CAN A GOOD MANIFOLD COME TO A BAD END?

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ABSTRACT. Two notions of cobordism are defined for compact CR-manifolds. The weaker notion, *complex cobordism* realizes two CR-manifolds as the boundary of a complex manifold; in the stronger notion, *strict complex cobordism* there is a strictly plurisubharmonic function defined on the total space of the cobordism with the boundary components as level sets of this function. We show that embeddability for a 3-dimensional, strictly pseudoconvex CR-manifold is a strict cobordism invariant. De Oliveira has recently shown that this is false for complex cobordisms. His construction is described in an appendix.

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1. INTRODUCTION

Let Y be a manifold of dimension $2n+1$. A CR-structure on Y is defined as a subbundle $T^{0,1}Y \subset TY \otimes \mathbb{C}$ which satisfies the following conditions

[dimension] fiber-dim$_{\mathbb{C}}$ $T^{0,1}Y = n$.

[non-degeneracy] $T^{0,1}Y \cap \overline{T^{0,1}Y} =$ zero section of TY.

[integrability] If $\overline{W}, \overline{Z} \in \mathscr{C}^\infty(Y; T^{0,1}Y)$ then their Lie bracket $[\overline{W}, \overline{Z}]$ is as well.

If we let $T^{1,0}Y = \overline{T^{0,1}Y}$ then there is a real hyperplane bundle $H \subset TY$ such that

$$T^{0,1}Y \oplus T^{1,0}Y = H \otimes \mathbb{C}. \tag{1}$$

Definition 1. If $T^{0,1}Y$ is a CR-structure on Y for which (1) holds then we say that $T^{0,1}Y$ is a CR-structure *supported* by H.

For θ a non-vanishing one form such that $H = \ker\theta$ we define the "Levi form" to be the Hermitian pairing defined on $T^{1,0}Y$ by $id\theta$,

$$(Z, W) \longrightarrow id\theta(Z, \overline{W}).$$

If θ' is another 1-form defining H then there is non-vanishing function f so that $\theta' = f\theta$ and therefore

$$d\theta'\big|_{T^{1,0}Y \oplus T^{0,1}Y} = f d\theta\big|_{T^{1,0}Y \oplus T^{0,1}Y}.$$

From this it is clear that, up to an overall sign, the signature of the Levi form is determined by the CR-structure. If the Levi form is definite then the CR-structure on Y is strictly pseudoconvex (if it is positive) or strictly pseudoconcave (if it is negative). For an abstract CR-manifold whether one wishes to regard the Levi form as positive or negative is simply a matter of convention. The choice of a sign for the Levi form is called a *transverse orientation* as it is fixed by choosing a non-vanishing vector field transverse to H.

Let X denote a complex manifold of dimension at least 2. A CR-structure is induced on a real hypersurface $Y \subset X$ by the rule

$$T^{0,1}Y = T^{0,1}X\big|_Y \cap TY \otimes \mathbb{C}.$$

If X is a complex manifold with boundary Y then the same construction induces a CR-structure on the boundary. Suppose that Y is a level set of the smooth function ρ and that $d\rho$ does not vanish along Y. The non-vanishing 1-form $-i\bar{\partial}\rho\big|_Y$ defines H and the Levi form is represented by the (1, 1)-form

$$\mathcal{L}_\rho = \partial\bar{\partial}\rho$$

restricted to Y. The boundary components of a complex manifold have induced transverse orientations. Suppose that Y is a connected component of the boundary of a complex manifold X. Let ρ be a smooth, *non-positive* function which vanishes on Y such that $d\rho \neq 0$ along Y. If $\mathcal{L}_\rho > 0$ on $T^{1,0}Y$ then Y is a strictly pseudoconvex boundary component of X, if $\mathcal{L}_\rho < 0$ then Y is a strictly pseudoconcave boundary component of X. It is easy to see that the sign of the Levi form is well defined under local biholomorphisms. Let J denote the almost complex structure on X. A direction T, transverse to $H \subset TY$ is determined by the condition that the JT is an *outward* pointing vector field along $Y = bX$. This better explains the terminology "transverse orientation."

Definition 2. Suppose that $(Y, T^{0,1}Y)$ is an compact CR-manifold. If there exists a compact, complex, connected manifold X with strictly pseudoconvex boundary $(Y, T^{0,1}Y)$ then we say that Y is a *fillable* CR-manifold.

It follows from results of Grauert that X is a holomorphically convex space which is a proper modification of a normal Stein space, X', see [15, 16]. The normal Stein space with boundary Y is uniquely determined, up to biholomorphism. Combining results of Kohn, Rossi, Boutet de Monvel and Harvey and Lawson one can show that any compact, strictly pseudoconvex CR-manifold of dimension at least 5 is fillable, see [22, 30, 5, 17]. On the other hand "most" strictly pseudoconvex 3-manifolds are not fillable, see [11, 14]. On a 3-manifold the integrability condition for a CR-structure is vacuous because the fiber dimension of $T^{0,1}Y$ is 1. The CR-structure is strictly pseudoconvex (or concave) if and only if the hyperplane field underlying the CR-structure is a contact structure. Thus if $H \subset TY$ is a contact structure then any choice of almost complex structure on the fibers of H defines a strictly pseudoconvex CR-structure on Y. From recent work of Eliashberg, et. el.. it follows that any 3-manifold has infinitely many inequivalent contact structures. It is also clear that most of these contact structures do not support any fillable CR-structures. It is then an interesting question to understand the set of fillable structures. In this note we investigate the problem of filling strictly pseudoconvex 3-manifolds from the point of view of cobordism.

In the following definitions we suppose that each connected component of a CR-manifolds is equipped with a transverse orientation, so that its pseudoconcavity or pseudoconvexity is fixed *a priori*. If X is a complex manifold and Y is a transversely oriented, CR-manifold then $bX = Y$ if

(1) The CR-structure induced on Y as the boundary of X agrees with the given CR-structure.
(2) The induced transverse orientation agrees with the given transverse orientation.

Definition 3. Suppose that Y_1 and Y_2 are (possibly disconnected) compact CR-manifolds. We say that Y_1 is *complex cobordant* to Y_2 if there exists a complex manifold with boundary X such that $bX = Y_1 \sqcup Y_2$.

Complex cobordism is, in general not an equivalence relation. A strictly pseudoconvex CR-manifold Y is never complex cobordant to itself. If it were then one could construct a compact, complex manifold with two strictly pseudoconvex ends, that impossible. Most

CR-manifolds are not complex cobordant to themselves with the transverse orientation reversed. A strengthening of this concept is also useful.

Definition 4. Suppose that Y_1 and Y_2 are (possibly disconnected) CR-manifolds and X defines a complex cobordism between Y_1 and Y_2. We say that Y_1 and Y_2 are *strictly complex cobordant* if there is a strictly plurisubharmonic function ρ defined on X so that the components of Y_1 and Y_2 are non-critical level sets of ρ.

It follows from the definition that all boundary components of X are either strictly pseudoconcave or strictly pseudoconvex. Well known approximations results imply that there is no loss in generality if we suppose that ρ is a Morse function, i.e. its critical points are non-degenerate.

These definitions suggest two questions. Suppose that Y_1 is a strictly pseudoconvex, compact 3-manifold and Y_2 is a union of strictly pseudoconcave components.

Question 1. If Y_1 is fillable and complex cobordant to Y_2 does it follow that the components of Y_2 are also fillable?

Question 2. If Y_1 is strictly complex cobordant to Y_2 and Y_1 is fillable are the components of Y_2 fillable as well?

Note that the hypothesis that Y_1 is fillable and strictly pseudoconvex implies, via the theorems of Grauert and of Kohn and Rossi that it is connected, see [23]. In this paper we show that the answer to the second question is yes, even if X is permitted to be complex space instead of complex manifold. In a recent preprint, Bruno De Oliveira has produced examples which show that the answer to the first question is no, see the appendix to this paper and [10]. If the dimension of the boundary is at least 5 then Rossi's theorem shows that the answer to the first question is always affirmative, see [30]. The non-fillability of complex cobordisms in the surface case is therefore another example of a purely 2-dimensional phenomenon. These concepts were defined in [13] and an analytic proof of the result below was sketched.

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2. Strict complex cobordisms

We prove the following theorem.

Theorem 1. *If Y_1 is a compact, fillable, strictly pseudoconvex 3-manifold and Y_2 is a union of strictly pseudoconcave components which is strictly complex cobordant to Y_1 then each component of Y_2 is also fillable.*

Corollary 1. *If under the hypotheses of Theorem 1 a complex manifold X defines complex cobordism between Y_1 and Y_2 then X is embeddable in $\mathbb{C}^N$.*

Remark 1. This Corollary implies in particular, the classical embedding results of Kodaira, Grauert and Andreotti and Tomassini for compact, pseudoconvex and pseudoconcave surfaces, respectively, see [21, 16, 3].

Proof. Let X denote a compact, complex manifold with boundary, $bX = Y_1 \sqcup Y_2$ which defines a strict complex cobordism between Y_1 and Y_2. Hence there is a strictly plurisubharmonic, Morse function ρ defined on X with the boundary components contained in sub-level sets. Without loss of generality we can assume that there are constants $c_0 > c_1 > \cdots > c_l$ such that

$$Y_1 = \rho^{-1}(c_0) \text{ and } Y_2^j \subset \rho^{-1}(c_j).$$

Indeed by adding the appropriately cut-off multiples of the functions $\log(d(x, Y_2^l) + \eta)$ and $-\log(d(x, Y_1) + \eta)$, for sufficiently small $\eta > 0$, to a large multiple of ρ it can be arranged that

$$c_0 = \sup\{\rho(x) : x \in X\}, \quad c_l = \inf\{\rho(x) : x \in X\}.$$

For any c let

$$X_c = \rho^{-1}((c, \infty)).$$

Since Y_1 is embeddable it follows from the Lempert approximation theorem, see [24] that there exists a normal projective variety V and an embedding $\Psi : Y_1 \to V \subset \mathbb{P}^N$ as a separating hypersurface. Let V_- denote the pseudoconcave part of $V \setminus \Psi(Y_1)$. Let $X' = X \sqcup_{Y_1} V_-$, this is a compact variety with a (possibly disconnected) strictly pseudoconcave boundary. The mapping Ψ is of course defined on $V_- \subset X'$ as the identity. For a $c < c_0$ we let $X'_c = X_c \sqcup_{Y_1} V_-$.

The variety V may fail to be smooth, but as it is two dimensional and normal, it is locally irreducible and its singular locus consists of a finite set of points. The image $\Psi(Y_1)$ can be assumed to lie in an affine chart $\mathbb{C}^N \simeq \mathbb{P}^N \setminus \mathbb{P}^{N-1}$. Henceforth we assume that linear coordinates are fixed on this affine chart and that the embedding of Y_1 into $\mathbb{C}^N$ is given by the coordinate functions

$$\Psi\big|_{Y_1} = (\psi_1, \ldots, \psi_N).$$

Step 1: The first step in the proof of the theorem is to extend the map Ψ to a holomorphic map of X into $\mathbb{C}^N$. As the map is holomorphic the coordinate functions satisfy

$$\bar{\partial}_b^Y \psi_j = 0$$

and therefore we can use the Lewy extension theorem to extend them as holomorphic functions to a small neighborhood of Y_1 in X. Using induction over the level sets of ρ and Lewy extension we can extend these functions up to the first critical level set of ρ. The critical points of ρ are isolated and therefore the following elementary result allows us to extend the coordinate functions to a neighborhood of a critical point of ρ.

Lemma 1. *Let φ be a plurisubharmonic function defined on a neighborhood U of $0 \in \mathbb{C}^n$. Suppose that $\varphi(0) = 0$ and that 0 is an isolated critical point of φ. Set*

$$U_+ = U \cap \varphi^{-1}((0, \infty)).$$

There exists a neighborhood W of 0 such that any holomorphic function defined in U_+ has a holomorphic extension to W.

Proof. A simple consequence of the Theorem 15 in [2], see also [19]. □

Using Lemma 1 we extend the coordinate functions across the critical points of ρ. By alternately inducting over the level sets of ρ and using Lemma 1 we extend the coordinate functions to all of X. We continue to use $\Psi = (\psi_1, \ldots, \psi_N)$ to denote the extended map. As $\Psi(Y_1) \subset V$, the permanence of functional relations implies that $\Psi(X) \subset V$. To complete the proof of the theorem we show that Ψ embeds X' into V. This is proved by induction over the level sets of ρ. It is clear that there for some $c < c_0$ the extended map Ψ embeds X'_c into V. We need to show that c can actually be taken to be equal to c_l. We show that one boundary component at a time can be filled.

Step 2: The $\{c_j\}$ are regular values for ρ, therefore there exists an $\epsilon > 0$ so that, for each j, the submanifolds $bX_{c_j+\delta}$, for $0 < \delta < \epsilon$ are disjoint unions of smooth manifolds diffeomorphic to bX_{c_j}. One component of $bX_{c_j+\delta}$ converges to Y_2^j as $\delta \to 0$. Denote this component by $Y_2^{j,\delta}$. If we can show that $Y_2^{1,\delta}$ is embeddable for sufficiently small $\delta > 0$ then it follows from relative index theory [11], see Lemma 5.1 in [14], that Y_2^1 is also embeddable. In this case there is an normal Stein space V_1 with $bV_1 = -Y_2^1$. The minus sign indicates that we take the opposite transverse orientation. Let $X^{(1)} = X' \sqcup_{Y_2^1} V_1$, as V_1 is normal the mapping Ψ extends to $X^{(1)}$ as well. If we can show that Ψ embeds a neighborhood, in $X^{(1)}$ of $\overline{X_{c_1}}$ then $\Psi(Y_2^1)$ is the boundary of a normal Stein domain in V. By uniqueness it is clear that Ψ embeds V_1 as an open subset of V with boundary $\Psi(Y_2^1)$.

We now show that bX_{c_1} is embedded by Ψ. Suppose that Ψ embeds X'_c for some $c_1 < c < c_0$ but Ψ does not embed $X'_{c+\delta}$ for any $\delta < 0$. First we show that the rank of $d\Psi(x) = 2$ for all $x \in \overline{X'_c}$. This would then imply that there is a pair of points

$$x_1 \neq x_2 \in \overline{X'_c} \text{ such that } \Psi(x_1) = \Psi(x_2). \tag{2}$$

After verifying that the rank of the differential cannot drop we show that (2) is also impossible. The claim as to the rank of differential is a consequence of the following lemma.

Lemma 2. *Let (W, p) be the germ of a normal surface and let ψ be the germ of a holomorphic mapping $\psi : (\mathbb{B}^2, 0) \to (W, p)$. Further suppose that there is a strictly plurisubharmonic function φ defined in a neighborhood of $0 \in \mathbb{B}^2$ such that*

(1) *The rank of $d\psi(z) = 2$ for $z \neq 0$.*
(2) $\varphi(0) = 0$.
(3) *The restriction of ψ to the set $\varphi^{-1}(0, \infty)$ is an embedding.*

Then the germ W is smooth at p and ψ is the germ of an embedding.

Proof. Using hypothesis (1), we apply a theorem of Prill to conclude that the map ψ is holomorphically conjugate to a quotient map $(\mathbb{B}^2, 0) \to (\mathbb{B}^2, 0)/G$. Here G is a finite group of germs of biholomorphic maps acting on $(\mathbb{B}^2, 0)$, see [29]. The maps act without fixed points on $\mathbb{B}^2 \setminus 0$. We then apply a theorem of H. Cartan to conclude that the action by the group G is holomorphically conjugate to a linear action by a finite subgroup of $U(2)$, which we continue to denote by G, see [7]. To prove the lemma it suffices to show that the group G must be trivial. Using the representation for the map germ ψ as a quotient map, the hypotheses of the lemma imply, after possibly scaling the normalized coordinates on $\mathbb{C}^2$, that there is a fundamental domain $\mathcal{F}_G$ for the action of the group G on $\mathbb{B}^2 \setminus \{0\}$ which contains the set $\{z : \varphi(z) > 0\}$.

Since $G \subset U(2)$ it preserves the unit sphere $\mathbb{S}^3$ and it follows that

$$|G| = \frac{\text{vol}(\mathbb{S}^3)}{\text{vol}(\mathcal{F}_G \cap \mathbb{S}^3)}. \tag{3}$$

If $d\varphi(0) \neq 0$ then it is clear that there exists a fundamental domain for the action by G which contains a half-space. Hence there is a linear function l such that $\mathcal{F}_G \supset \{z : l(z) > 0\}$. In this case, formula (3) implies that $|G| \leq 2$; either G is trivial or a group of order two. If $G \neq \{\text{Id}\}$ then it follows from the classification of finite subgroups of $U(2)$ that G is either the group $G_{\mathcal{A}} = (\text{Id}, \mathcal{A})$ where $\mathcal{A}(z, w) = (-z, -w)$ or a reflection group $G_v = (\text{Id}, R_v)$. Here R_v is the reflection

$$R_v(\zeta) = \zeta - 2 < \zeta, v > v.$$

Because $R_v(\zeta) = \zeta$ for any vector orthogonal to v, the later case is ruled out by the fact that G acts without fixed point on $\mathbb{S}^3$. We are therefore reduced to consideration of $G_{\mathscr{A}}$.

The group $G_{\mathscr{A}}$ is invariant under linear coordinate change and we can therefore choose linear coordinates (z_1, z_2) so that

$$\varphi(z_1, z_2) = \operatorname{Im} z_2 + 2\operatorname{Re}\sum a_{ij} z_i z_j + 2\sum b_{ij} z_i \bar{z}_j + O(|z|^3).$$

As φ is strictly plurisubharmonic the matrix b_{ij} is hermitian and positive definite. Let

$$\varphi_1(z_1, z_2) = \operatorname{Im} z_2 + 2\operatorname{Re}\sum a_{ij} z_i z_j + \sum b_{ij} z_i \bar{z}_j.$$

Define the analytic subvariety

$$Q_A = \{z : \sum a_{ij} z_i z_j = 0\}$$

of complex dimension at least 1. This implies that $Q_A \cap \{z : \operatorname{Im} z_2 = 0\}$ is of real dimension at least 1, and real-homogeneous. Let $z_0 \neq 0$ be a point in this intersection. Evidently both $\varphi_1(z_0) > 0$ and $\varphi_1(\mathscr{A}(z_0)) > 0$. By taking $|z_0|$ sufficiently small it follows that $\varphi(z_0) > 0$ and $\varphi(\mathscr{A}(z_0)) > 0$ as well. This shows that there is no fundamental domain for $G_{\mathscr{A}}$ that contains $\{z : \varphi(z) > 0\}$ and completes the analysis in the case that $d\varphi(0) \neq 0$.

We now consider the critical case, with $d\varphi(0) = 0$. In any system of linear coordinates

$$\varphi(z) = 2\operatorname{Re}\sum a_{ij} z_i z_j + 2\sum b_{ij} z_i \bar{z}_j + O(|z|^3),$$

with $A = a_{ij}$, a symmetric matrix and b_{ij} a positive definite hermitian matrix. As before we set

$$\varphi_1(z) = 2\operatorname{Re}\sum a_{ij} z_i z_j + \sum b_{ij} z_i \bar{z}_j.$$

If $A = 0$, then φ is positive in a deleted neighborhood of 0; therefore the hypothesis of the lemma already implies the conclusion. The analysis now divides into two further cases according to whether the rank A is one or two. Let $(v, z) = v_1 z_1 + v_2 z_2$; if rank $A = 1$ then there is a non-zero vector $v \in \mathbb{C}^2$ such that

$$(Az, z) = (v, z)^2$$

and therefore the set $\{z : \operatorname{Re}(v, z)^2 = 0\}$ is the union of two real hyperplanes

$$L_r = \{z : \operatorname{Re}(v, z) = 0\} \text{ and } L_i = \{z : \operatorname{Im}(v, z) = 0\}.$$

For any $g \in U(2)$ the intersections $L_r \cap gL_r$ and $L_i \cap gL_i$ are at least two (real) dimensional. Suppose that there exists a $g \in G \setminus \{\mathrm{Id}\}$, then we can choose a small, non-zero vector z_0 in one of these intersections. Evidently both $\varphi_1(z_0)$ and $\varphi_1(gz_0) > 0$. Therefore, by choosing $z_0 \neq 0$, of sufficiently small norm we obtain a contradiction to the assertions that $G \neq \{\mathrm{Id}\}$ and that there is a fundamental domain $\mathscr{F}_G \supset \{z : \varphi(z) > 0\}$.

We are left to consider the critical case with rank $A = 2$. We define the two open sets

$$S_A^{\pm} = \{z : \pm\operatorname{Re}(Az, z) > 0\}.$$

Observe that if $z \in S_A^+$ then $iz \in S_A^-$ and vice versa. Hence multiplication by i defines an isometric diffeomorphism of S_A^+ and S_A^-. Since the rank $A = 2$ the complement of $S_A^+ \cup S_A^-$ has empty interior and therefore

$$\operatorname{vol}(S_A^{\pm} \cap \mathbb{S}^3) = \frac{1}{2}\operatorname{vol}(\mathbb{S}^3). \tag{4}$$

That rank $A = 2$ implies that the signature of the quadratic form $\operatorname{Re}(Az, z)$ is $(2, 2)$. By a linear change of coordinates, it is equivalent to the quadratic form $x_1^2 + x_2^2 - y_1^2 - y_2^2$.

This implies that $S_A^\pm \cap \mathbb{S}^3$ are connected sets. Suppose that for some element $g \in U(2)$ the intersection

$$\{z \in \mathbb{S}^3 \ : \ \mathrm{Re}(Az, z) = 0\} \cap \{z \in \mathbb{S}^3 \ : \ \mathrm{Re}(Agz, gz) = 0\} \tag{5}$$

is empty, then $\{z \in \mathbb{S}^3 \ : \ \mathrm{Re}(Agz, gz) = 0\}$ is a subset of either $S_A^+ \cap \mathbb{S}^3$ or $S_A^- \cap \mathbb{S}^3$, say $S_A^+ \cap \mathbb{S}^3$. As the sets $S^\pm_{g^tAg} \cap \mathbb{S}^3$ are connected this implies that one of these sets is a relatively compact subset of $S_A^+ \cap \mathbb{S}^3$, say $S^+_{g^tAg} \cap \mathbb{S}^3$. This, in turn implies that

$$\mathrm{vol}(S^+_{g^tAg} \cap \mathbb{S}^3) < \mathrm{vol}(S^+_A \cap \mathbb{S}^3) = \frac{1}{2}\,\mathrm{vol}(\mathbb{S}^3).$$

However this contradicts (4) with g^tAg in place of A. Any other set of the possible choices for $\pm$ would of course lead to the same contradiction and therefore, for any $g \in U(2)$ the intersection in (5) is non-empty. Arguing as in the rank 1 case we again deduce that the group G must be trivial. This completes the proof of the lemma. □

We now return to the induction argument. Recall that we are assuming the Ψ embeds X'_c for a $c_1 < c < c_0$ but fails to embed $X'_{c+\delta}$ for any $\delta < 0$. From Lemma 2 it follows that $\mathrm{rank}\, d\Psi(x) = 2$ for all $x \in \overline{X_c}$.

Step 3: The only way that Ψ can fail to embed $X'_{c+\delta}$ for any $\delta < 0$ is if there exists a pair of points as in (2). There are two possibilities: 1. Both points lie on bX_c or 2. One point, which we denote by x_1 lies on bX_c and the other point x_2 lies in X'_c. Let us suppose that case 1 holds and case 2 does not hold. This implies that there is a point $p \in V \setminus \Psi(X'_c)$ such that $p = \Psi(x_1) = \Psi(x_2)$. Let U_1 and U_2 denote disjoint neighborhoods of x_1 and x_2 respectively. Choose the neighborhoods sufficiently small such that $U_1 \cup U_2$ is disjoint from the singular locus of X' and $\Psi\big|_{U_i}$, $i = 1, 2$ are embeddings. Let $U_i^+ = U_i \cap X_c$, and $U_i^- = U_i \setminus U_i^+$, from the induction hypothesis it follows that

$$\Psi(U_1^+) \cap \Psi(U_2^+) = \emptyset. \tag{6}$$

On the other, as $\Psi(x_1) = \Psi(x_2)$, (6) implies that the germ (V, p) is not locally reducible.

The only way that this could fail is that either $\Psi(U_1^+) \subset \Psi(U_2^-)$ or $\Psi(U_2^+) \subset \Psi(U_1^-)$. Suppose, without loss of generality, that the first inclusion holds. This would violate the maximum principle. We can find a holomorphic disk D which lies in $U_1^+ \cup \{x_1\}$ and meets x_1 at an interior point. This is easily seen whether or not x_1 is a critical point of φ. Using Ψ to pull back ρ from $\Psi(U_2^-)$ we would obtain a subharmonic function which assumes it maximum value, c at an interior point. This subharmonic function must therefore be constant, i.e. $\Psi(D) \subset bX_c$. However this is also impossible as bX_c is strictly pseudoconcave. Thus the germ (V, p) is not locally reducible. This is also not possible as V was assumed to be a normal surface.

We are reduced to consideration of case 2. The argument is similar, as before let $p = \Psi(x_1) = \Psi(x_2)$ and U_i, $i = 1, 2$ be disjoint neighborhoods of x_i, $i = 1, 2$ and suppose that $\Psi\big|_{U_1}$ is an embedding. With $U_1^\pm$ defined as above the induction hypothesis implies that

$$\Psi(U_1^+) \cap \Psi(U_2) = \emptyset. \tag{7}$$

It is an immediate consequence of (7) that the intersection $\Psi(U_1) \cap \Psi(U_2)$ is a proper subvariety and therefore the germ (V, p) is not locally reducible. This again violates the normality of V and thus completes the proof that Ψ embeds $\overline{X'_c}$ for any $c > c_1$.

Step 4: As noted above this implies that the boundary Y_2^1 is embeddable and therefore bounds a normal Stein space V_1. Following the outline in step 2, we set $X^{(1)} = X' \sqcup_{Y_2^1} V_1$.

The variety $X^{(1)}$ is smooth in a neighborhood of bX_{c_1} and therefore the argument in step 3 shows that Ψ extends to define an embedding of a neighborhood, in $X^{(1)}$ of X'_{c_1}. In particular $\Psi\big|_{Y_2^1}$ is an embedding and therefore the extension of Ψ to V_1 is an embedding. The variety V_1 has a strictly plurisubharmonic exhaustion function and therefore we can use the argument just presented to show that $\Psi\big|_{X'_{c_1}\sqcup V_1}$ is an embedding. Indeed as $\Psi\big|_{V_1}$ is an embedding, we only need to consider case 2, in step 3. The argument follows exactly as above.

We use this argument inductively for each of the remaining ends. Suppose that we have shown that the ends $\{Y_2^1,\dots,Y_2^{j-1}\}$ are embeddable and bound normal varieties $\{V_1,\dots,V_{j-1}\}$. For $i\le j-1$ we set

$$X^{(i)} = X'\sqcup_{Y_2^1} V_1\sqcup_{Y_2^2}\cdots\sqcup_{Y_2^i} V_i$$

and for $c\le c_i$ let

$$X_c^{(i)} = X'_c\sqcup_{Y_2^1} V_1\sqcup_{Y_2^2}\cdots\sqcup_{Y_2^i} V_i.$$

We suppose moreover that the extension of Ψ to $X_{c_{j-1}}^{(j-1)}$ is an embedding.

Using Lemma 2 and the argument in step 3 we show that Ψ embeds $X_{c_j}^{(j-1)}$. As noted in step 2, this implies that Y_2^j is embeddable and therefore bounds a normal Stein space V_j. Let $X^{(j)}=X^{(j-1)}\sqcup_{Y_2^j} V_j$, the mapping Ψ extends to $X^{(j)}$. As before $X^{(j)}$ is smooth in a neighborhood of bX_{c_j} and we can therefore repeat the argument in step 3 to conclude that Ψ embeds a neighborhood, in $X^{(j)}$ of $X_{c_j}^{(j-1)}$. As before this implies that $\Psi(Y_2^j)$ is embedded in V as the boundary of a normal Stein domain. Thus, by uniqueness $\Psi\big|_{V_j}$ is an embedding. Using a plurisubharmonic exhaustion of V_j and Step 3 we show that, in fact $\Psi\big|_{X_{c_j}^{(j)}}$ is an embedding. This completes the induction step and therefore the proof of the theorem. □

3. $\bar\partial$-EQUATION ON SINGULAR DOMAINS.

We now consider an extension of Theorem 1 which allows the bounding hypersurfaces, as well as the cobordism to have singularities. To prove these results we need versions of the regularity statements for the $\bar\partial$-equation on Stein subsets of complex *spaces*. Some of these statements are proved using L^2-methods, and others by kernel methods.

L^2-methods. Let W_+ be a relatively compact, open Stein subset in the Stein complex space W of dimension n. For a measurable function ψ defined on W_+ we denote by $H^{n,q}_{(2)}(W_+,e^{-\psi})$, $q=0,1,\dots,n$, the L_2-$\bar\partial$-cohomology spaces of W_+ with respect to the norm

$$\|f\|^2_{L^2(W_+,e^{-\psi})} = \int_{W_+} |f|^2 e^{-2\psi}dv.$$

Here dv is the volume form for the Kähler metric on Reg W_+, induced by an embedding of W in $\mathbb{C}^N$.

Definition 5. We say that $\alpha\in L^2_{0,n-q}(\text{Reg } W_+, e^{\psi})$ satisfies the (weak) Dirichlet boundary conditions for $\bar\partial$ if

$$\int_{\text{Reg } W_+} g\wedge\alpha = (-1)^{n+q}\int_{\text{Reg } W_+} f\wedge\bar\partial\alpha$$

for all $g\in L^2_{n,q}(\text{Reg } W_+, e^{-\psi})$ such that $g=\bar\partial f$ for some $f\in L^2_{n,q-1}(\text{Reg } W_+, e^{-\psi})$.

Let $H^{0,q}_{(2)\circ}(W_+, e^{\psi})$ denote L_2-$\bar\partial$-cohomology spaces of Reg W_+ with (weak) Dirichlet boundary conditions.

Proposition 1 (Version of Andreotti-Vesentini, L^2-estimate for $\bar\partial$). *For any pluri-subharmonic function ψ on W_+ we have*

(1) *The spaces $H^{n,q}_{(2)}(W_+, e^{-\psi})$ and $H^{0,n-q}_{(2)\circ}(W_+, e^{\psi})$ vanish for $q = 1, \dots, n$.*

(2) *The space $H^{0,n}_{(2)\circ}(W_+, e^{\psi})$ is dual to the space $H^{n,0}_{(2)}(W_+, e^{-\psi})$, the duality is realized by the pairing $\int_{W_+} g \wedge \alpha$, where $g \in H^{n,0}_{(2)}(W_+, e^{-\psi})$, $\alpha \in H^{0,n}_{(2)\circ}(W_+, e^{\psi})$.*

Remark 2. The first part of this proposition is an analogue, for Stein spaces of the L^2-version of the Kodaira vanishing theorem for projective varieties proved in [28].

Proof. Let ω be the (1,1)-form, associated with the Kähler metric on W. Let ρ be a continuous strictly plurisubharmonic function on W. Because $\bar W_+ \subset W$ there exists a constant $\sigma > 0$ such that, as currents, $i\partial\bar\partial\rho \geq \sigma\omega$ on W_+. Following Andreotti-Vesentini [4] and Demailly [9] we use the fact that Reg W_+ carries a complete Kähler metric and obtain that, for any $g \in L^2_{n,q}(W_+, e^{-\psi})$, $q = 1, \dots, n$, satisfying $\bar\partial g = 0$, there exists $f \in L^2_{n,q-1}(W_+, e^{-\psi})$ such that $\bar\partial f = g$ and

$$\int_{W_+} |f|^2 e^{-2(\psi+\rho)} dv \leq \frac{1}{\sigma} \int_{W_+} |g|^2 e^{-2(\psi+\rho)} dv.$$

Hence,

$$\|f\|_{L^2(W_+, e^{-\psi})} \leq \frac{1}{\sigma} \exp 2(\sup_{W_+} \rho - \inf_{W_+} \rho)\, \|g\|_{L^2(W_+, e^{-\psi})}.$$

This proves the vanishing statement for $H^{n,q}_{(2)}(W_+, e^{-\psi})$, $1 \leq q \leq n$.

The vanishing of $H^{n,q}_{(2)}(W_+, e^{-\psi})$, $q = 1, \dots, n$, and standard duality arguments (see, for example, §20 in [20]) $\alpha \in L^2_{0,n-q}(\text{Reg } W_+, e^{\psi})$, with $\bar\partial\alpha = 0$, $0 < n - q < n$, satisfying the (weak) Dirichlet boundary condition, there exists $\beta \in L^2_{0,n-q-1}(\text{Reg } W_+, e^{\psi})$ such that

$$\int_{\text{Reg } W_+} g \wedge \alpha = (-1)^{n+q+1} \int_{\text{Reg } W_+} \bar\partial g \wedge \beta$$

for all $g \in L^2_{n,q}(W_+, e^{-\psi})$ with $\bar\partial g \in L^2_{n,q+1}(W_+, e^{-\psi})$.

This means that $\alpha = \bar\partial\beta$, where $\beta \in L^2_{0,n-q-1}(\text{Reg } W_+, e^{\psi})$ and satisfies the (weak) Dirichlet boundary conditions. This shows that

$$H^{0,n-q}_{(2)\circ}(W_+, e^{-\psi}) = 0, \text{ for } 0 \leq n - q < n.$$

If $n - q = n$, then these arguments show that $\alpha = \bar\partial\beta$; $\beta \in L^2_{0,n-1}(\text{Reg } W_+, e^{\psi})$ and satisfies the Dirichlet boundary conditions, if and only if

$$\int_{\text{Reg } W_+} g \wedge \alpha = 0$$

for any L^2-holomorphic form: $g \in L^2_{n,0}(W_+, e^{-\psi})$, with $\bar\partial g = 0$. This implies that $H^{0,n}_{(2)\circ}(W_+, e^{\psi})$ is dual to $H^{n,0}_{(2)}(W_+, e^{-\psi})$. □

Kernel methods. Let X be an n-dimensional Stein space with at worst isolated singular points. Let ρ be a $\mathscr{C}^\infty$ strictly plurisubharmonic exhaustion function with at most isolated critical points defined on X, the definitions can be found in §1 of [15]. For a real number θ we let X_θ denote the strictly pseudoconvex, relatively compact subset of X defined by

$$X_\theta = \{x \in X \ : \ \rho(x) < \theta\},$$

As above we use the Riemannian metric induced on X by an embedding into $\mathbb{C}^N$.

Let $\pi \ : \ U_{x,\varepsilon} \to T_x(\text{Reg } X)$ denote the orthogonal projection of the ε-neighborhood $U_{x,\varepsilon}$ of the point $x \in \text{Reg } X$ on the tangent plane $T_x(\text{Reg } X)$. For $\alpha > 0$ let $C^\alpha_{0,q}(\bar{X}_\theta)$ denote the space of all those (0,q)-forms $f \in C^\alpha_{0,q}(\text{Reg } X_\theta)$, for which

$$\|f\|_{C^\alpha(\bar{X}_\theta)} = \sup_{x\in\text{Reg } X_\theta} \inf_{\varepsilon>0} \|\pi_* f\|_{C^\alpha_{0,q}(\pi(U_{x,\varepsilon}\cap X_\theta))} < \infty.$$

Let $A^\alpha_{0,q}(\bar{X}_\theta)$ denote the space of those forms $f \in C^\alpha_{0,q}(\bar{X}_\theta)$, which are $\bar\partial$-closed on Reg X_θ. For $Y \subset \bar{X}_\theta$ we denote by $C^\alpha_{0,q}(\bar{X}_\theta, Y)$ the space of those forms $f \in C^\alpha_{0,q}(\bar{X}_\theta)$, for which

$$\inf_{\varepsilon>0} \|\pi_* f\|_{C^\alpha_{0,q}(\pi(U_{x,\varepsilon}\cap X_\theta))} \to 0,$$

where $x \in \text{Reg } X_\theta$ and geodesic distance $(x, Y) \to 0$.

Let

$$A^\alpha_{0,q}(\bar{X}_\theta, Y) = C^\alpha_{0,q}(\bar{X}_\theta, Y) \cap A^\alpha_{0,q}(\bar{X}_\theta).$$

Proposition 2 (Regularity for $\bar\partial$ in strictly pseudoconvex domains)**.** *For any θ' and α' there exist $\gamma > 0$ and $\alpha > 0$ such that for all $\theta \leq \theta'$ and $q = 1, 2, \ldots, n$ one can construct a continuous linear operator*

$$R_{q,\theta} \ : \ A^\alpha_{0,q}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta) \to \mathscr{C}^{\alpha'}_{0,q-1}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta)$$

with the properties

$$\bar\partial R_{q,\theta} g = g \quad on \ \ \bar{X}_\theta \ \ \forall \ \ g \in A^\alpha_{0,q}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta)$$

and

$$\|R_{q,\theta} g\|_{\mathscr{C}^{\alpha'}(\bar{X}_\theta)} \leq \gamma \ \|g\|_{\mathscr{C}^{\alpha}(\bar{X}_\theta)}.$$

Remark 3. If X is smooth then in this Proposition one can take $\alpha = \alpha' - 1/2$ (see [19]).

Unfortunately, we can only prove Proposition 2 in parallel with the following Whitney type extension theorem.

A connected compact $K \subset \mathbb{R}^N$ is called (see [31]) ε-regular if $\exists\, c > 0$ and $\varepsilon > 0$ such that $\forall \ x, y \in K$ we have $|x - y|^\varepsilon \geq c\delta(x, y)$, where $\delta(\cdot, \cdot)$ is geodesic distance. From the classical Łojaciewicz inequality it follows that, for any θ, the compact set $\bar{X}_\theta$ is ε-regular for some $\varepsilon > 0$.

Proposition 3 (Version of Whitney extension theorem)**.** *Let the space X be properly embedded as a closed analytic set in $\mathbb{C}^N$, i.e.*

$$X = \{z \in \mathbb{C}^N \ : \ F_\nu(z) = 0, \ F_\nu \in \mathcal{O}(\mathbb{C}^N), \ \ \nu = 1, 2, \ldots, m\}.$$

Then for any $q = 0, 1, \ldots, n$ and any $\alpha' \geq 0$ there exists an $\alpha \geq 0$ and a continuous extension operator

$$E \ : \ A^\alpha_{0,q}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta) \to \mathscr{C}^{\alpha'}_{0,q}(\mathbb{C}^n, \text{Sing } \bar{X}_\theta)$$

such that $\bar\partial E g$ vanishes together with derivatives up to order α' on $\bar{X}_\theta \subset \mathbb{C}^N$ for any $g \in A^\alpha_{0,q}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta)$.

Proof. **Step 1.** Proposition 3 for given $q = r = 1, 2, \ldots, n$ implies Proposition 2 for the same $q = r$.

Let E be the extension operator from Proposition 3 for given q. Then for $g^0 \in A^{\alpha}_{0,r}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta)$ and for $Eg^0 \in \mathcal{C}^{\alpha'}_{0,r}(\mathbb{C}^n, \text{Sing } \bar{X}_\theta)$ we can apply the Propositions (2.2.1), (2.3.1) from [1], which show that $\forall \alpha'' \geq 0$ and $\forall \theta'$ there exist $\alpha' \geq \alpha''$, $\gamma > 0$ and a continuous linear operator

$$R : \mathcal{C}^{\alpha'}_{0,r}(\mathbb{C}^n, \text{Sing } \bar{X}_\theta) \longrightarrow \mathcal{C}^{\alpha''}_{0,r-1}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta) \tag{8}$$

such that

$$g^0\big|_{\bar{X}_\theta} = \bar{\partial} REg^0\big|_{\bar{X}_\theta} \text{ and} \tag{9}$$

$$\|REg^0\|_{\mathcal{C}^{\alpha''}_{0,r-1}(\bar{X}_\theta)} \leq \gamma \, \|g^0\|_{\mathcal{C}^{\alpha}_{0,r}(\bar{X}_\theta)}, \quad \theta \leq \theta'. \tag{10}$$

Step 2. Proposition 2 for given $q = r = 1, 2, \ldots, n$ implies Proposition 3 for $q = r - 1$.

To avoid non-essential technical details we will consider here only the case when X is embeddable in $\mathbb{C}^{n+1}$ as complex hypersurface, i.e. let

$$X = \{z \in \mathbb{C}^{n+1} : F(z) = 0\},\ F \in \mathcal{O}(\mathbb{C}^{n+1}), \tag{11}$$

$$X_\theta = \{z \in X : \rho(z) < \theta\}, \tag{12}$$

ρ is a strictly plurisubharmonic function and

$$\text{Sing } X = \{z \in X \ : \ dF(z) = 0\}.$$

We suppose that Sing $X \neq \emptyset$, otherwise the result is standard. Following the approach of Whitney, see [31, 32], we consider a locally uniformly, finite covering of Reg $X \subset \mathbb{C}^{n+1}$ by the polydiscs $D_j = D_j(z_j, r_j) \subset \mathbb{C}^{n+1}$ with centers at points $z_j \in$ Reg X and radii $r_j = \delta\,[\text{dist}\,(z_j, \text{Sing } X)]^\nu$.

Let

$$D_{j,\theta} = \{z \in D_j \ : \ \rho(z) < \theta\}.$$

If ν is large enough and δ is small then enough there is an orthogonal change of the coordinates $z \to \tilde{z}$ such that

$$D_{j,\theta} \cap X = \{\tilde{z} \in D_{j,\theta} \ : \ \tilde{z}_{n+1} = \tilde{F}(\tilde{z}_1, \ldots, \tilde{z}_n)\}.$$

In this open set $(\tilde{z}_1, \ldots, \tilde{z}_n)$ are local coordinates on X.

The form $g \in A^{\alpha}_{0,r-1}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta)$ restricted to $D_{j,\theta} \cap X$ can be represented in the form

$$g(\tilde{z}_1, \ldots, \tilde{z}_n) = \sum_{j_1 < j_2 < \ldots < j_{r-1} \leq n} g_{j_1,\ldots,j_{r-1}}(\tilde{z}_1, \ldots, \tilde{z}_n, \tilde{F}(\tilde{z}_1, \ldots, \tilde{z}_n))d\bar{\tilde{z}}_{j_1} \wedge \ldots \wedge \bar{\tilde{z}}_{j_{r-1}}.$$

We extend such g on $D_{j,\theta}$ to be independent of $\tilde{z}_{n+1}$, that is

$$(E_j g)(\tilde{z}_1, \ldots, \tilde{z}_{n+1}) = g(\tilde{z}_1, \ldots, \tilde{z}_n).$$

We obtain extension operators

$$E_j \ : \ A^{\alpha}_{0,r-1}(\bar{X}_\theta, \text{Sing } X_\theta) \to A^{\alpha}_{0,r-1}(\bar{D}_{j,\theta}),$$

with the properties

$$\|E_j g\|_{\mathcal{C}^{\beta}(\bar{D}_{j,\theta})} = O([\text{dist}\,(D_j, \text{Sing } X)]^{(\alpha-\beta)\varepsilon})\|g\|_{\mathcal{C}^{\alpha}(\bar{X}_\theta)}$$

for any $g \in A^{\alpha}_{0,r-1}(\bar{X}_\theta, \text{Sing } X_\theta)$, $\beta \le \alpha$. The constant $\varepsilon > 0$ corresponds to the ε-regularity of the compact subset $\bar{X}_\theta$.

Let $\{\chi_j\}$ be a partition of unity on a neighborhood $\mathcal{U}_\theta$ of Reg X_θ, subordinate to the covering $\cup B_j \supset \text{Reg } X_\theta$ and with the property $|D^\nu \chi_j| = O(r_j^{-\nu})$ for any derivative of any order ν of any function χ_j.

Following the standard cohomological construction and the fact that the ideal of X is generated by F, we introduce the following forms:

$$(E_j g - E_k g)\big|_{D_{j,\theta}\cap D_{k,\theta}} = g_{j,k}F,$$
$$\tilde{g}_j\big|_{D_{j,\theta}} = \sum_k \chi_k g_{j,k}\big|_{D_{j,\theta}} \text{and} \tag{13}$$
$$\psi = \{\bar\partial \tilde{g}_j = \sum_k \bar\partial \chi_k g_{j,k},\ z \in D_{j,\theta}\}.$$

We have obtained a form $\psi \in A^{\beta}_{0,r}(\bar{\mathcal{U}}_\theta, \text{Sing } \bar{X}_\theta)$ with $\beta = O(\alpha\varepsilon/\nu)$.

Proposition 2 implies that there exists a form $R\psi \in C^{\beta'}_{0,r-1}(\bar{\mathcal{U}}_\theta, \text{Sing } \bar{X}_\theta)$ such that $\psi = \bar\partial R\psi$ on $\bar{X}_\theta$ together with derivatives up to order β'. We can define now the necessary extension operator by the formula

$$Eg = \{E_j g - (\tilde{g}_j - R\psi)F,\ \ z \in D_{j,\theta}\}.$$

Step 3. Proof of Proposition 3 for $q = n$.

In this case Proposition 3 can be proved the same way as in Step 2. The reference to Proposition 2 need only be replaced by the classical statement (see, for example, §11 in [20]) about $\mathcal{C}^\beta$-solvability of the $\bar\partial$-equation $\bar\partial f = \psi$ on $\bar{\mathcal{U}}_\theta \subset \mathbb{C}^{n+1}$ for the case $\bar\partial$-closed (0,q)-form ψ of maximal degree $q = n + 1$.

Propositions 2 and 3 follow by recurrence: at first Step 3 + Step 1 + Step 2 for $q = n$, after Step 1 + Step 2 for $q = n - 1$, etc..., Step 1 + Step 2 for $q = 1$. □

Proposition 2 has several important consequences.

Proposition 4 (Versions of Hartogs-Lewy extension theorem)**.** *Under the hypotheses of Proposition 2 let $\tilde\rho$ be a plurisubharmonic function on $X \supset X_0$ and $Y = \{x \in X\ :\ \tilde\rho(x) < 0\}$. Then for given $\alpha' > 0$ there exists $\alpha > 0$ such that: If $\tilde f \in \mathcal{C}^\alpha(\bar Y)$ and $\bar\partial \tilde f$ vanishes on $\overline{Y \cap bX_0}$ together with all derivatives up to order α then there exists $\tilde F \in \mathcal{C}^{\alpha'}(\overline{Y \cap X_0})$ with the properties: $\tilde F \in \mathcal{O}(Y \cap X_0)$ and $\tilde F - \tilde f$ vanishes on $\overline{Y \cap bX_0}$ together with all derivatives up to order α'.*

Remark 4. If X and bX_0 are smooth then in this Proposition one can take $\alpha = \alpha'$ (see [8]).

Proof. Let χ be such C^∞-function on X that $\chi = 1$ on a neighborhood of bX_0 and $\chi = 0$ on a neighborhood of Sing X_0. Let

$$g = \begin{cases} \bar\partial(\chi \tilde f) & \text{on } \overline{Y \cap X_0} \\ 0 & \text{on } \bar Y \setminus (X_1 \setminus \bar X_0). \end{cases} \tag{14}$$

We have $g \in \mathcal{C}^{\alpha-1}_{0,1}(\overline{Y \cap X_1}, \text{Sing } \overline{Y \cap X_1})$ and $\bar\partial g = 0$.

Proposition 2 implies the existence of $u \in \mathcal{C}^{\alpha'}(\overline{Y \cap X_1})$ such that $\bar\partial u = g$. If we set

$$\tilde F_1 = \tilde f - u \text{ on } \overline{Y \cap X_0} \text{ and}$$
$$\tilde f_2 = -u \text{ on } \overline{Y \cap (X_1 \setminus X_0)}, \tag{15}$$

then

$$\tilde{F}_1 \in \mathscr{C}^{\alpha'}(\overline{Y \cap X_0}) \cap \mathcal{O}(Y \cap X_0) \text{ and}$$
$$\tilde{f}_2 \in \mathscr{C}^{\alpha'}(\overline{Y \cap (X_1 \setminus X_0)}) \cap \mathcal{O}(Y \cap (X_1 \setminus \bar{X}_0)). \tag{16}$$

Applying the Hartogs-Levi extension theorem on complex spaces ([2], Proposition 15) to the function $\tilde{f}_2$ we obtain a holomorphic function $\tilde{F}_2 \in \mathcal{O}(Y \cap X)$ such that $\tilde{F}_2 = \tilde{f}_2$ on $Y \cap (X_1 \backslash \bar{X}_0)$. Hence, function $\tilde{F} = \tilde{F}_1 - \tilde{F}_2$ has the necessary properties $\tilde{F} \in \mathscr{C}^{\alpha'}(\overline{Y \cap X_0}) \cap \mathcal{O}(Y \cap X_0)$ and $\tilde{F} - \tilde{f}$ vanishes on $\overline{Y \cap bX_0}$ together with all derivatives up to order α'.

□

Proposition 5 (Regularity for $\bar{\partial}$ in strictly pseudoconcave domains)**.** *Under the hypotheses of Proposition 2, let Ω_0 be strictly pseudoconvex neighborhood in X of a point $x^0 \in bX_0$. Then for some smaller neighborhood D_0 of x^0 and for any $\alpha' \geq 0$ there exist $\alpha \geq \alpha'$ and $\gamma > 0$ such that for any $h \in C(\overline{\Omega_0 \backslash X_0})$ with the property $f = \bar{\partial} h \in \mathscr{C}^{\alpha}_{0,1}(\overline{\Omega_0 \backslash X_0}, \text{Sing } \overline{\Omega_0 \backslash X_0})$ the following estimate is valid*

$$\|h\|_{\mathscr{C}^{\alpha'}(\overline{D_0 \backslash X_0})} \leq \gamma \ (\|\bar{\partial} h\|_{\mathscr{C}^{\alpha}_{0,1}(\overline{\Omega_0 \backslash X_0})} + \|h\|_{C(\overline{\Omega_0 \backslash X_0})}).$$

Proof. Let the space X be embedded as closed analytic subset in $\mathbb{C}^N$ such that

$$\begin{aligned} X &= \{z \in \mathbb{C}^N \ : \ F_\nu(z) = 0, \ \ \nu = 1, \dots, m\}, \\ \Omega_0 &= \{z \in X \ : \ \rho_0(z) < 0\}, \\ X_0 &= \{z \in X \ : \ \rho(z) < 0\}, \end{aligned} \tag{17}$$

where $F_\nu \in \mathcal{O}(\mathbb{C}^N)$, ρ and ρ_0 are smooth strictly plurisubharmonic functions on $\mathbb{C}^N$.

For the manifold $\overline{\Omega}_0 \backslash X_0$ we apply the integral formula, (2.2.16) from Proposition 2.2.1 of [1], using in it the barriers functions for domains X_0 and Ω_0 from Propositions 2.3.1, 3.3.1 of [1].

We obtain for $f = \bar{\partial} h \in \mathscr{C}^{\alpha}_{0,1}(\overline{\Omega}_0 \backslash X_0, \text{Sing } \overline{\Omega}_0 \backslash X_0)$ the integral representation of the form

$$f = \bar{\partial} \tilde{R} f + \tilde{K} f,$$

where $\tilde{R} f \in \mathscr{C}^{\alpha'}(\overline{\Omega}_0 \backslash X_0, \text{Sing } \overline{\Omega}_0 \backslash X_0)$ and $\tilde{K} f = \tilde{\tilde{K}} h \in \mathscr{C}^{\infty}_{0,1}(\bar{D}_0, \text{Sing } \bar{D}_0)$ for a sufficiently small neighborhood $D_0 \subset\subset \Omega_0$ of point $x^0 \in bX_0$ and $h \in \mathscr{C}_{0,1}(\overline{\Omega}_0 \backslash X_0)$.

Applying Proposition 2 to the $\bar{\partial}$-closed form $\tilde{\tilde{K}} h$ on D_0 we obtain the representation $f\big|_{D_0} = \bar{\partial} \tilde{\tilde{R}} f$, where $\tilde{\tilde{R}} f \in \mathscr{C}^{\alpha'}(\bar{D}_0)$.

To finish the proof we remark that $h\big|_{D_0} = \tilde{\tilde{R}} f + \tilde{h}$, where $\tilde{h} \in \mathcal{O}(D_0)$. □

4. Complex cobordisms on analytic spaces

Let ρ be a $\mathscr{C}^\infty$ strictly plurisubharmonic function with at most isolated critical points on the (almost) complex space X of dimension 2 with at most isolated singularities.

Definition 6. A compact oriented subset M in an (almost) complex space X of the form $M = bX_+ = -bX_-$, where $X_\pm = \{x \in X \ : \ \pm\rho(x) < 0\}$ will be called a strictly pseudoconvex CR-hypersurface. Such a CR-hypersurface will be called CR-embeddable in complex affine space $\mathbb{C}^N$ if for any $\alpha \geq 1$ there exists a real $\mathscr{C}^\alpha$-embedding $\Phi : X \to \mathbb{C}^N$ with the property: $\bar{\partial}\Phi$ vanishes on M together with all derivatives up to order $\alpha - 1$.

If the domains $X_{\pm}$ are relatively compact in X then they are called respectively strictly pseudoconvex and strictly pseudoconcave domains in X. By a real $\mathscr{C}^{\alpha}$-embedding $\Phi : X \to \mathbb{C}^N$ we mean the restriction to X of a $\mathscr{C}^{\alpha}$-embedding $\tilde{\Phi} : Z \to \mathbb{C}^N$ for some ambient smooth manifold $Z \supset X$.

Definition 7. A compact CR-hypersurface M_0 is called strictly CR-cobordant to a compact CR-hypersurface M_1 if there exists an (almost) complex space $\tilde{X}$ with at most isolated singularities and a $\mathscr{C}^{\infty}$-strictly plurisubharmonic function ρ with at most isolated critical points on $\tilde{X}$ such that the set $X = \{x \in \tilde{X} : 0 < \rho(x) < 1\}$ is a relatively compact, complex subspace in $\tilde{X}$ and $bX = M_1 - M_0$.

Theorem 2. *Let M_1 be embeddable strictly pseudoconvex CR-hypersurface. Then any (not necessary smooth) CR-hypersurface M_0, strictly cobordant to M_1, is also embeddable.*

Corollary 2. *If under the hypothesis of Theorem 2 a complex space X defines complex cobordism between M_1 and M_0, then X is embeddable in $\mathbb{C}^N$.*

Remark 5. This Corollary implies, in particular, the embeddings results of Grauert, R. Narasimhan, Andreotti and Y.-T.Siu for compact, pseudoconvex and pseudoconcave two-dimensional complex spaces, respectively, see [15, 27, 3].

The first small step in the proof of this theorem is the following.

Proposition 6. *Let X be a relatively compact, complex subspace in an (almost) complex space $\tilde{X}$ with at most isolated singularities such that*

$$X = \{x \in \tilde{X} : 0 < \rho(x) < 1\},$$

where ρ is a strictly plurisubharmonic function with at most isolated critical points on $\tilde{X}$. If the CR-hypersurface $M_1 = \{x \in \tilde{X} : \rho(x) = 1\}$ is CR-embeddable, then there exists $\theta_1 < 1$ such that the space $\{x \in X : \theta_1 < \rho(x) < 1\}$ is holomorphically embeddable in $\mathbb{C}^N$.

Proof. Let $\Phi : \tilde{X} \to \mathbb{C}^N$ be a real $\mathscr{C}^{\alpha}$-embedding with the property: $\bar{\partial}\Phi$ vanishes on M_1 together with derivatives up to order $\alpha - 1$. From Proposition 4 it follows that for any $\alpha' > 0$ there exist $\alpha \geq \alpha'$ and another mapping $\tilde{\Phi} : \tilde{X} \to \mathbb{C}^N$ such that $\tilde{\Phi} \in \mathscr{C}^{\alpha'}(\tilde{X})$, $\tilde{\Phi} = \Phi$ on $\{x \in \tilde{X} : \rho(x) > 1\}$ and $\tilde{\Phi}\big|_X$ is holomorphic.

From these properties it follows that if $\alpha' \geq 1$, then the mapping $\tilde{\Phi}$ is regular at any point of M_1 in the sense of §1 in [15] and is embedding on $\{x \in \tilde{X} : \rho(x) > 1\}$. Hence by Andreotti's proposition (see §1, [15]) it follows that there exists $\theta_1 < 1$ such that the mapping $\tilde{\Phi}$ is $\mathscr{C}^{\alpha'}$-real embedding of $\{x \in \tilde{X} : \rho(x) > \theta_1\}$ and holomorphic embedding of $\{x \in \tilde{X} : \theta_1 < \rho(x) < 1\}$. The Proposition is proved. □

Definition 8. A form $f \in C^{\alpha}_{0,q}(bX_\theta)$ is called a CR-form (on the given space) if

$$\bar{\partial}_\tau f\big|_{\text{Reg } bX_\theta} = 0,$$

where $\bar{\partial}_\tau$ is the tangential Cauchy-Riemann operator.

The second step in the proof of the Theorem 2 is the following.

Proposition 7. *[Version of Boutet de Monvel embedding theorem] Under the hypotheses of Proposition 6, let $M = \{x \in \tilde{X} : \rho(x) = 0\}$ and $C^{\perp\beta}_{0,1}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta)$ be the space of those $f \in C^{\beta}_{0,1}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta)$, which are $\bar{\partial}$-closed on* Reg X_θ *and $\bar{\partial}_\tau$-exact on M. Then M*

is embeddable in a complex affine space if for any $\theta_0 > 0$ *and for any* $\alpha > 0$ *there exists* $\theta < \theta_0$ *and* $\beta \geq \alpha$*, a constant* $\gamma > 0$ *and a linear operator*

$$T_\theta : C^{\perp\beta}_{0,1}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta) \to C^\alpha(\bar{X}_\theta)$$

such that $\bar{\partial} T_\theta f = f$ *on* Reg X_θ *and*

$$\|T_\theta f\|_{\mathcal{C}^\alpha(\bar{X}_\theta)} \leq \gamma \|f\|_{\mathcal{C}^\beta_{0,1}(\bar{X}_\theta)} \ \forall \ f \in C^{\perp\beta}_{0,1}(\bar{X}_\theta, \text{Sing } \bar{X}_\theta).$$

Remark 6. For a smooth hypersurface M this statement first appeared in [6] as an interpretation of the results in[5].

Proof. We only give the proof for the case of $\tilde{X}$ a complex space. Let $p \in M$ and Ω be a neighborhood of p in $\tilde{X}$, for which there exists a holomorphic embedding $Z : x \mapsto z(x) = \{z_1(x), \ldots, z_N(x)\}$ of Ω in a neighborhood $B = \{z \in \mathbb{C}^N : |z| < 1\}$ of $0 \in \mathbb{C}^N$.

The coordinates can be chosen so that $z(p) = 0$ and

$$\tilde{M} = Z(M \cap \Omega) = \{z \in \mathbb{C}^N : |z| < 1, \ \tilde{\rho}(z) = 0, \ F_\nu(z) = 0, \ \nu = 1, \ldots, m\},$$

where $F_\nu \in \mathcal{O}(\mathbb{C}^N)$, $\tilde{\rho}$ is a strictly plurisubharmonic function on B, $\tilde{\rho}(z(x)) = \rho(x)$, $x \in \Omega$.

Let g be the Levi polynomial for $\tilde{\rho}(z)$ at zero. The polynomial g has the properties:

$$g \in \mathcal{O}(B), \ g(0) = 0, \ \text{Re } g \geq c|z|^2, \ c > 0, \ \forall z \in \tilde{M}.$$

Let χ be a function with compact support in B such that $\chi \equiv 1$ on $(1/2)B = \{z \in \mathbb{C}^N : |z| < 1/2\}$ and $d\chi = 0$ in a neighborhood of Sing $\tilde{M}$.

Let us consider now the following sequence of smooth $\bar{\partial}$-exact $(0,1)$-forms on M

$$f_k(x) = \begin{cases} \bar{\partial}[\chi(z(x)) \exp(-kg(z(x)))], & x \in \Omega \cap X \\ 0, & x \in X \backslash \overline{\Omega}. \end{cases}$$

For sufficiently small θ_0 we have $f_k \to 0, k \to \infty$, in $\mathcal{C}^\infty(X_{\theta_0}, \text{Sing } X_{\theta_0})$.

For some $\theta < \theta_0$, the properties of the operator T_θ imply that the functions $u_k = T_\theta f_k \to 0$ in $\mathcal{C}^\alpha(\bar{X}_\theta)$ as $k \to \infty$, and $\bar{\partial} u_k = f_k$ on X_θ. Hence the functions $h_k = \chi \exp(-kg) - u_k$ are CR-functions on M with the properties $h_k(p) \to 1$, $k \to \infty$, and $h_k(x) \to 0$, $k \to \infty$, $\forall x \in M \backslash \{p\}$. Because p is arbitrary point of M, we have obtained that $\forall \alpha > 0$ the CR-functions of class $\mathcal{C}^\alpha(M)$ separate the points of M.

To finish the proof we must now find for every $\alpha \geq 1$ and for an arbitrary point $p \in M$ a CR-mapping $x \mapsto \tilde{z}(x)$, $x \in M$, which can be extended to a neighborhood D of p as real $C^{(\alpha)}$-embedding with the property: $\bar{\partial}\tilde{z}$ vanishes on $D \cap M$ together with all derivatives up to order $\alpha - 1$.

For this let us consider another sequence of $\bar{\partial}$-exact forms,

$$f_{k,j} = \begin{cases} \bar{\partial} z_j(x) \chi(z(x)) \exp(-kg(z(x)), & x \in \Omega \\ 0, & x \in X \backslash \overline{\Omega}, \end{cases}$$

$k = 1, 2, \ldots; \ j = 1, 2, \ldots, N$.

For sufficiently small θ_0 we have that $f_{k,j} \to 0$, $k \to \infty$, in $\mathcal{C}^\infty(\bar{X}_{\theta_0}, \text{Sing } \bar{X}_{\theta_0})$. For some $\theta < \theta_0$ the properties of the operator T_θ imply that the functions $u_{k,j} = T_\theta f_{k,j} \to 0$ in $\mathcal{C}^\alpha(\bar{X}_\theta)$ as $k \to \infty$.

Let us prove that for k large enough the CR-functions

$$z^{(k)}_j(x) = z_j(x) \chi(z(x)) \exp(-kg(z(x))) - u_{k,j}(x), \ \ x \in M,$$

$j = 1, 2, \ldots, N$, give the necessary CR-mapping $x \mapsto \tilde{z}^{(k)}(x)$, $x \in M$.

Because by construction for some $\Omega_0 \subset \Omega$, $p \in \Omega_0$, we have $f_{k,j}\big|_{X\cap\Omega_0} = 0$, the functions $u_{k,j}$ are holomorphic on $X \cap \Omega_0$ such that

$$\|u_{k,j}\|_{\mathscr{C}^\alpha(\overline{X\cap\Omega_0})} \to 0, \quad k \to \infty.$$

By the Hartogs-Levi theorem on a complex space see [2] there exists a smaller neighborhood $D_0 \subset \Omega_0$ of the point $p \in M$ such that functions the $u_{k,j}$ have holomorphic extensions

$$u^+_{k,j} \ \text{in} \ D^+_0 = \{x \in D_0 \ : \ \rho(x) < 0\} \quad \text{and} \tag{18}$$

$$\|u^+_{k,j}\|_{\mathscr{C}^{\alpha'}(\bar{D}^+_0)} \to 0, \quad \text{as } k \to \infty. \tag{19}$$

The compact set $\bar{D}^+_0$ is ε-regular for some $\varepsilon > 0$. By results of Whitney [32] and Tougeron [31] the functions $u^+_{k,j}$ can be extended as functions $\tilde{u}_{k,j}$, in the first instance defined D_0 and thence to the ambient domain $\tilde{D}_0 \subset \mathbb{C}^N$ such that for some $\varepsilon > 0$

$$\|\tilde{u}_{k,j}\|_{\mathscr{C}^{\varepsilon\alpha'}(\tilde{D}_0)} \to 0, \quad k \to \infty.$$

Let us suppose that α is so large that $\varepsilon\alpha' \geq 1$. Then for the functions

$$\tilde{z}^{(k)}_j = z_j \chi \exp(-kg) - \tilde{u}_{k,j}$$

we have

$$\frac{\partial \tilde{z}^{(k)}_j}{\partial z_i}(p) \to \delta_{ij}, \quad k \to \infty,$$

where $\delta_{ij} = 1$, if $i = j$, $\delta_{ij} = 0$, if $i \neq j$, $i, j = 1, 2, \ldots, N$.

Hence for k large enough the mapping $x \mapsto \tilde{z}^{(k)}(x)$ gives a real $\mathscr{C}^{\varepsilon\alpha'}$-embedding of some (sufficiently small) neighborhood of p in X and besides $\bar{\partial}\tilde{z}^{(k)}$ vanishes on $D_0 \cap M$ together with all derivatives up to order $\varepsilon\alpha' - 1$. □

The main step in the proof of Theorem 2 is the following.

Proposition 8. *Let X be a relatively compact domain in the complex space $\tilde{X}$ of dimension 2 with at most isolated singularities such that $bX = M_1 - M_0$, where M_0 is a strictly pseudoconvex CR*-variety *and M_1 is a strictly pseudoconvex CR-manifold. If there exists a holomorphic embedding $\varphi : X \cup M_1 \to \mathbb{C}^N$, then the variety M_0 is also CR-embeddable.*

For the proof of Proposition 8 we need several lemmas.

Let X be as in Proposition 8 and $\varphi : X \cup M_1 \to \mathbb{C}^N$ an holomorphic embedding. By the result of Rossi [30] there exists a Stein space W with isolated singularities and smooth strictly pseudoconvex boundary bM_1 embedded in $\mathbb{C}^N$ such that

$$W_- = \varphi(X) \subset W \quad \text{and} \quad W_+ = W \backslash \bar{W}_-$$

is relatively compact domain in W.

From the concavity of X near M_0 and the Hartogs-Levi extension theorem on complex spaces [2] it follows that the holomorphic mapping φ has a holomorphic extension to the neighborhood $\mathscr{U}_0$ of M_0 in $\tilde{X}$.

Let ρ_0 be a smooth, strictly plurisubharmonic function, defined in $\mathscr{U}_0$ such that

$$M_0 = \{x \in \mathscr{U}_0 \ : \ \rho_0(x) = 0\} \quad \text{and} \quad \rho_0 > 0 \quad \text{on} \quad \mathscr{U}_0 \cap X.$$

Let G be the analytic, exceptional subset of those $x \in X \cup \mathscr{U}_0$, for which $x \in \operatorname{Sing} X$ or $x \in \operatorname{Reg} X$, but rank of $d\varphi(x) < 2$. From injectivity of φ on X and from maximum principle for $\rho_0\big|_G$, it follows that exceptional set G must be a finite set.

Lemma 3. *W_+ is a Stein open subset in the space W.*

Proof. For $\varepsilon > 0$ we consider

$$X_\varepsilon^- = X \backslash \{x \in \mathcal{U}_0 \ : \ \rho_0(x) \le \varepsilon\},$$
$$W_{\varepsilon-} = \varphi(X_\varepsilon^-) \ \text{and} \ W_{\varepsilon+} = W \backslash \bar{W}_{\varepsilon-}. \tag{20}$$

For all sufficiently small $\varepsilon > 0$, the set $W_{\varepsilon+}$ is a domain with strictly pseudoconvex boundary in the Stein space W. Hence, $W_{\varepsilon+}$ and also $W_+ = \operatorname{int} \cap_{\varepsilon>0} W_{\varepsilon+}$ are also Stein. □

Define the functions

$$\rho(z) = \sum_{z^* \in \varphi(G)} \ln |z - z^*| \ \text{and} \ r = e^\rho.$$

Let $\Lambda^1_{0,q}(\bar{W}_\pm)$ be the spaces of (0,q)-forms on $\bar{W}_\pm$ with coefficients in the space of Lipschitz functions. For real numbers $\nu_\pm$ we define the spaces

$$\Lambda^{1,\nu_\pm}_{0,q}(\bar{W}_\pm) = r^{-\nu_\pm} \Lambda^1_{0,q}(\bar{W}_\pm).$$

Lemma 4. *For the given mapping $\varphi \ : \ X \to W_- \subset \mathbb{C}^N$ there exists $\nu_- \ge 0$ such that the operator*

$$\varphi_* \ : \ C^{(1)}_{0,1}(\bar{X}) \to \Lambda^{1,\nu_-}_{0,1}(\bar{W}_-)$$

is continuous.

Proof. Using the Łojasiewicz inequality the authors obtained in [12] the following estimate: There are positive constants c, A such that

$$|\varphi(x) - \varphi(y)| \ge c d(x,y)[d(x,G) + d(y,G)]^A, \ \ x, y \in \bar{X}.$$

Here $d(\cdot,\cdot)$ denotes the distance on $\tilde{X}$, measured with respect to a Riemannian metric on $\tilde{X}$.

It follows from this estimate that there exists a $\nu > 0$ such that if $f \in \mathcal{C}^1(\bar{X})$ and $|\nabla f(p)| \le c[d(p,G)]^\nu$, then $\varphi_* f(w) = f(\varphi^{-1}(w))$ belongs to $\Lambda^1(\varphi(\bar{X}))$. So for any $f \in \mathcal{C}^1(\bar{X})$ we have obtained an estimate of the form

$$|\varphi_* f|_{\Lambda^1(\varphi(\bar{X}))} \le c\big(\|f\|_{C(X)} + \sup_{p\in X} \frac{|\nabla f(p)|}{[d(p,G)]^\nu}\big).$$

Using Cramer's rule and the Łojasiewicz inequality one can show that if $f \in \mathcal{C}^1_{0,1}(\bar{X})$ vanishes to high enough order on G then we can represent f in terms of $\varphi^*(d\bar{z}_j)$, $j = 1, 2, \dots, N$, with $\mathcal{C}^1$-coefficients, vanishing to any specified order on G. □

Using the Lipschitz extension theorem from [25] and [32] it follows that, for the given $W_\pm$ and $\nu_- \ge 0$, there exists $\nu_+ \ge 0$ and a continuous linear extension operator

$$\mathcal{E}_+ \ : \ \Lambda^{1,\nu_-}_{0,1}(\bar{W}_-) \to \Lambda^{1,\nu_+}_{0,1}(\bar{W}_+). \tag{21}$$

There exists also $\mu_+ \ge 0$ such that the operator

$$\bar{\partial} \ : \ \Lambda^{1,\nu_-}_{0,1}(\bar{W}_+) \to L^2_{0,2}(W_+, e^{\mu_+\rho})$$

is continuous.

Let $C^{\perp s}_{0,1}(\bar{X})$ be the space of s-times differentiable $(0,1)$-forms on $\bar{X}$, which are $\bar{\partial}$-closed on $\bar{X}$ and $\bar{\partial}_\tau$-exact on M_0.

Lemma 5. *For any* $f \in C^{\perp 1}_{0,1}(\bar{X})$*, the form* $b_+ = \bar{\partial}\mathscr{E}_+\varphi_* f$ *belongs to* $L^2_{0,2}(W_+, e^{\mu+\rho})$ *and satisfies the following orthogonality property*

$$\int_{W_+} b_+ \wedge h = 0 \;\; \forall\, h \in H^{2,0}_{(2)}(W_+, e^{-\mu+\rho}). \tag{22}$$

Proof. Let us fix $\varepsilon > 0$ and $h_\varepsilon \in H^{2,0}_{(2)}(W_{\varepsilon +}, e^{-\mu+\rho})$. We have

$$\begin{aligned}&\int_{W_{\varepsilon+}} b_+ \wedge h_\varepsilon = \\ &\int_{W_{\varepsilon+}} \bar{\partial}\mathscr{E}_+ g_- \wedge h_\varepsilon \overset{L^2\text{Stokes}}{=} \int_{bW_{\varepsilon+}} \mathscr{E}_+ g_- \wedge h_\varepsilon = \\ &\int_{bW_{\varepsilon+}} g_- \wedge h_\varepsilon = \int_{M_\varepsilon} \varphi^* g_- \wedge \varphi^* h_\varepsilon = \int_{M_\varepsilon} f \wedge \varphi^* h_\varepsilon .\end{aligned} \tag{23}$$

To prove that the last integral is equal to zero we remark that the property $h_\varepsilon \in H^{2,0}_{(2)}(W_{\varepsilon+}, e^{-\mu+\rho})$ implies that the form $\varphi^* h_\varepsilon \in H^{2,0}_{(2)}((X \backslash X^-_\varepsilon)\backslash G)$. An L^2-holomorphic form of maximal degree on a complex space has an holomorphic extension through analytic singularities.

From Proposition 1, using the approximation arguments in §10 of [20], which are in turn based on the solution of Cousin's problem with estimates in $W_{\varepsilon+}$, it follows that $\forall\, h \in H^{2,0}_{(2)}(W_+, e^{-\mu+\rho})$ one can find $h_{\varepsilon_j} \in H^{2,0}_{(2)}(W_{\varepsilon_j +}, e^{-\mu+\rho})$ such that $h_{\varepsilon_j} \to h$ in $H^{2,0}_{(2)}(W_+, e^{-\mu+\rho})$, $\varepsilon_j \to 0$. Hence $\varphi^* h\big|_{M_0}$ is $\bar{\partial}_\tau$-closed form in the distribution sense on M_0. This means that

$$\int_{M_0} f \wedge \varphi^* h = \int_{M_0} \bar{\partial}_\tau \alpha \wedge \varphi^* h = 0.$$

□

Let $L^{\perp 2}_{0,2}(W_+, e^{\mu+\rho})$ denote the subspace in $L^2_{0,2}(W_+, e^{\mu+\rho})$ consisting of the forms with the property (22). Proposition 1 and Lemma 3 imply the following lemma.

Lemma 6. *There exists an operator*

$$T_+ \; : \; L^{\perp 2}_{0,2}(W_+, e^{\mu+\rho}) \to L^2_{0,1}(W_+, e^{\mu+\rho})$$

such that for any $b_+ \in L^{\perp 2}_{0,2}(W_+, e^{\mu+\rho})$ *we have*

$$\|T_+ b_+\|_{L^2(W_+, e^{\mu+\rho})} \le const\, \|b_+\|_{L^2(W_+, e^{\mu+\rho})}, \tag{24}$$

$$T_+ b_+\big|_{bW_+} = 0 \quad \text{in the} \quad L^2 \quad \text{distribution sense}, \tag{25}$$

$$\bar{\partial} T_+ b_+ = b_+ \quad \text{on} \quad W_+. \tag{26}$$

We also need the following version of L^2-solvability for the $\bar{\partial}$-equation on a Stein space with isolated singularities.

Lemma 7. *For any* $\mu \ge 0$ *there exists a continuous operator*

$$T \; : \; L^2_{0,1}(W, e^{\mu\rho}) \to L^2(W, e^{\mu\rho})$$

such that $f = \bar{\partial} T f$ *on* Reg W *for any* f *from a finite codimensional subspace in the space*

$$\{ f \in L^2_{0,1}(W, e^{\mu\rho}) \; : \; \bar{\partial} f = 0 \quad \text{on} \quad \operatorname{Reg} W\}. \tag{27}$$

Proof. From Proposition 1 it follows that the space $H^{0,1}_{(2)\circ}(W, e^{\mu\rho}) = 0$. To prove the lemma it is sufficient to check the following statement:

(28) For the elements f of a finite-codimensional subspace of the space (27),

the restrictions $f\big|_{bW}$ are $\bar{\partial}_\tau$ − exact on bW.

By Theorem 10.3 from [18], the form $f\big|_{bW}$ is $\bar{\partial}_\tau$-exact if and only if

$$\int_{bW} f \wedge h = 0 \;\; \forall\, h \in H^{2,0}(bW) \cap \mathscr{C}^\infty_{2,0}(bW).$$

From the generalized Hartogs-Levi theorem [2] it follows that the space

$$H^{2,0}_{(2)}(W, e^{-\mu\rho}) \cap \mathscr{C}^\infty_{2,0}(\bar{W})$$

can be considered as a finite co-dimensional subspace of the space $H^{2,0}(bW) \cap \mathscr{C}^\infty_{2,0}(bW)$. For any $h \in H^{2,0}_{(2)}(W, e^{-\mu\rho}) \cap \mathscr{C}^\infty_{2,0}(\bar{W})$ the equality $\int_{bW} f \wedge h = 0$ follows from Stokes formula. The statement mentioned above is verified and Lemma 7 is proved. □

Proof. Now we complete the proof of Proposition 8. Let $\tilde{f} \in C^{\perp\alpha}_{0,1}(\bar{X}, \text{Sing } \bar{X})$. Lemma 4 implies that $g_- = \varphi_* \tilde{f} \in \Lambda^{1,\nu_-}_{0,1}(\bar{W}_-)$. Applying Lemma 5 for $g_- \in \Lambda^{1,\nu_-}_{0,1}(W_-)$ we obtain that form $\bar{\partial}\mathscr{E}_+ g_-$ belongs to $L^{\perp 2}_{0,2}(W_+, e^{\mu_+\rho})$, where operator $\mathscr{E}_+$ is defined by (21).

Let T_+ be an operator defined in Lemma 6. Then the following operator $g_- \mapsto E_+ g_- = \mathscr{E}_+ g_- - T_+(\bar{\partial}\mathscr{E}_+ g_-)$ has the properties

$$E_+ g_- \in L^2_{0,1}(W_+, e^{\mu_+\rho}), \quad E_+ g_-\big|_{bW_-} = g_-\big|_{bW_-}$$

and $\bar{\partial} E_+ g_- = 0$ on W_+. Let us set

$$g = \begin{cases} E_+ g_- & \text{for } \; z \in W_+ \\ g_- & \text{for } \; z \in W_-. \end{cases} .$$

Then $g \in L^2_{0,1}(W, e^{\mu\rho})$, where $\mu = \max(\mu_+, \mu_-)$ and $\bar{\partial} g = 0$ on Reg W.

By Lemma 7 for any g from the finite-codimensional subspace $B^2_{0,1}(W, e^{\mu\rho})$ of the space

$$\{g \in L^2_{0,1}(W, e^{\mu\rho}) \; : \; \bar{\partial} g = 0 \text{ on Reg } W\}$$

we have $g = \bar{\partial} T g$, where $T : L^2_{0,1}(W, e^{\mu\rho}) \to L^2(W, e^{\mu\rho})$ is continuous linear operator. Hence for $\tilde{f}$ from the finite-codimensional subspace

$$\varphi^* B^2_{0,1}(W, e^{\mu\rho}) \cap C^{\perp 1}_{0,1}(\bar{X}, \text{Sing } \bar{X}) \subset C^{\perp 1}_{0,1}(\bar{X}, \text{Sing } \bar{X})$$

we have

$$\tilde{f} = \varphi^* g = \bar{\partial} T \varphi^* g = \bar{\partial} R \tilde{f},$$

where the function $R\tilde{f}(x) = Tg(\varphi(x))$ is continuous on $X \backslash G$ and has at most polynomial growth near $M_0 \cup G$.

From the concavity of the variety X in a neighborhood $\mathscr{U}_0$ of $M_0 \cup G$ it follows that there exists a smooth family of holomorphic Levi-discs $S_b \subset X \backslash G$, $S_b \ni b$, parametrized by the points $b \in \overline{\mathscr{U}_0 \cap X}$ such that there exists a compact set $K \subset X \backslash G$, containing all the closed curves bS_b.

Applying Proposition 2 to the restrictions $\tilde{f}\big|_{S_b}$, $b \in \mathcal{U}_0 \cap X \setminus G$, we can find for large enough α a family of functions $R_b \tilde{f} \in \mathcal{C}(\bar{S}_b)$ depending continuously on the parameter b such that

$$\bar{\partial} R_b \tilde{f}\big|_{S_b} = \tilde{f}\big|_{S_b} \quad \text{and} \quad \|R_b \tilde{f}\|_{\mathcal{C}(\bar{S}_b)} \leq \gamma \, \|\tilde{f}\|_{\mathcal{C}^\alpha(\bar{X})},$$

where γ does not depend on b. Hence, for the restrictions $R\tilde{f}\big|_{S_b}$ we have a representation

$$R\tilde{f}\big|_{S_b} = R_b \tilde{f} + K_b \tilde{f}, \quad \text{where} \quad K_b \tilde{f} \in \mathcal{O}(S_b).$$

Now allowing b to tend to $M_0 \cup G$ we obtain from this representation and the maximum principle for $K_b \tilde{f}$ on S_b the inequality

$$\sup_{x \in \mathcal{U}_0 \cap X \setminus G} |R\tilde{f}(x)| \leq \tilde{\gamma}\, (\|f\|_{C^{(\alpha)}(\bar{X})} + \sup_{x \in K} |R\tilde{f}(x)|).$$

This implies that $R\tilde{f} \in \mathcal{C}(X \cup M_0)$.

From Proposition 5 it follows that for every α' there exists $\alpha \geq \alpha'$ such that $R\tilde{f} \in \mathcal{C}^{\alpha'}(X \cup M_0)$ if $\tilde{f} \in \mathcal{C}^{\alpha}_{0,1}(\bar{X}, \text{Sing } \bar{X})$ and $R\tilde{f} \in \mathcal{C}(X \cup M_0)$. We have therefore constructed a continuous linear operator $R \ : \ C^{\perp\alpha}_{0,1}(\bar{X}, \text{Sing } \bar{X}) \to C^{\alpha'}(\bar{X})$ such that $\bar{\partial} Rf = f$ on Reg X for a finite co-dimensional subspace of $f \in C^{\perp\alpha}_{0,1}(\bar{X}, \text{Sing } \bar{X})$.

Because, in the argument above, one can take X_θ instead of X, Proposition 7 implies the embeddability of M_0 in affine space. □

Proof of Theorem 2. Let $X^-_\theta = \{x \in X \ : \ \rho(x) > \theta\}$ and $M_\theta = \{x \in M \ : \ \rho(x) = \theta\}$. Let $\varphi_1 \ : \ M_1 \to \mathbb{C}^N$ be a CR-embedding. By Proposition 6 the mapping φ_1 admits an holomorphic extension as a holomorphic embedding $\psi_\theta \ : \ X^-_\theta \to \mathbb{C}^N$ for some $\theta < 1$. Let θ_1 be the infimum of numbers θ such that there exists an embedding $\psi_\theta \ : \ X^-_\theta \to \mathbb{C}^N$.

Using Rossi's "filling of holes" result, [30] we deduce the existence of a holomorphic embedding of $X^-_{\theta_1}$ in a normal Stein space. Applying the Remmert embedding theorem to this Stein space we obtain an embedding $\psi_{\theta_1} \ : \ X^-_{\theta_1} \to \mathbb{C}^N$. From the Hartogs type extension theorem on complex spaces [2] it follows that the holomorphic mapping ψ_{θ_1} admits holomorphic extension to X.

From Proposition 8 we obtain the existence of a CR-embedding $\varphi_{\theta_1} \ : \ M_{\theta_1} \to \mathbb{C}^N$. To finish the proof it is sufficient to show that $\theta_1 = 0$. Suppose that $\theta_1 > 0$. From Proposition 6 it follows that the mapping φ_{θ_1} admits a holomorphic extension as a holomorphic embedding

$$\tilde{\psi}_{\theta_2} \ : \ (X^-_{\theta_2} \setminus X^-_{\theta_1}) \to \mathbb{C}^N \quad \text{for some} \quad \theta_2 < \theta_1.$$

From Hartogs-Levi extension theorem and Oka-Weil approximation theorem on complex spaces [2] it follows that the holomorphic embedding $\tilde{\psi}_{\theta_2}$ can be chosen to be holomorphic on X.

Hence holomorphic functions on $X^-_{\theta_2}$ separate all points of $X^-_{\theta_2}$ and we can again apply Rossi's and Remmert's results to obtain the existence of an embedding $\psi_{\theta_2} \ : \ X^-_{\theta_2} \to \mathbb{C}^N$ with $\theta_2 < \theta_1$. This contradicts the minimality of θ_1 and proves Theorem 2. □

5. Appendix: Embeddability is not a complex cobordism invariant

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In this Appendix we state some results on embeddability and complex-cobordisms of strictly pseudoconvex 3-manifolds that will appear in full detail in [10]. In that paper we also study the non-extendibility of CR-functions from the pseudoconvex component of the boundary of a complex manifold X.

Theorem 3. *Fillability of strictly pseudoconvex 3-manifolds is not a complex cobordism invariant.*

Theorem 4. *Let M_0 be an embeddable strictly pseudoconvex 3-manifold. The embeddability of strictly pseudoconvex 3-manifolds complex cobordant to M_0 is not stable, for small deformations of the CR-structure preserving the property of being complex cobordant to M_0.*

These results follow from the construction sketched below. In the paper [10] more general examples of this type are described.

Let C_1 and C_2 be two distinct linear $\mathbb{P}^1$'s in $\mathbb{P}^2$ and $x_0 = C_1 \cap C_2$. Construct an open covering of a neighborhood U of $C = C_1 \cup C_2$, consisting of U_0, U_1 and U_2 such that $x_0 \in U_0$, $C_1 \subset U_0 \cup U_1$, $C_2 \subset U_0 \cup U_2$, and $U_1 \cap U_2 = \emptyset$. By [26] one has a smooth family of gluings of U_0 and U_2, such that the initial gluing is the given one and all other gluings give rise to open surfaces containing C_2 with the same normal bundle but non-equivalent embeddings. In the cited paper it is shown that the only surface germ containing $\mathbb{P}^1$ with the standard normal bundle, which is fillable is that of the linear embedding of $\mathbb{P}^1$ into $\mathbb{P}^2$.

Let $\omega : \mathcal{V} \to \Delta$ be a family of surfaces, with $= V_t = \omega^{-1}(t)$ and $V_0 = U$, obtained by fixing the gluing of U_0 to U_1 but changing the gluing of U_0 with U_2, using a family gluings dependent on t as described in the previous paragraph. Each member, V_t of the family contains the curve C embedded with the same normal bundle. Hence the tubular neighborhoods of C in all the V_t are diffeomorphic to the tubular neighborhood W of C in V_0. There is a smooth map $\phi : W \times \Delta \to \mathcal{V}$, with each $\phi_t : W \to W_t = \phi(W \times t) \subset V_t$ a diffeomorphism from W to a tubular neighborhood of C in V_t. The family of surfaces $\{W_t \ : \ t \in \Delta\}$ can be therefore described as the deformation of the complex structure on W, induced by the diffeomorphisms ϕ_t.

One can construct a strictly plurisubharmonic exhaustion function $f : V_0 \setminus C \to \mathbb{R}$. For large enough c

$$S_c = \{x \in V_0 : f(x) \geq c\} \subset\subset W.$$

Fix some $c \gg 0$, after possibly shrinking Δ, one can assume that for all $t \in \Delta$, $f : W \setminus C \to \mathbb{R}$ is strictly plurisubharmonic on a neighborhood of S_c for the complex structures on W induced by ϕ_t.

As a consequence, for each sufficiently small $t \neq 0$ the surface W_t contains a pseudoconcave neighborhood, Y_{t-} of C. We denote its boundary by $M_{1t} = \phi_t(S_c)$. The strictly pseudoconcave manifold Y_{t-} contains both the neighborhood germ of a linear $\mathbb{P}^1 \subset \mathbb{P}^2$ and the neighborhood germ of a nontrivial deformation of the linear embedding of $\mathbb{P}^1$. Any nontrivial deformation of the neighborhood germ of a linear $\mathbb{P}^1$ in $\mathbb{P}^2$ cannot be contained in a embeddable pseudoconcave surface. This implies that the pseudoconcave surfaces $Y_{t-} \subset W_t$, $t \neq 0$ are not embeddable and hence the strictly pseudoconvex 3-manifolds, M_{1t} are not embeddable. On the other hand each M_{1t} is complex-cobordant to an embeddable strictly pseudoconvex 3-manifold M_0 contained in the neighborhood germ of the linear $\mathbb{P}^1$. Because this neighborhood contains a subset biholomorphic to a neighborhood of infinity in $\mathbb{C}^2$, each of the CR-manifolds M_{1t} is, in fact complex-cobordant to a round S^3.

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