avoid confusion with operator adjoints we use * to denote the Hodge star operator. For each $x \in X_-$, it is the conjugate linear map $^\star : \Lambda^{p,q}_x \to \Lambda^{n-p,n-q}_x$ defined by

$$\eta \wedge {}^\star\eta = \|\eta\|^2_x \, \mathrm{dV}_x .$$

If η is a (p, q)-form then

$$^{\star\star}\eta = (-1)^{p+q}\eta;$$

the formal adjoint of $\bar\partial^{p,q}$ is given by

$$[\bar\partial^{p,q}]^* = -{}^\star\bar\partial^{p,q\star}.$$

If ξ is a (p, q) form on X_- then ξ_b denotes the restriction of $\xi\big|_{bX_-}$ to

$$[T^{1,0}X\big|_{bX_-}]^p \otimes [T^{0,1}bX_-]^q.$$

It is important to note that

$$\mathcal{T}bX_- \overset{d}{=} T^{1,0}X\big|_{bX_-} \simeq TbX_- \otimes \mathbb{C}/T^{0,1}bX_- \tag{13}$$

has the natural structure of holomorphic vector bundle over bX_-. It contains $T^{1,0}bX_-$ as a smooth subbundle of codimension 1. According to this definition, if ξ is a $(n, n-1)$-form then ξ_b does not have to be zero, on the other hand if ξ is a $(n-1, n)$-form then $\xi_b \equiv 0$.

The $\bar\partial$-operator defines maps

$$\bar\partial^{p,q} : \mathcal{C}^\infty(X_-;\Lambda^{p,q}) \longrightarrow \mathcal{C}^\infty(X_-;\Lambda^{p,q+1}), \tag{14}$$

for $0 \le p \le n, 0 \le q \le n$. For each such (p,q) define an L^2-closeable, hermitian form

$$Q^{p,q}(\omega) = \int_{X_-} \left[\|\bar\partial^{p,q}\omega\|^2 + \|[\bar\partial^{p,q+1}]^*\omega\|^2\right] \mathrm{dV}$$

with form domain

$$\mathrm{Dom}(Q^{p,q}) = \omega \in \mathcal{C}^\infty(X_-;\Lambda^{p,q}) \text{ such that } i_\nu\omega\big|_{bX_-} = 0.$$

Using Friedrichs' extension, the closures of these forms define self adjoint operators $\Box^{p,q}$ with domains $\mathrm{Dom}(\Box^{p,q}) \subset L^2(X_-;\Lambda^{p,q})$. These are the $\bar\partial$-Neumann operators; this is well trodden ground and we direct the reader to [11] for a detailed discussion of these matters.

For the remainder of this section X_- denotes a strictly pseudoconcave surface. Using the estimates above we now describe the analytic properties of the $\bar\partial$-Neumann problem in this case. The $\bar\partial$-Neumann operators are given formally by

$$\Box^{p,q} = \begin{cases} [\bar\partial^{p,0}]^*\bar\partial^{p,0} & \text{if } q=0, \\ \bar\partial^{p,0}[\bar\partial^{p,0}]^* + [\bar\partial^{p,1}]^*\bar\partial^{p,1} & \text{if } q=1, \\ \bar\partial^{p,1}[\bar\partial^{p,1}]^* & \text{if } q=2. \end{cases} \tag{15}$$

As

$$\Lambda^{p,q}X_- = \Lambda^p(T^{1,0}X)' \otimes \Lambda^q(T^{0,1}X)'$$

and $\Lambda^p(T^{1,0}X)'$ is a holomorphic vector bundle, it is clear that the gross analytic properties of the $\bar\partial$-Neumann operator, i.e. closedness of the range and finite dimensionality of the kernel, do not depend on p. From Lempert's estimates it follows that $\Box^{p,0}$ is subelliptic. It is a classical result of Kohn and Rossi that $\Box^{p,2}$ is the standard Dirichlet Laplacian which is actually elliptic. This leaves only $\Box^{p,1}$.

Proposition 6. *If X_- is a smooth, strictly pseudoconcave surface then the $\bar\partial$-Neumann operator $\Box^{p,1}$ has a closed range for $p = 0,1,2$ and an infinite dimensional kernel.*

Proof. Because $\Box^{p,1}$ is an unbounded, self adjoint operator it has a closed range if and only if its spectrum does not accumulate at 0. The identities

$$\begin{aligned} \Box^{p,1}\bar\partial^{p,0} &= \bar\partial^{p,0}\Box^{p,0}, \\ \Box^{p,1}[\bar\partial^{p,1}]^* &= [\bar\partial^{p,1}]^*\Box^{p,2}. \end{aligned} \tag{16}$$

imply that $\lambda \neq 0$ is an eigenvalue of $\Box^{p,1}$ if and only if it is also an eigenvalue of either $\Box^{p,0}$ or $\Box^{p,2}$. Indeed if $\{(u_j,\lambda_j)\}$ and $\{(\omega_j,\mu_j)\}$ are orthogonal eigenbases for $\Box^{p,0}$ and $\Box^{p,2}$ respectively then $\{(\bar\partial u_j,\lambda_j),(\bar\partial^*\omega_j,\mu_j)\}$ is an orthogonal eigenbasis for the orthocomplement of $\ker\Box^{p,1}$. This shows that its range is closed. The statement that $\ker \Box^{p,1}$ is infinite dimensional is proved in [14, §18]. $\Box$

Since $\Box^{p,q}$ has a closed range for all (p,q) there are (bounded) partial inverses, $G^{p,q}$ and orthogonal projections onto the null spaces, $P^{p,q}$ which satisfy

$$\Box^{p,q}G^{p,q}\omega = (\mathrm{Id} - P^{p,q})\omega \text{ and } P^{p,q}G^{p,q}\omega = G^{p,q}P^{p,q}\omega = 0$$

for all forms $\omega \in L^2(X_-;\Lambda^{p,q})$ and

$$G^{p,q}\Box^{p,q}\omega = (\mathrm{Id} - P^{p,q})\omega$$

for all forms in $\mathrm{Dom}(\Box^{p,q})$. In particular we get the Hodge decompositions, if $\omega \in L^2(X_-;\Lambda^{p,q})$ then

$$\omega = \bar{\partial}^*\bar{\partial}G^{p,q}\omega + \bar{\partial}\bar{\partial}^*G^{p,q} + P^{p,q}\omega. \tag{17}$$

The summands on the right hand side are pairwise orthogonal and the operators $\bar{\partial}^*\bar{\partial}G^{p,q}$ and $\bar{\partial}\bar{\partial}^*G^{p,q}$ are orthogonal projections.

We let $\mathcal{H}^{p,q}(X_-) = \ker \Box^{p,q}$ denote the groups of harmonic (p,q)-forms. For a pseudoconcave surface we have the isomorphisms

$$\begin{aligned} &H^{p,q}(\bar{X}_-) \simeq \mathcal{H}^{p,q} \text{ for } q = 0,1, \quad p = 0,1,2, \\ &H^{p,0}(X_-) \simeq \mathcal{H}^{p,0} \text{ for } p = 0,1,2, \end{aligned} \tag{18}$$

see [11, §4.3]. Corollary 2 implies bounds on the dimensions of these groups, provided $q \neq 1$.

Corollary 3. *Let (X_-, g) be a smooth strictly pseudoconcave surface. There are constants $N_{p,q}$ for $p = 0,1,2$ and $q = 0,2$ which depend on finite geometric bounds on (X_-, g) so that*

$$\dim \mathcal{H}^{p,q}(X_-) \leq N_{p,q}. \tag{19}$$

The group $\mathcal{H}^{0,0}(X_-) = \mathbb{C}$ because a holomorphic function on a pseudoconcave manifold is constant.

We close this section with some further consequences of the Hodge decomposition.

Proposition 7. *Suppose that $\eta \in L^2(X_-;\Lambda^{2,2})$ satisfies*

$$\int_{X_-} \eta = 0 \tag{20}$$

then the $(2,1)$-form

$$\xi = -^\star\bar{\partial}G^{0,0\star}\eta \tag{21}$$

satisfies

$$\begin{aligned} \bar{\partial}\xi &= \eta, \\ \xi_b &= 0. \end{aligned} \tag{22}$$

Proof. If η is a $(2,2)$-form then ${}^\star\eta$ is a function and therefore the Hodge decomposition reads

$$^\star\eta = -^\star\bar{\partial}^\star\bar{\partial}G^{0,0\star}\eta + P^{0,0\star}\eta.$$

The operator $P^{0,0}$ is an orthogonal projection onto the constant function, so (20) implies that

$$\eta = \bar{\partial}\xi.$$

The form ${}^\star\xi = \bar{\partial}G^{0,0\star}\eta$ belongs to the domain of $\bar{\partial}^*$, that is

$$i_\nu{}^\star\xi = 0.$$

This is equivalent to the condition

$$\xi_b = 0,$$

see [11]. □

Proposition 8. *Suppose that η is a $(2,1)$-form which satisfies $\bar{\partial}\omega = 0$ then*

$$\theta = \bar{\partial}^* G^{2,1}\omega$$

solves

$$\bar{\partial}\theta = (\mathrm{Id} - P^{2,1})\omega.$$

Proof. Using (17) it suffices to show that

$$\bar{\partial}^*\bar{\partial}G^{2,1}\omega = 0.$$

The operator $\bar{\partial}^*\bar{\partial}G^{2,1}$ is an orthogonal projection and therefore

$$\|\bar{\partial}^*\bar{\partial}G^{2,1}\omega\|^2_{L^2} =< \omega, \bar{\partial}^*\bar{\partial}G^{2,1}\omega > .$$

As $\bar{\partial}G^{2,1}\eta \in \mathrm{Dom}(\bar{\partial}^*)$ for any $\eta \in L^2$, the fact that $\bar{\partial}\omega = 0$ implies that $\omega \in \mathrm{Dom}(\bar{\partial})$. As a result we can integrate by parts to obtain

$$< \omega, \bar{\partial}^*\bar{\partial}G^{2,1}\omega >=< \bar{\partial}\omega, \bar{\partial}G^{2,1}\omega >= 0.$$

□

6. Proof of the main theorem

Let Y denote a strictly pseudoconvex, 3-dimensional CR-manifold which bounds a pseudoconcave surface X_-. Recall that

$$\mathcal{T}Y = TY \otimes \mathbb{C}/T^{0,1}Y$$

has the natural structure of a holomorphic bundle. The Hodge star operator on the boundary defines conjugate linear maps

$$\begin{aligned} &\star \quad \mathcal{C}^\infty(Y; \Lambda_b^{0,1}) \longrightarrow \mathcal{C}^\infty(Y; \Lambda_b^{2,0}), \\ &\star \quad \mathcal{C}^\infty(Y; \Lambda_b^{2,1}) \longrightarrow \mathcal{C}^\infty(Y; \Lambda_b^{0,0}). \end{aligned} \tag{23}$$

The adjoint of $\bar{\partial}_b^{0,1}$ is given by

$$[\bar{\partial}_b^{0,1}]^* = -^\star\bar{\partial}_b^{2,0\star}. \tag{24}$$

Also note that if β is a section of $\Lambda_b^{2,0}$ then

$$\bar{\partial}_b\beta = d\beta$$

can also be computed by smoothly extending β to $B \in \mathcal{C}^\infty(X_-;\Lambda^{2,0})$ and setting

$$\bar{\partial}_b\beta = [\bar{\partial}B]_b. \tag{25}$$

To prove the main theorem we follow the outline of Kohn's treatment in the pseudoconvex case in section 4 of [18]. The analytic details are much simpler in the present case. Let $\alpha \in \mathcal{C}^\infty(Y;\Lambda_b^{0,1})$; Corollary 1 shows that we need to find constants C and N, depending on finite geometric bounds, so that if $\bar{\partial}_b^*\alpha$ satisfies N linear conditions then there is a solution, $\beta \in \mathcal{C}^\infty(Y;\Lambda_b^{0,1})$ to

$$\bar{\partial}_b^*\beta = \bar{\partial}_b^*\alpha \text{ with } \|\bar{\partial}_b^*\beta\|_{L^2} \leq C\|\bar{\partial}_b^*\alpha\|_{L^2}.$$

Let $\widetilde{\alpha} = {}^\star\bar{\partial}_b^*\alpha$. From (24) it is clearly sufficient to find a $(2,0)$-form $\widetilde{\beta} = {}^\star\beta$ satisfying

$$\bar{\partial}_b\widetilde{\beta} = \widetilde{\alpha}$$

and the estimates above. In light of (25) this can be done much as in [18].

The first step is to extend $\widetilde{\alpha}$ to A a $(2,1)$-form on $\bar{X}_-$ with an estimate of the form

$$\|A\|_{(1,-\frac{1}{2})} \leq C_1\|\widetilde{\alpha}\|_{L^2};$$

as usual the constant depends on finite geometric bounds on (X_-,g). That this can be done is a standard result which can be found in [18, pg. 541]. Next we need to correct A so that it is a closed form. To do that we use Proposition 7 and set

$$B = -{}^\star\bar{\partial}G^{0,0\star}\bar{\partial}A. \tag{26}$$

We need to check that (20) holds for $\bar{\partial}A$. This is an elementary application of Stokes' formula. Since A is a $(2,1)$-form $\bar{\partial}A = dA$ and therefore

$$\begin{aligned}\int_{X_-}\bar{\partial}A &= \int_{X_-} dA\\ &= \int_Y \widetilde{\alpha}\\ &= -\int_Y d^\star\alpha = 0.\end{aligned} \tag{27}$$

In the second to last line we use the fact that for a $(2,0)$-form η

$$\bar{\partial}_b\eta = d\eta.$$

Thus B defined in (26) satisfies

$$\begin{aligned}\bar{\partial}B &= \bar{\partial}A,\\ B_b &= 0,\end{aligned} \tag{28}$$

and therefore

$$\bar\partial(A-B)=0 \text{ and } [A-B]_b=\widetilde{\alpha}.$$

Below we discuss the estimate satisfied by B.

The next step is to solve

$$\bar\partial\vartheta=(A-B).$$

We apply Proposition 8 setting

$$\vartheta=\bar\partial^*G^{2,1}(A-B).$$

This form satisfies

$$\bar\partial\vartheta=(A-B)+P^{2,1}(A-B),$$

which implies that

$$\bar\partial_b[\vartheta_b-{}^\star\alpha]=[P^{2,1}(A-B)]_b$$

and therefore

$$[P^{2,1}(A-B)]_b\in\ker r_1. \tag{29}$$

This explains why it was necessary to prove Proposition 2, if Y is embeddable then $\ker r_1$ is finite dimensional. We now turn to the estimates satisfied by B and ϑ.

Lemma 2. *There is a constant C_3 depending on finite geometric bounds on (X_-,g) so that for all $v\in H_{(0,-\frac{1}{2})}(X_-)$*

$$\|\bar\partial G^{0,0}v\|_{L^2}\leq C_3\|v\|_{(0,-\frac{1}{2})}. \tag{30}$$

Proof. The space $H_{(0,-\frac{1}{2})}(X_-)$ is canonically dual to $H_{(0,\frac{1}{2})}(X_-)$ with respect to the $L^2(X_-)$ pairing. If $\xi\in\operatorname{Dom}([\bar\partial^{0,1}]^*)$ and v is an arbitrary smooth function then

$$<\bar\partial G^{0,0}v,\xi>_{L^2}=<v,G^{0,0}\bar\partial^*\xi>_{L^2}.$$

As $\operatorname{Dom}([\bar\partial^{0,1}]^*)$ is dense in L^2 this implies that the adjoint of $\bar\partial G^{0,0}$ with respect to this pairing is $G^{0,0}\bar\partial^*$. It therefore suffices to prove that for all $\xi\in\operatorname{Dom}([\bar\partial^{0,1}]^*)$ we have

$$\|G^{0,0}\bar\partial^*\xi\|_{(0,\frac{1}{2})}\leq C_3\|\xi\|_{L^2}. \tag{31}$$

This estimate is a consequence of (10) which states that there is a constant C', depending on finite geometric bounds, so that

$$\|G^{0,0}\bar\partial^*\xi\|^2_{(1,-\frac{1}{2})}\leq C'[\|\bar\partial G^{0,0}\bar\partial^*\xi\|^2_{L^2}+\|G^{0,0}\bar\partial^*\xi\|^2_{L^2}]. \tag{32}$$

The operator $\bar\partial G^{0,0}\bar\partial^*$ is an orthogonal projection, so the first term on the r.h.s of (32) is bounded by $\|\xi\|^2_{L^2}$. If λ_1 denotes the smallest non-zero eigenvalue of $\Box^{0,0}$ then it is easy to show that

$$\|G^{0,0}\bar\partial^*\xi\|^2_{L^2}\leq\frac{1}{\lambda_1}\|\xi\|^2_{L^2}.$$

Combining this with (32) gives

$$\|G^{0,0}\bar{\partial}^*\xi\|^2_{(1,-\frac{1}{2})} \leq C'[1+\frac{1}{\lambda_1}]\|\xi\|^2_{L^2} \tag{33}$$

As the $\ker \Box^{0,0} = \mathbb{C}$, Corollary2 implies that there is a lower bound for λ_1 which depends on finite geometric bounds on (X_-, g). As

$$H_{(0,\frac{1}{2})}(X_-) \supset H_{(1,-\frac{1}{2})}(X_-)$$

and $\|\xi\|_{(0,\frac{1}{2})} \leq \|\xi\|_{(1,-\frac{1}{2})}$ this completes the proof of the lemma. □

Using this lemma we obtain the estimate

$$\|B\|_{L^2} \leq C_4\|\bar{\partial}A\|_{(0,-\frac{1}{2})} \leq C_4'\|A\|_{(1,-\frac{1}{2})} \tag{34}$$

where again, C_4' is a constant depending only on finite geometric bounds on (X_-, g).

Now we estimate ϑ.

Lemma 3. *Let $\{\chi_j\}$ denote an orthonormal basis for $[\ker \Box^{2,0}]^\perp$ consisting of eigenfunctions with*

$$\Box^{2,0}\chi_j = \mu_j\chi_j \text{ and } \mu_j \leq \mu_{j+1}.$$

There is a constant C_5 depending on finite geometric bounds such that

$$\|\bar{\partial}^*G^{2,1}\xi\|_{(1,-\frac{1}{2})} \leq C_5\sqrt{1+\mu_k^{-1}}\|\xi\|_{L^2} \tag{35}$$

provided that

$$\bar{\partial}\xi = 0 \text{ and } <\xi, \bar{\partial}\chi_j>= 0 \text{ for } 1 \leq j < k. \tag{36}$$

Proof. As $\bar{\partial}^*G^{2,1}\xi$ is a $(2,0)$-form we can use the estimate (10) to conclude that there is a constant C' depending on finite geometric bounds such that

$$\|\bar{\partial}_b^*G^{2,1}\xi\|^2_{(1,-\frac{1}{2})} \leq C'\left[\|\bar{\partial}^*G^{2,1}\xi\|^2_{L^2} + \|\bar{\partial}\bar{\partial}^*G^{2,1}\xi\|^2_{L^2}\right]. \tag{37}$$

The operator $\bar{\partial}\bar{\partial}^*G^{2,1}$ is an orthogonal projection so the second term in (37) is bounded by $\|\xi\|^2_{L^2}$.

In light of (36), the Hodge decomposition of ξ is given by

$$\begin{aligned}\xi &= \bar{\partial}\bar{\partial}^*G^{2,1}\xi + P^{2,1}\xi \\ &= \sum_{j=k}^{\infty} a_j\bar{\partial}\chi_j + P^{2,1}\xi,\end{aligned} \tag{38}$$

for a complex sequence $\{a_j\}$. On the other hand

$$\bar{\partial}^*G^{2,1}\xi = \sum_{j=k}^{\infty} a_j\chi_j.$$

These identities imply that

$$\|\xi\|^2_{L^2} = \sum_{j=k}^{\infty}\mu_j|a_j|^2 + \|P^{2,1}\xi\|^2_{L^2}$$

and

$$\begin{aligned}\|\bar{\partial}^* G^{2,1}\xi\|_{L^2}^2 &= \sum_{j=k}^{\infty} |a_j|^2 \\ &\leq \frac{1}{\mu_k}\|\xi\|_{L^2}^2.\end{aligned} \tag{39}$$

□

If $<(A-B), \bar{\partial}\chi_j>_{L^2}=0$ for $j<k$ then this lemma and (34) imply that

$$\begin{aligned}\|\vartheta\|_{(1,-\frac{1}{2})} &\leq C_5\sqrt{1+\mu_k^{-1}}\|A-B\|_{L^2} \\ &\leq C_5'\sqrt{1+\mu_k^{-1}}\|A\|_{(1,-\frac{1}{2})}.\end{aligned} \tag{40}$$

The constant C_5' depends only on finite geometric bounds on (X_-, g). It is interesting to note that this estimate holds whether or not bX_- is embeddable.

We are now in a position to complete the proof of Theorem 1.

Proof on Theorem 1. Recall that $(Y, \bar{\partial}_b)$ is an embeddable strictly pseudoconvex CR-manifold which also bounds a pseudoconcave surface X_-. There is a smooth, compact curve $Z \hookrightarrow X_-$ such that (2) is satisfied. Let ω be an embeddable deformation of the CR-structure. The hypotheses (2) implies that if ω is a sufficiently small deformation of the CR-structure on Y then it extends to X_- as Ω, an integrable deformation of the complex structure on X_-. The size of Ω is bounded by that of ω, see [7, pg. 66]. Under the first assumption, Z remains holomorphic. Under the second, we apply Kodaira's stability theorem to conclude that there is a small deformation Z' of Z which is holomorphic in the deformed complex structure, see [17, pg. 80]. In either case the genus of the curve and the degree of its normal bundle are unchanged.

Let μ_1 denote the smallest non-zero eigenvalue of $\Box^{2,0}$ with respect to the reference structure and let

$$d = \dim \mathcal{H}^{2,0}(X_-),$$

again with the respect to the reference structure. From the estimates in section 5 it is clear that, for sufficiently small deformations, the operator ${}^{\omega}\Box^{2,0}$ has at most d eigenvalues less than $\mu_1/2$. Let $\{\chi_1^{\omega}, \dots, \chi_k^{\omega}\}$ be the eigenforms of ${}^{\omega}\Box^{2,0}$ for eigenvalues less than $\mu_1/2$.

For $\alpha \in \mathcal{C}^{\infty}(Y; \Lambda_b^{0,1})$ let A, B, ϑ be the forms constructed above so that

$${}^{\omega}\bar{\partial}_b^{*\star}\vartheta_b = {}^{\omega}\bar{\partial}_b^{*}\alpha + {}^{\star}[P^{2,1}(A-B)]_b.$$

Lemmas 3 and 2 and (9) imply that

$$\|\vartheta_b\|_{L^2} \leq C\sqrt{1+2\mu_1^{-1}}\|{}^{\omega}\bar{\partial}_b^{*}\alpha\|_{L^2}$$

provided that $A-B$ is orthogonal to $\{\bar{\partial}\chi_1^\omega, \dots, \bar{\partial}\chi_k^\omega\}$. For a small enough deformation ω, the complex manifold X_-^ω with its deformed complex structure certainly satisfies finite geometric bounds and therefore C can be taken to be independent of the deformation. Let X_+^ω denote a strictly pseudoconvex, complex manifold bounded by $(Y, {}^\omega\bar{\partial}_b)$ and set

$$X^\omega = X_+^\omega \sqcup_Y X_-^\omega.$$

In light of (29) the form $P^{2,1}[A-B]$ belongs to the $\ker r_1$. If $m = \dim\ker r_1$ then Proposition 2 implies that $m \leq \dim H^{0,1}(X^\omega)$. Let

$$\{\eta_1, \dots, \eta_m\} \subset \mathcal{H}^{2,1}(X_-)$$

be an orthonormal basis for $\ker r_1$. If $(A-B)$ satisfies the m additional linear conditions

$$< (A-B), \eta_j >_{L^2} = 0 \text{ for } j = 1, \dots, m \tag{41}$$

then in fact

$$\bar{\partial}\vartheta = A - B \text{ and therefore } {}^\omega\bar{\partial}_b^{*\star}\vartheta_b = {}^\omega\bar{\partial}_b^*\alpha.$$

The map ${}^\omega\bar{\partial}_b^*\alpha \mapsto A - B$ is a bounded linear map from $L^2(Y; \Lambda_b^{0,1})$ to $L^2(X_-; \Lambda^{2,1})$ and therefore the conditions in (41) are defined by bounded linear functionals on $L^2(Y; \Lambda_b^{0,1})$. We have therefore shown that for ${}^\omega\bar{\partial}_b^*\alpha$ satisfying at most $d + m$ linear conditions there is a solution β to

$${}^\omega\bar{\partial}_b^*\beta = {}^\omega\bar{\partial}_b^*\alpha$$

satisfying an estimate

$$\|\beta\|_{L^2} \leq C''\|{}^\omega\bar{\partial}_b^*\alpha\|_{L^2}$$

for a constant which is independent of the (sufficiently small) deformation ω. Hence

$$\text{R-Ind}(\bar{\partial}_b, {}^\omega\bar{\partial}_b) \geq -d - m. \tag{42}$$

The following lemma completes the proof of the main theorem .

Lemma 4. *If X is a smooth compact complex surface and $Z \hookrightarrow X$ is a smooth holomorphic curve with a positive normal bundle then*

$$m = \dim\ker r_1 \leq \dim H^{0,1}(X) \leq \dim H^1(Z; \mathcal{O}_Z).$$

Proof. The first inequality is a consequence of Proposition 2. Let $\mathcal{I}_Z$ be the sheaf of ideals defined by Z. The following sequence of sheaves is exact

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

and therefore we get the long exact sequence in cohomology

$$\longrightarrow H^1(X; \mathcal{I}_Z) \longrightarrow H^1(X; \mathcal{O}_X) \longrightarrow H^1(Z; \mathcal{O}_Z) \longrightarrow H^2(X; \mathcal{I}_Z) \longrightarrow .$$

The group $H^1(X; \mathcal{I}_Z) \simeq H^1(X; [-Z])$. Because Z is positively embedded we can use the Pardon-Stern-Kodaira vanishing theorem to conclude that

$$H^1(X; \mathcal{I}_Z) = 0,$$

see [25, pg. 605] and [8, pg. 167]. As $H^1(X;\mathcal{O}_X)\simeq H^{0,1}(X)$, the long exact sequence implies the conclusion. □

Theorem 1 is a consequence of (42), where $d=\dim H^{2,0}(X_-)$ and Lemma 4. □

Remark 8. Suppose that ω is a sufficiently small deformation of the CR-structure on Y which extends to define an integrable almost complex structure on X_-. The proof of the main theorem shows that if $(Y, {}^{\omega}\bar{\partial}_b)$ bounds a strictly pseudoconvex manifold then we have a bound on the relative index

$$\text{R-Ind}(\bar{\partial}_b, {}^{\omega}\bar{\partial}_b)\geq -[\dim H^{2,0}(X_-)+\dim H^{0,1}(X)].$$

This suggests the following question: Under the hypothesis $H^2_c(X_-;\Theta)=0$ is there an *a priori* bound on $\dim H^{0,1}(X)$? That is, can we obtain the conclusion of the main theorem without assuming that the holomorphic curve $Z\subset X_-$ also deforms?

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