Chapter 2

Elementary properties of holomorphic functions in several variables

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2.1 Holomorphy for functions of several variables

In this chapter we introduce holomorphic functions of several variables and deduce their simpler properties. Much is routine generalization from the one–variable case via the Cauchy integral formula. Though even the elementary theory of the $\overline{\partial}$–equation is more involved. The extension theorems in several variables are quite different from the single variable case; there is a straightforward analogue of Riemann's removable singularities but Hartogs' theorem is truly a multi–variable result.

We consider the theory of power series in many variables, their convergence properties are quite different from the single variable case. This follows in part from the Hartogs' result mentioned above. To begin to understand these new phenomena we consider various notions of 'convexity'.

We begin with the real vector space $\mathbb{R}^m$. An automorphism, J of $T\mathbb{R}^m$, that may depend smoothly on the point and satisfies

$$J^2 = -\operatorname{Id} \tag{2.1.1}$$

is called an almost complex structure. Note that (2.1.1) implies that the minimal polynomial for J is t^2+1. Since this has simple roots it follows that J is diagonalizable

Using J we can split $\mathbb{R}^m \otimes \mathbb{C}$ into the two eigenspaces of J corresponding to eigenvalues $i, -i$. We denote these by $T_J^{1,0}\mathbb{R}^m$ and $T_J^{0,1}\mathbb{R}^m$. The complex vector space $\mathbb{R}^m \otimes \mathbb{C}$ has a natural conjugation defined by

$$\overline{v \otimes \alpha} = v \otimes \bar{\alpha}.$$

Since J is a real transformation (i. e. it commutes with the conjugation defined above) it follows that

$$T_J^{0,1}\mathbb{R}^m = \overline{T_J^{1,0}\mathbb{R}^m}. \tag{2.1.2}$$

From (2.1.2) it follows that m is even, we'll denote it by $m = 2n$. An almost complex structure defines a complex structure provided the following 'integrability' condition is satisfied

$$\text{If } X, Y \text{ are sections of } T_J^{1,0}, \text{ then so is } [X, Y]. \tag{2.1.3}$$

If J is real analytic and satisfies (2.1.3) then the Frobenius theorem can be applied to find local coordinates $x_i, y_i, i = 1, \dots, n$ such that

$$T_J^{1,0}\mathbb{R}^{2n} = \operatorname{sp}\{\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})\}. \tag{2.1.4}$$

Even if J is only finitely differentiable then a deep theorem of Newlander and Nirenberg states that (2.1.3) is necessary and sufficient for the existence of local coordinates which satisfy (2.1.4).

Exercises 2.1.5**.**

(1) Show that if J is a linear transformation of $\mathbb{R}^{2n}$ satisfying (2.1) then the almost complex structure it defines is integrable.

(2) Prove that the space

$$\mathcal{J} = Gl(2n,\mathbb{R})/Gl(n,\mathbb{C})$$

parametrizes the almost complex structures defined by linear transformations. Note that if $A = X + iY \in Gl(n,\mathbb{C})$ then

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \in Gl(2n,\mathbb{R}).$$

(3) Show that the following condition on J is equivalent to (2.1.3): for every pair of vector fields, X, Y the quadratic form

$$N(X,Y) = [JX,Y] + [X,JY] - J[X,Y] + J[JX,JY]$$

vanishes.

(4) Show that for $f, g \in \mathcal{C}^\infty$, $N(fX, gY) = fgN(X,Y)$. This shows that N is a tensor; it is called the Nijenhaus tensor.

For the time being we will only consider almost complex structures defined by linear transformations. The 'canonical' almost complex structure is defined by the matrix

$$J_0 = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

We use $\mathbb{C}^n$ to denote $\mathbb{R}^{2n}$ with its canonical complex structure. It is a consequence of exercise (2.1.5)b that every linear complex structure is linearly equivalent to this one. The eigenspaces are given by

$$\begin{aligned} T^{1,0}\mathbb{C}^n &= \mathrm{sp}\{\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i}), i = 1,\dots,n\} \\ T^{0,1}\mathbb{C}^n &= \mathrm{sp}\{\partial_{\bar{z}_i} = \frac{1}{2}(\partial_{x_i} + i\partial_{y_i}), i = 1,\dots,n\} \end{aligned} \tag{2.1.6}$$

The dual spaces are denoted by

$$\begin{aligned} \Lambda^{1,0}\mathbb{C}^n &= \mathrm{sp}\{dz_i, i = 1,\dots,n\} \\ \Lambda^{0,1}\mathbb{C}^n &= \mathrm{sp}\{d\bar{z}_i, i = 1,\dots,n\}. \end{aligned} \tag{2.1.7}$$

As in the case of one variable we can express any derivative of a function in terms of these vector fields. Thus

$$df = \partial f + \overline{\partial} f, \text{ where } \partial f = \sum_{i=1}^n \partial_{z_i} f dz_i, \quad \overline{\partial} f = \sum_{i=1}^n \partial_{\bar{z}_i} f d\bar{z}_i. \tag{2.1.8}$$

As before we call the $1,0$–part of df ∂f and the $0,1$–part $\overline{\partial} f$.

Definition 2.1.9**.** If $\Omega \subset \mathbb{C}^n$ is open and $u \in C^1(\Omega)$ then u is holomorphic provided that

$$\overline{\partial} u = 0. \tag{2.1.10}$$

We denote the set of such functions by $H(\Omega)$.

Note that (2.1.10) is a system of $2n$ equations for 2 unknown functions.

It is a very important fact in one dimension that the composition of two holomorphic functions is holomorphic. To generalize this to several variables we need to define a 'holomorphic mapping'

Definition 2.1.12. A mapping from $\Omega \subset \mathbb{C}^n$ to $\mathbb{C}^m$ is holomorphic provided the coordinate functions are. That is if $U = (u_1, \dots, u_m)$ then $\overline{\partial} u_i = 0, i = 1, \dots, m$. We denote such mappings by $H(\Omega; \mathbb{C}^m)$. If a holomorphic mapping between two open subsets is invertible with holomorphic inverse then the mapping is said to be *biholomorphic.*

Proposition 2.1.13. *Suppose that* $f \in H(\Omega'), \Omega' \subset \mathbb{C}^m$ *and* $U \in H(\Omega; \mathbb{C}^m)$ *has range contained in* Ω' *then*

$$U^* f(z) = f(U(z)) \in H(\Omega). \tag{2.1.14}$$

Proof. We compute $dU^* f$:

$$dU^* f = \sum_{i=1}^{n} \sum_{j=1}^{m} \partial_{u_j} f \partial_{z_i} u_j dz_i.$$

All other terms are absent as df and $du_j, j = 1, \dots, m$ are of type $1, 0$. This proves the claim.

The implicit and inverse function theorems also extend easily to holomorphic mappings

Theorem 2.1.15. *Let* $f_j(w, z)$ *be analytic functions in a neighborhood of the point* $(w_0, z_0) \in \mathbb{C}^m \times \mathbb{C}^n$ *and assume that* $f_j(w_0, z_0) = 0$, $j = 1, \dots, m$. *Finally suppose that*

$$\det \partial_{w_j} f_k(w_0, z_0) \neq 0, \tag{2.1.16}$$

then there is a neighborhood V *of* z_0 *and a holomorphic mapping* $w(z) \in H(V, \mathbb{C}^m)$ *with*

$$w(z_0) = w_0 \text{ and } f_j(w(z), z) = 0. \tag{2.1.17}$$

Proof. Identifying $Gl(n, \mathbb{C})$ with a subgroup of $Gl(2n, \mathbb{R})$ as in (2.1.5), reinterpreting (2.1.16) allows us to apply the standard implicit function theorem to the system of $2m$ real equations $\operatorname{Re} f_j = 0, \operatorname{Im} f_j = 0$ with respect to the variables $\operatorname{Re} w_j, \operatorname{Im} w_j, \operatorname{Re} z_i, \operatorname{Im} z_i$. This uniquely determines differentiable functions $\operatorname{Re} w_j(\operatorname{Re} z_i, \operatorname{Im} z_i), \operatorname{Im} w_j(\operatorname{Re} z_i, \operatorname{Im} z_i)$ which satisfy (2.1.17).

At this point we can reexpress everything in terms of $z_i, \bar{z}_i$ and differentiate the equations in (2.1.17) to obtain

$$\sum_{k=1}^{m} \partial_{w_k} f_j dw_j + \sum_{i=1}^{n} \partial_{z_i} f_j = 0 dz_i, j = 1, \dots m. \tag{2.1.18}$$

In light of (2.1.16) we can use (2.1.18) to solve for the dw_k in some neighborhood of (w_0, z_0). It is clear that these one forms are of type $1, 0$ and therefore $w_k(z)$ are holomorphic functions.

The exterior algebra $\Lambda^* \mathbb{R}^{2n}$ can be split into *p,q–types* using the complex covectors $dz_i, d\bar{z}_i$ as a basis. Let

$$I = \{1 \leq i_1 < \dots < i_p \leq n\}; J = \{1 \leq j_1 < \dots < j_q \leq n\},$$

be multiindices, then we define

$$dz^I = dz_{i_1} \wedge \dots \wedge dz_{i_p}, d\bar{z}^J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}.$$

As usual we define $|I| = p, |J| = q$.

Definition 2.1.19. A differential form, ω is said to be of type p, q if it has a local representation as

$$\omega = \sum_{I, J; |I| = p, |J| = q} f_{IJ} dz^I \wedge d\bar{z}^J. \tag{2.1.20}$$

The set of such forms defined on $U \subset \mathbb{C}^n$ is denoted by $\Lambda^{p,q}(U)$.

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Exercise 2.1.21. Prove that the notion of p, q–type is invariant under biholomorphic mappings.

The operators $\partial, \overline{\partial}$ extend to define differential operators on $\Lambda^{p,q}(U)$. In fact

$$
\begin{aligned}
&\partial : \Lambda^{p,q}(U) \longrightarrow \Lambda^{p+1,q}(U) \text{ is defined by } \partial\omega = \sum_{I,J} \partial f_{IJ} \wedge dz^I \wedge d\bar{z}^J, \\
&\overline{\partial} : \Lambda^{p,q}(U) \longrightarrow \Lambda^{p,q+1}(U) \text{ is defined by } \overline{\partial}\omega = \sum_{I,J} \overline{\partial} f_{IJ} \wedge dz^I \wedge d\bar{z}^J.
\end{aligned} \tag{2.1.22}
$$

It is trivial to see that $\partial^2 = \overline{\partial}^2 = 0$; since $d^2 =$ as well, this implies that

$$\partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Exercise 2.1.23.

(1) Show that one can define the $1,0$ and $0,1$–parts of df with respect to any almost complex structure,
(2) Use the previous part to define ∂_J and $\overline{\partial}_J$,
(3) Show that the almost complex structure defined by J is integrable if and only if $\partial_J^2 = 0$.

The main topic of interest is the solution of

$$\overline{\partial} u = \alpha,\ \alpha \in \Lambda^{0,1}.$$

This equation is overdetermined and has compatibility conditions:

$$\overline{\partial}\alpha = 0.$$

We see that consideration of $0,1$–forms leads inevitably to $0,2$–forms and so to consideration of $0,3$–forms, etc. Though the case of $0,1$–forms is most important for applications, it presents no essential difficulty to consider p, q–forms from the start.

2.2 The Cauchy formula for polydiscs and its elementary consequences

The unit disk in $\mathbb{C}$ is a model domain for the study of the local properties of holomorphic functions. In several variables there are two, quite different analogues of the unit disk, the ball and the polydisk. The ball is

$$\mathbb{B}(w; R) = \{z \in \mathbb{C}^n;\ |z - w| < R\}.$$

A ball has a smooth boundary. A polydisk is defined by an n–tuple of positive numbers $r_1, \dots, r_n$ and by a point $w \in \mathbb{C}^n$ by

$$D(w; r) = D(w; r_1, \dots, r_n) = \{(z_1, \dots, z_n); |z_i - w_i| < r_i, i = 1, \dots, n\}.$$

As we shall soon see the local analysis of holomorphic functions on polydisks is very similar to the one variable case. Note however that a polydisk does not have a smooth boundary. It has a distinguished boundary component defined by

$$\partial_0 D(w; r) = \{(z_1, \dots, z_n) : |z_i - w_i| = r_i\}.$$

This is the lowest dimensional boundary component but as we shall see it is also the most important. Later on we will show that the complex analytic geometry of the unit ball has more in common with the unit disk, though the analysis on the ball is quite a bit more involved.

For functions that are holomorphic in a polydisc we have a direct generalization of the Cauchy integral formula.

Cauchy Integral Formula 2.2.1. *Let $u(z)$ be continuous is $\overline{D}(w;r)$ and holomorphic separately in each variable then*

$$u(z) = \left(\frac{1}{2\pi\imath}\right)^n \underset{\partial_0 D}{\int \cdots \int} \frac{u(w) dw_1 \dots dw_n}{(w_1 - z_1)\dots(w_n - z_n)}. \tag{2.2.2}$$

Proof. We can prove this inductively. This is simply (1.3.16) if $n = 1$. Suppose that it is proved for $n-1$. The polydisk $D(w;r)$ can be written as a product

$$D(w;r) = D(w_1;r_1) \times D'.$$

The inductive hypothesis implies that for each fixed $z \in D(w_1;r_1)$ we have the representation

$$u(z, z_2, \dots, z_n) = \left(\frac{1}{2\pi\imath}\right)^{n-1} \underset{\partial_0 D'}{\int \cdots \int} \frac{u(z,\zeta') d\zeta_2 \dots d\zeta_n}{(\zeta_2 - z_2)\dots(\zeta_n - z_n)}. \tag{2.2.3}$$

On the other hand the continuity hypothesis and (1.3.16) imply that for each $\zeta' \in \partial_0 D'$

$$u(z,\zeta') = \frac{1}{2\pi\imath} \int_{|\zeta_1 - w_1| = r_1} \frac{u(\zeta,\zeta') d\zeta}{\zeta - z}. \tag{2.2.4}$$

We can put the integral in (2.2.4) into (2.2.3), for $(z, z') \in D(w;r)$ the combined integrand is continuous and therefore we can apply the Fubini theorem to identify the iterated integral with the (2.2.2).

Note that in the hypothesis of (2.2.1) we did not assume that u was C^1 in $D(w;r)$ but rather continuous and separately holomorphic. The Cauchy formula has the following corollary

Corollary 2.2.5. *If u satisfies the hypotheses of* (2.2.1) *then $u \in \mathcal{C}^\infty(D(w;r))$ and therefore $u \in H(D(w;r))$.*

The assertions follow by differentiating the integral representation. The maximum modulus principle follows from

Corollary 2.2.6. *If $u \in C^0(D(w;r)) \cap H(D(w;r))$ then*

$$|u(w)| \le \max_{z \in \partial_0 D(w;r)} |u(z)| \tag{2.2.7}$$

with equality only if $u \restriction_{\partial_0 D(w;r)}$ is constant.

Proof. Exactly the same argument as in the one variable case.

In the case of equality we can use the Cauchy integral representation to deduce that u is actually constant in $\overline{D}(w;r)$.

The Maximum Modulus Principle 2.2.8. *If $\Omega \subset \mathbb{C}^n$ is an open set and $u \in C^0(\overline{\Omega}) \cap H(\Omega)$ then $|u|$ does not assume its maximum at an interior point unless u is constant.*

Proof. This follows from (2.2.6) and the fact that $\forall\, w \in \Omega$ we can find a positive n–tuple r such that $D(w;r) \subset\subset \Omega$.

The several variables result is actually a bit stronger than the one variable case. If two holomorphic functions on a disk in $\mathbb{C}$ agree on the boundary then they agree in the whole disk. In many variables we see that if two holomorphic functions in a polydisk agree on the distinguished boundary then they agree in the polydisk as well. The boundary of the polydisk is a manifold with corners of dimension $2n-1$ whereas the distinguished boundary has dimension n.

We also have a generalization of (1.3.7) to the case at hand. Estimates for the derivatives of a holomorphic function follow from this representation. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be an n–tuple of non–negative integers. Then we define the differential operator

$$\partial^\alpha = \partial_{z_1}^{\alpha_1} \dots \partial_{z_n}^{\alpha_n}.$$

We also define

$$|\alpha| = \alpha_1 + \dots + \alpha_n; \quad \alpha! = \alpha_1! \dots \alpha_n!$$

The operator $\overline{\partial}^\alpha$ is defined analogously. The estimates for the derivatives of a holomorphic function are

Corollary 2.2.9. *Suppose that $u \in H(\Omega)$ and that K is a compact subset with $K \subset\subset \omega \subset\subset \Omega$ then for each multiindex α there is a constant C_α such that*

$$\sup_{z\in K} |\partial^\alpha u(z)| \le C_\alpha \|u\|_{L^1(\omega)}.$$

Proof. We simply cover K by a finite collection of polydisks and apply the argument used to prove (1.4.2).

These estimates imply compactness properties for holomorphic functions with respect to locally uniform convergence.

Theorem 2.2.10. *If $u_n \in H(\Omega)$ converges locally uniformly to a function u then the limit is holomorphic. If $u_n \in H(\Omega)$ is locally uniformly bounded then u_n has a locally uniformly convergent subsequence.*

Proof. Exactly as in the one variable case.

Using the Cauchy integral formula we can deduce the existence of convergent power series expansions for holomorphic functions. The notion of convergence that is appropriate in this context is that of *normal convergence.* If $\{a_\alpha(z) : \quad z \in \Omega\}$ is a collection of functions defined on a open set Ω then we say that

$$\sum_\alpha a_\alpha(z)$$

converges normally if

$$\sum_\alpha \sup_K |a_\alpha|$$

converges for every $K \subset\subset \Omega$. Evidently we can rearrange a normally convergent series and obtain the same limit. If the functions $a_\alpha(z)$ are holomorphic then clearly the limit is holomorphic as well.

Theorem 2.2.11. *If u is holomorphic in a polydisk, $D(0; r)$ then we have*

$$u(z) = \sum_\alpha \frac{\partial^\alpha u(0) z^\alpha}{\alpha!}. \tag{2.2.12}$$

With normal convergence for $z \in D(0; r)$.

Proof. We observe that

$$[(\zeta_1 - z_1)\dots(\zeta_n - z_n)]^{-1} = \sum_\alpha \frac{z^\alpha}{\zeta^\alpha \zeta_1 \dots \zeta_n}. \tag{2.2.13}$$

The series in (2.2.13) converges normally for $(z, \zeta) \in D \times \partial_0 D$. If $u \in C^0(D(0; r))$ then we can interchange the order of integration and summation in the Cauchy integral formula to obtain

$$u(z) = \sum_\alpha z^\alpha \left(\frac{1}{2\pi\imath}\right)^n \int_{\partial_0 D} \frac{u(\zeta) d\zeta_1 \dots d\zeta_n}{\zeta^\alpha \zeta_1 \dots \zeta_n}. \tag{2.2.14}$$

This is again normally convergent in $D(0; r)$. A simple calculation using Cauchy's formula shows that

$$\partial^\alpha u(0) = \frac{\alpha!}{(2\pi i)^n} \int_{\partial_0 D} \frac{u(\zeta) d\zeta}{\zeta^\alpha \zeta_1 \dots \zeta_n}. \tag{2.2.15}$$

Putting (2.2.15) into (2.2.14) implies (2.2.12) in this special case. Otherwise we simply apply this argument to polydisks $D(0; r')$ with $r'_i < r_i, i = 1, \dots, n$.

Once again (2.2.10) and (2.2.11) imply that a function is holomorphic if and only if it has a normally convergent power series expansion about every point. As in the one variable case we obtain estimates on the derivatives of u from the Cauchy formula.

Cauchy Estimates 2.2.16. *If $u \in H(D(0,r))$ and $|u| \leq M$ then*

$$|\partial^\alpha u(0)| \leq M\alpha! r^{-\alpha}.$$

Proof. These estimates follow by applying (2.2.15) to smaller polydisks.

In addition we also have an analogue of Schwarz's lemma for functions of several variables

Schwarz's Lemma 2.2.17. *If u is holomorphic in a neighborhood of a closed ball $B(0;R)$, $|u(z)| \leq M$ in the ball and*

$$\partial^\alpha u(0) = 0 \text{ if } |\alpha| < k$$

for some positive integer k then

$$|u(z)| \leq MR^{-k}|z|^k. \tag{2.2.18}$$

Proof. Let $Z \in \mathbb{C}^n$ with $|Z| = 1$. Then the function $f(t) = u(tZ)$ is holomorphic for $t \in B(0;R) \subset \mathbb{C}$ and satisfies $|f(t)| \leq M$ and $f^{[j]}(0) = 0, j = 0, \dots, k-1$. The classical Schwarz lemma implies that

$$|f(t)| \leq MR^{-k}|t|^k. \tag{2.2.19}$$

Since Z is an arbitrary point on the unit sphere (2.2.19) implies (2.2.18).

Exercise 2.2.20. What can you conclude if equality holds at some point in (2.2.18).

Holomorphic functions in several variables have remarkable property: if a function is separately holomorphic in each variable then it is holomorphic in the sense of definition (2.2.1). We proved a weak version of this above. Note that this sort of a property fails if we replace separately holomorphic with separately real analytic. For example

$$f(x,y) = \frac{xy}{x^2+y^2}$$

is real analytic when restricted to any vertical or horizontal line, however this function fails to be continuous at $(0,0)$.

In order to study this problem we need to establish a few elementary facts about sub–harmonic functions.

Definition 2.2.21. A function $u(z)$ defined in Ω an open subset of $\mathbb{C}$ is subharmonic provided

$$\begin{aligned} &u(z) \text{ is bounded from above,} \\ &u(z) \text{ is upper semicontinuous,} \\ &\pi r^2 u(z) \leq \iint\limits_{B(z,r)} u\,dx\,dy, \text{ whenever } B(z,r) \subset\subset \Omega. \end{aligned} \tag{2.2.23}$$

The assumption that u is upper semicontinuous, implies that the integral in (2.2.23) is well defined.

Exercise 2.2.24.

(1) If $f(z) \in H(\Omega)$ show that $\log|f(z)|$ is subharmonic,
(2) If φ is a convex monotone increasing function defined on $\mathbb{R}$ and u is subharmonic then so is $\varphi(u)$.

Lemma 2.2.25. *Suppose that $v_n(z)$ is a sequence of subharmonic functions defined in Ω such that $v_n(z) \leq M$, $n = 1, 2, \ldots$, and $\limsup_{n\to\infty} v_n(z) \leq c$. Then given a compact subset K of Ω and an $\epsilon > 0$ there exists an N so that*

$$v_n(z) < c + \epsilon \text{ provided } z \in K,\ n > N.$$

Proof. We prove this by contradiction. Suppose that conclusion is false then we can choose a subsequence n_k and a sequence $z_{n_k} \in K$ such that $v_{n_k}(z_{n_k}) > c + \epsilon$. Since K is compact there is no loss in generality in assuming that this subsequence converges to $z^* \in K$. Since $K \subset\subset \Omega$ we can choose an $r > 0$ such that $B(z_{n_k}; 2r) \subset\subset \Omega$. Using (2.2.24) we observe that $f_k(z) = e^{u_{n_k}(z)}$ is a sequence of bounded, non–negative subharmonic functions. Since these functions are non–negative and the sequence $\{z_{n_k}\}$ converges to z^* it is clear that for sufficiently large k and $\delta > 0$

$$\int\limits_{B(z_{n_k}, r)} f_k(w)dxdy \leq \int\limits_{B(z^*, r+\delta)} f_k(w)dxdy. \tag{2.2.26}$$

Fixing a $\delta > 0$ a simple application of Fatou's Lemma implies that

$$\begin{aligned}\limsup_{k\to\infty} \int\limits_{B(z^*; r+\delta)} f_k(W)dxdy \leq & \int\limits_{B(z^*; r+\delta)} \limsup_{k\to\infty} f_k(w)dxdy \\ \leq & e^c\pi(r+\delta)^2.\end{aligned} \tag{2.2.27}$$

On the other hand since f_k is subharmonic

$$e^{c+\epsilon}\pi r^2 \leq \pi r^2 f_k(z_{n_k}) \leq \int\limits_{B(z_{n_k}; r)} f_k(w)dxdy. \tag{2.2.28}$$

Letting $k \to \infty$ in (2.2.29) and applying (2.2.26)–(2.2.27) we obtain

$$e^{c+\epsilon} r^2 \leq e^c(r+\delta)^2. \tag{2.2.30}$$

Since $\delta > 0$ is arbitrary and $\epsilon > 0$ is fixed (2.2.30) leads to a contradiction.

Though the generalization of Runge's theorem to several variables is in general quite complicated, there is one case which is essentially trivial. If $D_1 \subset\subset D_2 \subset\subset D$ are polydisks and $f \in H(D_2)$ then given $\epsilon > 0$ we can find $F \in H(D)$ which satisfies

$$\sup_{z\in D_1} |F(z) - f(z)| < \epsilon. \tag{2.2.31}$$

One simply expands f in a Taylor series about the center of D_2, a sufficiently large partial sum satisfies (2.2.31). For latter reference we formulate this as a proposition

Proposition 2.2.32. *If D is a polydisk then the uniform closure on D of functions holomorphic in a neighborhood of D equals the uniform closure on D of the holomorphic polynomials.*

2.3 Hartogs' Theorem on separately holomorphic functions

In the previous section we defined a holomorphic function as a function which is continuously differentiable and satisfies the Cauchy–Riemann equations in each variable separately. In the course of deriving the Cauchy integral formula we established that a function which is merely continuous and satisfies the Cauchy–Riemann equations in each variable, with the other variables regarded as constant is actually holomorphic in the previous sense. In this section we prove the Theorem of Hartogs' which states that if a function is separately holomorphic in each variable then it is holomorphic in the above sense. No additional assumption needs to be made about the regularity of the function as a function of several variables.

Hartogs' Theorem 2.3.1. *If u is a complex valued function defined in an open subset $\Omega \subset \mathbb{C}^n$ which is holomorphic in each variable z_j when the other variables are given fixed arbitrary values then $u \in H(\Omega)$.*

Proof. The theorem is proved inductively using several lemmas and the result on subharmonic functions proved in §2.2. As a preliminary step we prove

Lemma 2.3.2. *Suppose that u satisfies the hypotheses of the previous theorem and in addition u is locally uniformly bounded then $u \in H(\Omega)$.*

Remark. The subtlety is that u might fail to be measurable as a function of $2n$–real variables so we cannot simply apply the Cauchy formula and Fubini's theorem.

Proof. Since u is uniformly bounded on compact subsets and holomorphic in each variable separately, the one–variable Cauchy estimates imply that $\partial_{z_i} u$ are locally uniformly bounded for each i. Since $\partial_{\bar{z}_i} u = 0$ it follows that all first partial derivatives of u are locally uniformly bounded and thus the mean value theorem implies that u is continuous. The conclusion then follows from (2.2.1).

Now we will show that u is bounded on some open set. We make an inductive hypothesis that (2.3.1) is proved for $n-1$. It is trivial for $n = 1$.

Lemma 2.3.3. *Let u satisfy the hypotheses of* (2.3.1) *and let $D = D_1 \times \cdots \times D_n \subset\subset \Omega$ be a closed polydisk Then there exist disks $D_i' \subset D_i, i = 1, \dots, n-1$, with non–empty interior, such that u is bounded in the polydisk $D_1' \times \cdots \times D_{n-1}' \times D_n$.*

Proof. We prove this using the Baire category theorem and the inductive hypothesis . Let

$$E_M = \{z' : z' \in \prod_{j=1}^{n-1} D_j \text{ and } |f(z', z_n)| \leq M \text{ when } z_n \in D_n\}.$$

Since $f(z', z_n)$ is holomorphic for $z_n \in D_n$ it follows that

$$\prod_{j=1}^{n-1} D_j = \bigcup_{M \in \mathbb{N}} E_M.$$

The inductive hypothesis implies that, for fixed z_n, $f(z', z_n)$ is holomorphic in the first $n-1$ coordinates. This implies that $f(z', z_n)$ is continuous as a function of these coordinates from which we conclude that E_M is a closed set for each M. Since $\prod_{j=1}^{n-1} D_j$ is a complete metric space it follows from the Baire category theorem that E_M must have nonempty interior for some M. If z_0' lies in the interior then we can find a polydisk D' centered at z_0' so that $D' \times D_n$ satisfies the conclusion of the lemma.

A final lemma is required to finish the proof.

Lemma 2.3.4. *Let u be a complex valued function in a polydisk $D = \{z;\ |z_j - z_j^0| < R, j = 1 \dots, n\}$, assume that u is analytic in $z' = (z_1, \dots, z_{n-1})$ if z_n is fixed and that u is analytic and bounded in*

$$D' = \{z;\ |z_j - z_j^0| < r, j = 1 \dots, n-1, |z_n - z_n^0| < R\}$$

for some $r > 0$. Thenu is analytic in D.

Proof. For simplicity we assume that $z^0 = 0$. Choose R_1, R_2 with $0 < R_1 < R_2 < R$. By (2.2.12) we have a power series expansion for $u(z)$

$$u(z) = \sum_{\alpha} a_\alpha(z_n) {z'}^{\alpha}, \quad z \in D, \tag{2.3.5}$$

where the sum extends over $(n-1)$–multiindices. The coefficients are given by

$$a_\alpha(z_n) = \frac{\partial_z^\alpha u(0, z_n)}{\alpha!}.$$

The coefficients are analytic in z_n since $u(z)$ is analytic in D'. Because $u(z', z_n)$ is holomorphic in z' for $|z_j - z_j^0| < R, j = 1, \ldots, n-1$

$$\limsup_{|\alpha|\to\infty} |a_\alpha(z_n)| R_2^{|\alpha|} = 0, \text{ for fixed } z_n \text{ with } |z_n| < R.$$

On the other hand since $u(z)$ is holomorphic in D' the Cauchy inequalities apply to give

$$|a_\alpha(z_n)| r^{|\alpha|} \le M, \tag{2.3.6}$$

if M is a bound for $|u|$ in D'.

The functions $z_n \longrightarrow \frac{\log |a_\alpha(z_n)|}{|\alpha|}$ are subharmonic for $|z_n| < R$. From (2.3.6) we conclude that these functions are uniformly bounded when $|z_n| < R$ and the $\limsup_{|\alpha|\to\infty}$ is at most $-\log R_2$ for each fixed z_n. We can therefore apply (2.2.25) to conclude that, for sufficiently large $|\alpha|$,

$$\frac{\log |a_\alpha(z_n)|}{|\alpha|} \le -\log R_1 \text{ if } z_n < R_1,$$

or in other words

$$|a_\alpha(z_n)| R_1^{|\alpha|} \le 1 \text{ for large } \alpha \text{ if } |z_n| < R_1.$$

This proves that the series in (2.3.5) converges normally in D. Since the terms are analytic the sum must also be.

Now we can complete the proof of the theorem. Given $\zeta \in \Omega$ we choose an $R > 0$ so that the polydisk $\{z;\, |z_j - \zeta_j| \le 2R, j = 1, \ldots, n\}$ is contained in Ω. By Lemma (2.3.3) applied to the polydisk

$$D = \{|z_j - \zeta_j| \le R, j = 1, \ldots, n-1,\ |z_n - \zeta_n| \le 2R\},$$

we can find a point z^0 with $|z_j^0 - \zeta_j| < R$ so that the hypotheses of Lemma (2.3.4) are satisfied with this z^0 and R as above. The lemma implies that u is holomorphic in a neighborhood ζ. This completes the proof of the theorem.

2.4 Solving the $\overline{\partial}$–equation in a polydisc, extension theorems

In this section we consider the inhomogeneous Cauchy–Riemann equation:

$$\overline{\partial} u = f,\ f \in \mathcal{C}_c^\infty(D; \Lambda^{p,q}). \tag{2.4.1}$$

As noted above this equation has a nontrivial integrability condition

$$\overline{\partial} f = 0. \tag{2.4.2}$$

We first consider the fundamental case of $(0,1)$–forms.

Proposition 2.4.3. *Suppose that $D \subset \mathbb{C}^n$, $n > 1$ is a polydisk and $f \in \mathcal{C}_c^k(D; \Lambda^{0,1})$ satisfying* (2.4.2). *Then there is a function $u \in \mathcal{C}_c^k(D)$ such that*

$$\overline{\partial} u = f.$$

Remark. In several variables note the interesting difference in the support properties of solutions to (2.4.1). In general the solution to (2.4.1) does not have compact support in one dimension.

Exercise 2.4.4.

(1) Find necessary and sufficient conditions on f for the solution to (2.4.1) to be compactly supported when $n = 1$.

(2) If $n = 1$ and the solution to (2.4.1) is compactly supported, what is the support of u?

Proof. We simply apply the one variable formula to obtain

$$u(z) = \frac{1}{2\pi\imath} \int \frac{f_1(w, z')dw \wedge d\bar{w}}{w - z_1}. \tag{2.4.5}$$

The regularity follows as in the single variable case. Applying Proposition (1.3.10) we obtain

$$\partial_{\bar{z}_1} u(z) = f_1(z).$$

We can differentiate with respect to $\bar{z}_2, \ldots, \bar{z}_n$ under the integral sign to obtain

$$\partial_{\bar{z}_j} u(z) = \frac{1}{2\pi\imath} \int \frac{\partial_{\bar{z}_j} f_1(w, z')dw \wedge d\bar{w}}{w - z_1}. \tag{2.4.6}$$

The integrability condition implies that

$$\partial_{\bar{z}_j} f_1 = \partial_{\bar{z}_1} f_j. \tag{2.4.7}$$

Using this in (2.4.6) and taking account of the compact support of f we can apply (1.3.7) to conclude that

$$\partial_{\bar{z}_j} u(z) = f_j(z).$$

Outside of the support of f, u is a holomorphic function. We can write $D = D_1 \times D'$. If z' lies close enough to the boundary of D' then $f(w, z') = 0$ for $w \in D_1$. Thus $u(z)$ vanishes in an open subset of the unbounded component of its domain of holomorphy and therefore must vanish identically in that component.

Using this result we deduce the basic extension result for holomorphic functions due to Hartogs :

Theorem 2.4.8. *Let $\Omega \subset \mathbb{C}^n$, $n > 1$ be an open set and let $K \subset\subset \Omega$ be a compact subset such that $\Omega \setminus K$ is connected. Every $u \in H(\Omega \setminus K)$ has an extension to a $U \in H(\Omega)$.*

Proof. Choose a polydisk D containing Ω in its interior and a function $\psi \in \mathcal{C}^\infty(D)$ with $\psi \leq 1$. We suppose that $\psi = 0$ in a neighborhood of K and the set $C = \{z; \psi(z) < 1\}$ is a relatively compact subset of Ω. The function ψu can be continued to all of Ω and $f = \overline{\partial}(\psi u)$ can be continued by zero to $D \setminus C$. It is clear that f so extended is compactly supported in D and $\overline{\partial} f = 0$. Thus we can apply (2.4.3) to obtain $v \in \mathcal{C}_c^\infty(D)$ such that $\overline{\partial} v = f$. The function $U = \psi u - v \in H(\Omega)$.

Since C is a relatively compact subset of Ω it follows that v vanishes in an open subset of $\Omega \setminus K$. Thus U agrees with u on some open subset of $\Omega \setminus K$, as this set is connected the uniqueness of analytic continuation implies that $U = u$ in all of $\Omega \setminus K$.

We consider two other extension theorems. The first is a straightforward extension of the Riemann removable singularities in one dimension while the second lies somewhere between the Hartogs' and the Riemann theorem.

Riemann removable Singularities Theorem 2.4.9. *Suppose that $\Omega \subset \mathbb{C}^n$ is an open subset and $X \subset \Omega$ is the zero set of a holomorphic function then a bounded function $u \in H(\Omega \setminus X)$ has an extension to a function $U \in H(\Omega)$.*

Proof. As we have not covered all the prerequisites we only prove this theorem in the special case that $X = \Omega \cap \{z_n = 0\}$.

Let $(w', 0) \in X$, choose a number $R > 0$ so that

$$D = \{z;\ |z_i - w_i| \leq R, i = 1, \ldots n - 1, |z_n| \leq R\} \subset\subset \Omega.$$

Then the function defined by

$$U(z', z_n) = \frac{1}{2\pi\imath} \int\limits_{|\zeta|=R} \frac{u(z', \zeta)d\zeta}{\zeta - z_n} \tag{2.4.10}$$

is holomorphic in D. On the other hand we can apply the single variable removable singularities theorem to $u(z', z_n)$ to conclude that as a function of z_n this has an analytic extension to $z_n = 0$. Therefore $U(z', z_n) = u(z', z_n)$ for $z_n \neq 0$.

The last continuation result is a purely several variables theorem.

Theorem 2.4.11**.** *Let Ω be an open subset of $\mathbb{C}^n$ and set $Y = \Omega \cap \{z;\ z_{n-1} = z_n = 0\}$ If $u \in H(\Omega \setminus Y)$ then there is a function $U \in H(\Omega)$ extending u.*

Remark. This theorem is not a consequence of (2.4.8) because Y is not a compact subset of Ω and it is not a consequence of (2.4.9) because u is not assumed to be bounded. The previous theorem says, in effect, that for a holomorphic function to be singular on a variety of codimension 1 it must blow up there. The present theorem says that a holomorphic function cannot be singular on a variety of codimension 2.

Proof. As before suppose that $(w'', 0, 0) \in Y$ and that the polydisk with this center and radii all equal to $R > 0$ is a relatively compact subset of Ω. Denote it by D. The function defined by

$$U(z', z_n) = \frac{1}{2\pi\imath} \int\limits_{|\zeta_n|=R} \frac{u(z', \zeta_n)d\zeta_n}{\zeta_n - z_n}, \tag{2.4.12}$$

is holomorphic in D.

If $z_{n-1} \neq 0$ then $u(z'', z_{n-1}, z_n)$ is holomorphic for $|z_n| < R$. Thus it follows from (2.4.12) That

$$U(z) = u(z) \text{ if } z_{n-1} \neq 0. \tag{2.4.13}$$

Since $U(z)$ is holomorphic and $\{z_{n-1} = 0\}$ does not separate the polydisk it follows from (2.4.13) that

$$U \restriction_{D \setminus D\cap Y} = u \restriction_{D\setminus D\cap Y} .$$

This proves the theorem.

The mechanism behind this argument is two pronged. Firstly a holomorphic function is determined by its values on the distinguished boundary. While the putative singular locus does intersect the boundary of any polydisk centered at a point on this locus, it can be arranged to be disjoint from the distinguished boundary. The other basic fact is that a disk, which is of $\mathbb{R}$–codimension two generically does not intersect a subvariety of $\mathbb{R}$–codimension four. A more general statement than (2.4.11) replaces $z_{n-1} = z_n = 0$ with a $\mathbb{C}$–codimension two subvariety of Ω.

2.5 Local solution of the $\bar\partial$–equation for p, q–forms

For the sake of completeness we include an argument that the $\bar\partial$–equation can be locally solved for p, q–forms. Suppose that $D \subset \mathbb{C}^n$ is an open polydisk and $f \in \mathcal{C}^\infty(D; \Lambda^{p,q})$, $q > 0$, with $\bar\partial f = 0$ then we want to find a form $u \in \mathcal{C}^\infty(D; \Lambda^{p,q-1})$ such that

$$\bar\partial u = f. \tag{2.5.1}$$

At present we will content ourselves with finding u is $D' \subset\subset D$. As contrasted with the previous result we do not assume that f has compact support. To obtain u in all of D we need an Runge type approximation result.

Theorem 2.5.2. *Let $D \subset \mathbb{C}^n$ be an open polydisk and $f \in \mathcal{C}^\infty(D; \Lambda^{p,q})$ with $q > 0$ satisfy $\overline{\partial} f = 0$. If $D' \subset\subset$ is polydisk then we can find a form $u \in \mathcal{C}^\infty(D'; \Lambda^{p,q-1})$ such that*

$$\overline{\partial} u = f. \tag{2.5.3}$$

Proof. The argument is by induction. We assume that if f is independent of $d\bar{z}_{k+1}, \ldots, d\bar{z}_n$ then we can solve (2.5.3) in D'. Since $q > 0$ it follows that the case $k = 0$ corresponds to $f = 0$ and therefore the claim is trivially true. If we can verify the claim for $k = n$ then we've proved the theorem. Assume it for k. Suppose that f is of the form

$$f = d\bar{z}_{k+1} \wedge g + h,$$

where g, h are independent of $d\bar{z}_{k+1}, \ldots, d\bar{z}_n$. We can write

$$g = \sum_{I,J} g_{IJ} dz^I \wedge d\bar{z}^J,$$

the sum extends over increasing $p, q-1$–multiindices and J varies between 1 and k. The hypothesis that $\overline{\partial} f = 0$ is easily seen to imply that

$$\partial_{\bar{z}_j} g_{IJ} = 0, j = k+2, \ldots, n. \tag{2.5.4}$$

That is because these are, up to a sign, the coefficients of $dz^I \wedge d\bar{z}^{k+1} \wedge d\bar{z}^j$.

Using these facts and the standard Cauchy theorem we can remove the $d\bar{z}^{k=1}$ term in f. Choose a function $\psi \in \mathcal{C}_c^\infty(D)$ such that $\psi = 1$ on D'. We set

$$G_{IJ}(z) = \frac{1}{2\pi\imath} \int \frac{\psi g_{IJ}(z_1, \ldots, z_k, \tau, z_{k+2}, \ldots, z_n) d\tau \wedge d\overline{\tau}}{\tau - z_k}.$$

The regularity of G follows immediately from the integral formula.

It is clear that

$$\partial_{\bar{z}_j} G_{IJ}(z) = 0, z \in D', j = k+2, \ldots, n. \tag{2.5.5}$$

From the Cauchy formula it follows that

$$\partial_{\bar{z}_{k+1}} G_{IJ} = g_{IJ} \text{ in } D'. \tag{2.5.6}$$

We let

$$G = \sum_{I,J} G_{IJ} dz^I \wedge d\bar{z}^J.$$

The formulæ (2.5.5)–(2.5.6) imply that

$$h' = d\bar{z}_{k+1} g - \overline{\partial} G$$

depends only on $d\bar{z}_1, \ldots d\bar{z}_k$. We can therefore apply the inductive hypothesis to $h + h' = f - \overline{\partial} G$. This form is $\overline{\partial}$–closed and only depends on $d\bar{z}_1, \ldots, d\bar{z}_k$ thus there exists $v \in \mathcal{C}^\infty(D'; \Lambda^{p,q-1})$ such that

$$\overline{\partial} v = h + h'.$$

Setting $u = v + G$ completes the proof of the theorem.

2.6 Power series and Reinhardt Domains

A important topic in the theory of one complex variable is the study of the convergence properties of power series. There are many criteria, a very simple one is the following: the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolutely in the set B defined by the condition

$$z \in B \text{ provided there exists a constant } C \text{ such that } |a_n z^n| \le C,\ \forall\, n \in \mathbb{N}.$$

A moments thought shows that B is a disk and its diameter is precisely the radius of convergence of the series.

This criterion has a simple generalization to several variables. If

$$\sum_{\alpha} a_\alpha z^\alpha \tag{2.6.1}$$

is a series then we define the set B as those points $z \in \mathbb{C}^n$ such that there exists a constant C such that

$$|a_\alpha z^\alpha| \le C,\ \forall\, \alpha. \tag{2.6.2}$$

If we denote by D the domain of normal convergence for the series in (2.6.1) then clearly D is contained in the interior of B. In fact it is easy to show that $D = \overset{\circ}{B}$.

Proposition 2.6.3. *The set D is the interior of B.*

Proof. Suppose that $z \in B$ then the series converges normally in the polydisk defined by $|w_i| < |z_i|$. For suppose the w lies in a compact subset of this polydisk, then we can find constants $k_i < 1$ so that

$$|w_i| \le k_i |z_i|.$$

By assumption there exists a constant C such that

$$|a_\alpha z^\alpha| \le C\,\forall, \alpha.$$

Therefore

$$\begin{aligned} \sum_{\alpha} |a_\alpha w^\alpha| &\le \sum_{\alpha} C k^\alpha \\ &= \prod_{i=1}^{n} (1-k_i)^{-1}. \end{aligned} \tag{2.6.4}$$

From (2.6.4) the conclusion is immediate.

From (2.6.3) it is clear that if $z \in D$ then

$$(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D,\ \text{ if } |\lambda_i| \le 1, i = 1, \ldots, n. \tag{2.6.5}$$

A domain that satisfies this condition is called a complete Reinhardt domain. If a domain satisfies (2.6.5) with $|\lambda_i| = 1, i = 1, \ldots, n$ it is called a Reinhardt domain. A little consideration shows that actually the domain of convergence of a power series satisfies a further property.

Suppose that z, w are two points in B then we can rewrite the condition (2.6.2) as follows

$$\begin{aligned} \log|a_\alpha| + |\alpha_1| \log|z_1| + \cdots + |\alpha_n| \log|z_n| &\le \log C \\ \log|a_\alpha| + |\alpha_1| \log|w_1| + \cdots + |\alpha_n| \log|w_n| &\le \log C \end{aligned} \tag{2.6.6}$$

From (2.6.6) it is clear that if ζ is a third point that satisfies

$$\log|\zeta_i| = \lambda \log|z_i| + (1-\lambda)\log|w_i|,\ \text{ for some } 0 < \lambda < 1,$$

then $\zeta \in B$ as well. From this we see that the set defined by

$$\mathbb{R}^n \supset D^* = \{(\log|z_1|, \ldots, \log|z_n|);\, z \in D\}$$

is convex. We have proved

Theorem 2.6.7. *If D is the domain of convergence of a power series then the set D^* is an open convex subset of $\mathbb{R}^n$. Furthermore if $(\xi_1, \dots, \xi_n) \in D^*$ and $\eta_i \leq \xi_i, i = 1, \dots n$ then $\eta \in D^*$ as well.*

A set which satisfies the hypotheses of (2.6.7) is called a log–convex Reinhardt domain. Now we show that if Ω is a Reinhardt domain containing zero then every function $u \in H(\Omega)$ is represented by a power series that converges normally in Ω.

Theorem 2.6.8. *Suppose that Ω is a bounded Reinhardt domain which contains zero and $u \in H(\Omega)$ then the power series for u about 0 converges normally in Ω.*

Proof. Let

$$\Omega'_\epsilon = \{z \in \Omega;\ d(z, \Omega^c) > \epsilon|z|\}.$$

Let Ω_ϵ be the component of Ω'_ϵ which contains 0. Evidently Ω_ϵ is a Reinhardt domain for each ϵ and since Ω is path connected it is clear that

$$\Omega = \bigcup_{\epsilon > 0} \Omega_\epsilon. \tag{2.6.9}$$

Define the function

$$U_\epsilon(z) = \left(\frac{1}{2\pi\imath}\right)^n \int \cdots \int_{|t_i| = 1-\epsilon} \frac{u(t_1 z_1, \dots, t_n z_n) dt_1 \dots dt_n}{(t_1 - 1) \dots (t_n - 1)}. \tag{2.6.10}$$

If $z \in \Omega_\epsilon$, and $w \in \Omega^c$ then

$$\begin{aligned} |(1-\epsilon)w - z| &\geq |w - z| - \epsilon|z| \\ &\geq d(z, \Omega^c) - \epsilon|z| \\ &> 0. \end{aligned} \tag{2.6.11}$$

The final inequality in (2.6.11) follows because $z \in \Omega_\epsilon$. Since Ω_ϵ is a Reinhardt domain, this shows that the integral in (2.6.10) is defined for $z \in \Omega_\epsilon$; differentiating under the integral sign shows that U_ϵ is analytic.

Using the series expansion for the denominator in (2.6.10) we can expand U_ϵ as a normally convergent series

$$U_\epsilon(z) = \sum_\alpha f_\alpha(z). \tag{2.6.12}$$

If δ is small enough then the polydisk $|z_i| \leq \delta$ is contained in Ω. For z is this polydisk we can use the Cauchy formula to compute that

$$f_\alpha(z) = \frac{\partial_z^\alpha u(0) z^\alpha}{\alpha!}. \tag{2.6.13}$$

Since f_α is holomorphic for each α and Ω_ϵ is connected it follows that (2.6.13) holds in all of Ω_ϵ. Thus

$$U_\epsilon(z) = u(z) \restriction_{\Omega_\epsilon}$$

and the power series for u is normally convergent in Ω_ϵ for all $\epsilon > 0$. The theorem follows from (2.6.9).

It follows from Theorem (2.6.7) that the domain of convergence of a power series is always a complete log–convex Reinhardt domain. It is clear that there is a smallest such domain containing any Reinhardt domain, Ω which contains 0. Call it $\widetilde{\Omega}$. From Theorem (2.6.8) it follows that any function which belongs to $H(\Omega)$ actually extends, via its power series about 0, to a function in $H(\widetilde{\Omega})$. Thus we have another extension theorem in several variables which has no one–variable analogue. Recall that a domain of holomorphy is an open subset $\Omega \subset \mathbb{C}^n$ such that there is a function $u \in H(\Omega)$ which cannot be extended to any open set U which contains Ω as a proper subset. We have proved the following

Theorem 2.6.14. *In order for a Reinhardt domain which contains 0 to be a domain of holomorphy it must be complete and log–convex.*

We shall soon see that this is, in fact, also a sufficient condition.

2.7 Domains of holomorphy and holomorphic convexity

As remarked above a domain of holomorphy is defined in $\mathbb{C}^n$ exactly as in $\mathbb{C}$:

Definition 2.7.1. A domain $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy if there exists a function $u \in H(\Omega)$ such that for any open set V which contains Ω as a proper subset there is no function $U \in H(V)$ with $U \restriction_\Omega = u$.

Loosely speaking, the function cannot be extended across any boundary point of Ω.

We saw that any open set in $\mathbb{C}$ is a domain of holomorphy. If $V_i \subset \mathbb{C}$ are open sets then the open subset of $\mathbb{C}^n$ defined by

$$\Omega = \prod_{i=1}^{n} V_i$$

is a domain of holomorphy. Let $f_i \in H(V_i)$ be a function with no holomorphic extension beyond V_i. Defining

$$f(z) = \prod_{i=1}^{n} f_i(z_i),$$

we obtain a function in $H(\Omega)$ which does not extend to any open set properly containing Ω.

As in the case of one variable we can define a notion of holomorphic convexity.

Definition 2.7.2. An open set $\Omega \subset \mathbb{C}^n$ is holomorphically convex if for every compact subset, $K \subset\subset \Omega$, the holomorphic convex hull

$$\widehat{K}_\Omega = \{z \in \Omega;\ |f(z)| \leq \sup_K |f|\, \forall\, f \in H(\Omega)\},$$

is a compact subset of Ω.

To study the relationship between these two concepts it is useful to define a notion of distance to the complement of Ω in terms of polydisks. Let $R = (r_1, \ldots, r_n)$ be a polyradius then for a point $z \in \Omega$ we define

$$\delta_{\Omega,R} = \sup\{\epsilon;\ D(z; \epsilon R) \subset \Omega\}.$$

For a subset $K \subset \Omega$ we define

$$\delta_{\Omega,R}(K) = \inf_{z \in K} \delta_{\Omega,R}(z).$$

The utility of this concept is illustrated by the following proposition

Proposition 2.7.3. *If K is a compact subset of Ω and $\delta = \delta_{\Omega,R}(K)$, for some polyradius, then any function in $H(\Omega)$ extends to be holomorphic in $D(z; \delta R)$ for any $z \in \widehat{K}_\Omega$.*

Proof. Let $0 < \epsilon < \delta$, then the set

$$K_\epsilon = \bigcup_{z \in K} D(z; \epsilon R).$$

is a relatively compact subset of Ω. Thus, the Cauchy inequalities imply that for any function $f \in H(\Omega)$ there is a constant M such that

$$|\partial_z^\alpha f(z)| \leq M \alpha! (\epsilon R)^{-\alpha}, z \in K. \tag{2.7.4}$$

Since any derivative of f also belongs to $H(\Omega)$ it follows from the definition of holomorphic convex hull that the estimates in (2.7.4) are valid for $z \in \widehat{K}_\Omega$. From this it is immediate that the power series for f about a point $z \in \widehat{K}_\Omega$ converges normally in $D(z; \delta R)$.

With this lemma we can show that being a domain of holomorphy is equivalent to being a holomorphically convex.

Theorem 2.7.5. *An open set $\Omega \subset \mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy*

Proof. First we show that if Ω is holomorphically convex then we can find a function which does not extend across any boundary point of Ω. Arguing as in the one dimensional case, choose a nested sequence of compact subsets, K_j such that $\widehat{K}_j = K_j$. Then we choose a sequence of points $\mathcal{A} = \{A_j\}$ such that

(1) The boundary of Ω equals the set of cluster points of $\mathcal{A}$,
(2) The intersections $K_j \cap \mathcal{A}$ are finite.

We can relabel the set $\mathcal{A}$ so that there is a monotone sequence $\{n_j\}$ such that

$$K_j \cap \mathcal{A} = A_1, \dots a_{n_j}.$$

Since $\widehat{K}_j = K_j$, for each $j \in n_j + 1, \dots n_{j+1}$ we can find functions $f_k \in H(\Omega)$ such that $|f_k| < 1$ on K_j but $f_k(A_k) = 1$. By taking sufficiently high powers of these functions, which we also denote by f_k we can arrange that

$$\sup_{z \in K_j} \sum_{k=n_j+1}^{n_{j+1}} |f_k(z)| \leq (j2^j)^{-1}. \tag{2.7.6}$$

If we fix an m then it follow from (2.7.6) that

$$\sum_{j=1}^{\infty} j|f_j(z)|$$

converges uniformly on K_m. Therefore the function defined by

$$F(z) = \prod_{j=1}^{\infty}(1 - f_j(z))^j$$

is nonconstant and belongs to $H(\Omega)$. This function cannot be extended to any polydisk $D(w; R)$ not contained in Ω.

To prove this we observe that F has a zero of order j at A_j. If we could extend F beyond Ω then we could extend it to some polydisk with center $w \in \partial\Omega$. By the construction of the sequence $\mathcal{A}$ we can find a subsequence j_k such that $lim A_{j_k} = w$. From this it follows easily that for any multiindex α

$$\partial\alpha F(w) = \lim_{k\to\infty} \partial^\alpha F(A_{j_k}) = 0.$$

This would imply that F is identically zero.

To prove the converse we assume that Ω is a domain of holomorphy but is not holomorphically convex. Let $f \in H(\Omega)$ be a function which cannot be extended across any boundary point. Let K be a compact subset of Ω with a non–compact holomorphic convex hull, $\widehat{K}$. Let R be some polyradius and let $\delta = \delta_{\Omega,R}(K) > 0$. Since $\widehat{K}$ is non–compact we can choose a point $w \in \widehat{K}$ such that $D(w; \delta R)$ is not contained in Ω. According to Proposition (2.7.3) the function f can be extended to $D(w; \delta R)$. But this contradicts the fact that f cannot be extended across any boundary point of Ω.

This turns out to be a useful criterion for deciding if an open set is a domain of holomorphy. The following theorem illustrates why.

Theorem 2.7.7. *In order for an open set Ω to be a domain of holomorphy it is necessary and sufficient that given any discrete sequence of points $\{A_j\} \subset \Omega$ there is a function $f \in H(\Omega)$ such that*

$$\limsup_{k\to\infty} |f(A_k)| = \infty. \tag{2.7.8}$$

Proof. First suppose that such a function can be found for any sequence but that Ω is not holomorphically convex this means that there is a compact subset K such that $\widehat{K}$ is non–compact. We can choose a sequence of points $\{A_j\} \subset \widehat{K}$ such that A_j tend to $\partial\Omega$. Let $f \in H(\Omega)$ satisfy (2.7.8) relative to this sequence. Finally let

$$M = \sup_{z \in K} |f(z)|.$$

Clearly we can find some j_0 such that $|f(A_{j_0}| > M$. but this contradicts the fact that $A_{j_0} \in \widehat{K}$. Thus $\widehat{K}$ must also be a compact set.

To construct a function we argue much as in the previous proof. We can exhaust Ω by a nested sequence of compact subsets K_j which satisfy

$$\widehat{K}_j = K_j. \tag{2.7.9}$$

Possibly after choosing a subsequence we can assume that $A_k \notin K_j, k \geq j$. In light of (2.7.9) we can find a sequence of functions $f_j \in H(\Omega)$ such that

$$\sup_{z \in K_j} |f_j(z)| < 1 \text{ and } |f_j(A_j)| > 1. \tag{2.7.9}$$

By raising this function to a sufficiently high power we can assume that

$$\begin{aligned} &\sup_{z \in K_j} |f_j(z)| < 2^{-j} \text{ and} \\ &|f_j(A_j)| > j + \sum_{k=1}^{j-1} |f_k(A_j)|. \end{aligned} \tag{2.7.10}$$

It is easy to see that the series

$$f = \sum_{j=1}^{\infty} f_j$$

converges locally uniformly to a function in $H(\Omega)$. Moreover, using the conditions in (2.7.10) we obtain that

$$|f(A_j)| > j - 2^{-j}. \tag{2.7.11}$$

From (2.7.11) it follows easily that

$$\lim_{j \to \infty} |f(A_j)| = \infty.$$

As an application of this theorem we obtain

Theorem 2.7.12. *If $\Omega \subset \mathbb{C}^n$ is linearly convex then it is holomorphically convex and therefore a domain of holomorphy.*

Proof. If $p \in \partial\Omega$ then there is a real valued linear function $l_p(z)$ such that $l_p(p) = 0$ and $l_p(z) > 0$ for $z \in \Omega$. Observe that any real linear function can be written in the form

$$l_p(z) = \text{Re}[z \cdot a + b] \text{ for some } a \in \mathbb{C}^n, b \in \mathbb{C}. \tag{2.7.12}$$

If we set $\lambda_p(z) = z \cdot a + b$ then $\lambda_p(z)$ is a holomorphic function whose real part vanishes at p and is positive in Ω. Thus

$$\Lambda_p(z) = (\lambda_p(z) - \lambda_p(p))^{-1} \in H(\Omega)$$

but

$$\lim_{z \to p} |\Lambda_p(z)| = \infty.$$

From this it is evident that for any compact set, K the holomorphic convex hull $\widehat{K}$ avoids some neighborhood of p. Since $p \in \partial\Omega$ is arbitrary it follows that $\widehat{K}$ avoids some neighborhood and is therefore compact. This proves the theorem.

In fact it is clear that the method used in the proof above give another criterion for holomorphic convexity

Proposition 2.7.13. *Suppose that Ω is an open subset of $\mathbb{C}^n$ such that for every boundary point p there is a function $f \in H(\Omega)$ such that $\operatorname{Re} f(z) > 0$ for every $z \in \Omega$ and*

$$\lim_{z \to p} \operatorname{Re} f(z) = 0$$

then Ω is holomorphically convex and therefore a domain of holomorphy.

This condition come very close to being a local condition. That is, we can easily give a local condition which will ensure that we can find a holomorphic function defined in some neighborhood of a given boundary point which satisfies the hypotheses of (2.7.13) in that neighborhood. The problem then is to show that we can actually find a global holomorphic function with desired properties. This is what is usually called the Levi problem. More precisely it asks whether there is a local condition which implies that a domain is a domain of holomorphy. Being a domain of holomorphy is clearly a biholomorphically invariant notion. We might try, as a local condition, that some neighborhood of each boundary is biholomorphically equivalent to a linearly convex set. This turns out to be almost correct.

For latter applications we need a somewhat more general version of Proposition (2.7.3):

Theorem 2.7.14. *Let R be a positive polyradius and Ω be an open subset of $\mathbb{C}^n$. Suppose that $K \subset\subset \Omega$ and $f(z) \in H(\Omega)$ satisfies:*

$$|f(z)| < \delta_{R,\Omega}(z), z \in K. \tag{2.7.15}$$

Let $\zeta \in \widehat{K}_\Omega$, then if $u \in H(\Omega)$ then the power series of u converges in the polydisk center at ζ of polyradius $f(\zeta)|R$.

Proof. The argument is essentially identical to the proof of (2.7.3). Let D denote the polydisk centered at zero with polyradius R. For $w \in K$, the power series for u:

$$u(z) = \sum_\alpha \frac{\partial^\alpha u(w)(z-w)^\alpha}{\alpha},$$

converges in the polydisk $w + \delta_{\Omega,R}(w)D$. If $t < 1$ it follows from (2.7.15) that the union of the polydisks

$$K_t = \bigcup_{w \in K} w + t|f(w)|D$$

is a relatively compact subset of Ω and therefore we can find a constant M such that $|u(z)| \leq M, z \in K_t$. We therefore have the Cauchy estimates

$$|\partial^\alpha u(w)|t^{|\alpha|}R^\alpha|f(w)|^\alpha \leq M\alpha! \tag{2.7.16}$$

hold for $w \in K$. Since $\partial^\alpha u(w)f(w)^\alpha \in H(\Omega)$ for each α it follows that the estimates in (2.7.16) hold for $w \in \widehat{K}_\Omega$ From this it follows that the series expansion for u converge in $w + t|f(w)|D$ for every $t < 1, w \in /hK_\Omega$. This proves the theorem.

This theorem has a important corollary

Corollary 2.7.17. *If Ω is a domain of holomorphy, $K \subset\subset \Omega$, R a positive polyradius and $f \in H(\Omega)$ satisfies*

$$|f(z)| \leq \delta_{\Omega,R}(z),\ z \in K,$$

then

$$|f(z)| \leq \delta_{\Omega,R}(z),\ z \in \widehat{K}_\Omega.$$

Proof. If this were not the case then (2.7.14) would imply that we could extend every function in $H(\Omega)$ to some polydisk not entirely contained in Ω. This violates the assumption that Ω is a domain of holomorphy.

This corollary in turn has a corollary which is obtained by using the special functions f =constant:

Corollary 2.7.18. *If Ω is a domain of holomorphy then for any polyradius R and any compact set K*

$$\delta_{\Omega,R}(K) = \delta_{\Omega,R}(\widehat{K}_\Omega). \tag{2.7.19}$$

Exercise 2.7.19. The purpose of this exercise is to show that a complete log–convex Reinhardt domain is a domain of holomorphy. We know from Theorem (2.6.14) that this is a necessary condition, we will show that it is also sufficient. The idea is to show that a complete, log–convex is holomorphically convex.

(1) If $K \subset\subset \Omega$ then there is a finite set, $\mathcal{K} \subset \Omega$ such that

$$K \subset \bigcup_{\zeta\in\mathcal{K}} \{z, |z_i| \leq |\zeta_i|, \, i = 1, \dots n\} \subset\subset \Omega.$$

(2) If α is a multiinidex with $\alpha_1 \dots \alpha_j \neq 0$ and $\alpha_m = 0, m = j+1, \dots, n$ then

$$|z_1^{\alpha_1} \dots z_j^{\alpha_j}| \leq \sup_{\zeta\in\mathcal{K}} \{|\zeta_1^{\alpha_1} \dots \zeta_j^{\alpha_j}|; \, \zeta \in \mathcal{K}\}.$$

(3) Show that this implies that if $\lambda_i, i = 1, \dots, j \in [0,1]$ then

$$\sum \lambda_i \log |z_i| \leq \sup_{\zeta\in\mathcal{K}} \sum \lambda_i \log |\zeta_i|.$$

(4) Conclude from this that if Ω is complete, log–convex Reinhardt domain then $\widehat{K}_\Omega \subset\subset \Omega$. hint: show that

$$(\log |z_1|, \dots, \log |z_j|), z \in \widehat{K}_\Omega$$

is in the convex hull of the $\eta \in \Omega^*$ with

$$\eta_i \leq \log |\zeta_i|, i = 1, \dots j$$

for some $\zeta \in \mathcal{K}$.

2.8 Pseudoconvexity, the ball versus the polydisc

In this section we will consider a local condition, pseudoconvexity, which is satisfied by any domain of holomorphy. This is geometric condition which is more closely related to the holomorphic geometry of the unit ball than to that of the polydisk. After discussing pseudoconvexity and its relationship to holomorphic convexity we will consider complex analysis on the unit ball.

We need to consider a generalization of subharmonic functions to several variables.

Definition 2.8.1. A function u defined in an open set $\Omega \subset \mathbb{C}^n$ taking values in $[-\infty, \infty)$ is call plurisubharmonic provided

(1) u is upper semicontinuous, that is

$$u(z) \geq \limsup_{w\to z} u(w),$$

(2) for any $w, z \in \mathbb{C}^n$ the function $f(\tau) = u(z + \tau w)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.

The collection of such functions is denoted by $P(\Omega)$.

Briefly, a function is plurisubharmonic if its restriction to any complex line is subharmonic. Note that if $u \in H(\Omega)$ then $\log |u(z)| \in P(\Omega)$. If $u_1, u_2 \in P(\Omega)$ then so is $au_1 + bu_2$ is a, b are positive, and

$$u(z) = \sup\{u_1(z), u_2(z)\}$$

is also in $P(\Omega)$.

As with subharmonic functions, plurisubharmonic functions may not be smooth, however if a plurisubharmonic function has two derivatives then this condition can be described in terms of a differential inequality.

Proposition 2.8.2. *A function $u \in C^2(\Omega)$ is plurisubharmonic if and only if*

$$\sum_{i,j} \partial_{z_i}\partial_{\bar{z}_j} u(w)\xi_i\overline{\xi}_j \geq 0, \, \forall\, w \in \Omega, \, \xi \in \mathbb{C}^n. \tag{2.8.3}$$

Proof. A twice differentiable function $f(\tau), \tau \in \mathbb{C}$ is subharmonic if and only if

$$\Delta f = \frac{1}{4}\partial_\tau \overline{\partial}_\tau f \geq 0. \tag{2.8.4}$$

Let $z, w \in \mathbb{C}^n$ and set $f(\tau) = u(z + \tau w)$ then

$$\Delta f(\tau) = \sum_{i,j} \partial_{z_i}\partial_{\bar{z}_j} u(z + \tau w) w_i \bar{w}_j. \tag{2.8.5}$$

The proposition follows from (2.8.4)–(2.8.5).

Exercise 2.8.6. Suppose that $u \in P(\Omega)$ and suppose that $\phi \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ has support in the ball of radius 1, depends only on $|z_1|, \ldots, |z_n|$ and satisfies

$$\int \phi \, \mathrm{dVol} = 1.$$

(1) The functions

$$u_\epsilon(z) = \int u(z + \epsilon w)\phi(w) \, \mathrm{dVol},$$

are smooth and plurisubharmonic in

$$\Omega_\epsilon = \{z \in \epsilon; (z, \Omega^c) > \epsilon\},$$

(2) $u_\epsilon(z) \geq u_\delta(z)$ if $\epsilon > \delta$,

(3) $\lim_{\epsilon \to 0} u_\epsilon = u$.

We can now establish a connection between domains of holomorphy and plurisubharmonic functions

Theorem 2.8.7. *If Ω is a domain of holomorphy and R is a positive polyradius then $-\log \delta_{\Omega,R}(z)$ is plurisubharmonic and continuous*

Proof. The continuity is clear from the definition. We use the following characterization of subharmonic functions: a function $f(\tau)$ is subharmonic in a domain $D \subset \mathbb{C}$ if for every disk $B(w;r) \subset\subset D$ and every harmonic function h defined on $\overline{B}(w;r)$ which satisfies

$$f(w + re^{i\theta}) \leq h(w + re^{i\theta})$$

also satisfies

$$f(w + \rho e^{i\theta}) \leq h(w + \rho e^{i\theta}), 0 \leq \rho < r. \tag{2.8.9}$$

Fix a point $z_0 \in \Omega$ and a $w \in \mathbb{C}^n$. Choose an $r > 0$ so that

$$D = \{(z_0 + \tau w) \in \Omega; |\tau| \leq r\}.$$

Let $f(\tau)$ be an analytic polynomial which satisfies

$$-\log \delta_{\Omega,R}(z_0 + \tau w) \leq \operatorname{Re} f(\tau), \text{ for } |\tau| = r.$$

Any harmonic function on $|\tau| \leq r$ which satisfies such an estimate can be approximated uniformly by real parts of holomorphic polynomials. This is so because any harmonic function in a disk is the real part of a holomorphic function and any holomorphic function is uniformly approximated by polynomials. Thus it suffices to consider polynomials.

Let $F(z)$ be a polynomial defined in $\mathbb{C}^n$ which satisfies

$$F(z_0 + \tau w) = f(\tau).$$

The maximum principle implies that the holomorphic convex hull of ∂D with respect to Ω must contain D and therefore we can apply Theorem (2.7.14) an (2.8.9) to conclude that

$$|e^{-F(z)}| \leq \delta_{\Omega,R}(z), z \in D. \tag{2.8.10}$$

Rewriting this we obtain

$$-\log \delta_{\Omega,R}(z_0 + \tau w) \leq \operatorname{Re} f(\tau).$$

Thus $-\log \delta_{\Omega,R}(z_0 + \tau w)$ is a subharmonic function, where it is defined and thus $-\log \delta_{\Omega,R}(z) \in P(\Omega)$.

The converse of this result is true but it will take some effort to prove.

In analogy with the holomorphic convex hull we define the define a plurisubharmonic convex hull.

Definition 2.8.11. If $K \subset\subset \Omega \subset \mathbb{C}^n$ then we define the plurisubharmonic hull of K relative to Ω by

$$\widehat{K}_\Omega^P = \{z \in \Omega;\ u(z) \leq \sup_K u \text{ for all } u \in P(\Omega)\}.$$

Obviously the $P(\Omega)$–hull of K is contained inside the $H(\Omega)$–hull. From this it is immediate that a holomorphically convex domain is also pseudoconvex.

Definition 2.8.12. An open subset $\Omega \subset \mathbb{C}^n$ is said to be pseudoconvex if for every $K \subset\subset \Omega$ we have

$$\widehat{K}_\Omega^P \subset\subset \Omega.$$

This turns out to be quite a flexible concept. It has several alternative characterizations .

Theorem 2.8.13. *An open subset $\Omega \subset \mathbb{C}^n$ is pseudoconvex if and only if either*

(2.8.14) *For any positive polyradius R the function $-\log \delta_{\Omega,R}(z)$ is plurisubharmonic*

or

(2.8.15) *There exists a function $u \in P(\Omega)$ such that the sets $\{z;\ u(z) < c\} \subset\subset \Omega$.*

Proof. It is clear that (2.8.14) implies (2.8.15). Furthermore (2.8.15) clearly implies that Ω is pseudoconvex. All that remains is to show that if Ω is pseudoconvex then (2.8.14) holds. Let D denote the polydisk centered at zero with polyradius R. We need to show that $-\log \delta_{\Omega,R}(z)$ is plurisubharmonic. To simplify the notation we denote this function by $-\log \delta$.

Choose $z_0 \in \Omega$, $w \in \mathbb{C}^n$ and an positive number r such that

$$B = \{z_0 + \tau w;\ |\tau| < r\} \subset \Omega$$

and a holomorphic polynomial $f(\tau)$ which satisfies

$$-\log \delta(z_0 + \tau w) \leq \operatorname{Re} f(\tau), |\tau| = r. \tag{2.8.16}$$

We need to show that (2.8.16) holds throughout B. We can rewrite this inequality as

$$\delta(z_0 + \tau w) \geq |e^{-f(\tau)}|. \tag{2.8.16'}$$

To extend this inequality to $|\tau| < r$ we let $a \in D$ and consider the mapping for $0 \le \lambda \le 1$

$$\tau \longrightarrow z_0 + \tau w + \lambda a e^{-f(\tau)}.$$

Denote the image of $|\tau| \le r$ by B_λ. If we can show that $B_1 \subset \Omega$ then, since $a \in D$ is arbitrary, (2.8.16') would follow for $|\tau| \le r$. Let

$$\Lambda = \{\lambda \in [0,1];\, B_\lambda \in \Omega\}.$$

It is clear that $0 \in \Lambda$ and furthermore that Λ is an open set. We will show that pseudoconvexity implies that it is also closed. Let

$$K = \{z_0 + \tau w + \lambda a e^{-f(\tau)};\, |\tau| = r, \lambda \in [0,1]\}.$$

The inequality (2.8.16') implies that $K \subset \Omega$, it is clearly compact. Let $u \in P(\Omega)$ and suppose that $\lambda \in \Lambda$ then

$$\tau \longrightarrow u(z_0 + \tau w + \lambda a e^{-f(\tau)})$$

is subharmonic in a neighborhood of the disk $|\tau| < r$. Thus we have the inequality

$$u(z_0 + \tau w + \lambda a e^{-f(\tau)}) \le \sup_K u \text{ if } |\tau| < r.$$

Since $u \in P(\Omega)$ is arbitrary this implies that $B_\lambda \subset \widehat{K}^P_\Omega$ for every $\lambda \in \Lambda$. This in turn implies that Λ is closed as $\widehat{K}^P_\Omega$ is assumed to be a relatively compact subset of Ω.

Since $[0,1]$ is connected this implies that $B_1 \subset \Omega$ and therefore

$$z_0 + \tau w + a e^{-f(\tau)} \in \Omega \text{ if } a \in D,\ |\tau| \le r.$$

But this implies that

$$\delta(z_0 + \tau w) \ge |e^{-f(\tau)}|,\ |\tau| \le r$$

which implies that

$$-\log \delta \in P(\Omega).$$

In this theorem the function $\delta_{\Omega,R}$ does not have to be defined by a polydisk. In fact the theorem is true if we choose any continuous, non–negative function δ defined on $\mathbb{C}^n$ which satisfies

(2.8.17)
$$\delta(tz) = |t|\delta(z),\ t \in \mathbb{C}$$

and define

$$\delta_\Omega(z) = \inf_{w \in \Omega^c} \delta(z - w).$$

For example we could take $\delta(z) = |z|$.

One of the important features of pseudoconvexity is that it is a local property of the boundary:

Theorem 2.8.18. *Let $\Omega \subset \mathbb{C}^n$ be an open set such that to every point in $\overline{\Omega}$ there is an open set ω such that $\omega \cap \Omega$ is pseudoconvex then Ω is also pseudoconvex.*

Proof. We will show that Ω has a plurisubharmonic exhaustion function. For a $z_0 \in \partial\Omega$ choose a neighborhood ω such that $\omega \cap \Omega$ is pseudoconvex. The for some $\delta(z)$ satisfying (2.8.17) the function $-\log \delta_{\omega\cap\Omega}(z)$ is plurisubharmonic. There is a smaller open set $\omega' \subset \omega$ in which we have

$$\delta_\Omega(z) = \delta_{\omega\cap\Omega}(z), z \in \omega'.$$

From this we conclude that $-\log \delta_\Omega(z)$ is itself plurisubharmonic in some neighborhood of $\partial\Omega$. This means that there is a closed subset $F \subset \Omega$ such that $-\log \delta_\Omega$ is plurisubharmonic in $\Omega \setminus F$. It follows from (2.8.17) that we can choose a convex increasing function $\phi(x)$ such that

(2.8.19)
$$-\log \delta_\Omega(z) < \phi(|z|^2) \text{ for } z \in F.$$

If we set $u(z) = \sup\{-\log \delta_\Omega(z), \phi(|z|^2)\}$ then $u \in P(\Omega)$ as $u = \phi(|z|^2)$ in a neighborhood of F and the supremum of two plurisubharmonic functions is subharmonic. This function clearly satisfies the condition (2.8.15) and therefore Ω is plurisubharmonic.

We consider various properties of pseudoconvex sets. First we consider the intersection of pseudoconvex sets.

Theorem 2.8.20. *If Ω_1, Ω_2 are pseudoconvex open subsets of $\mathbb{C}^n$ then so is $\Omega_1 \cap \Omega_2$.*

Proof. This follows easily from the properties of the plurisubharmonic hull. Let $K \subset\subset \Omega_1 \cap \Omega_2$. Since

$$P(\Omega_i) \subset P(\Omega_1 \cap \Omega_2), i = 1, 2$$

it follows that

$$\widehat{K}_{\Omega_1 \cap \Omega_2} \subset \widehat{K}_{\Omega_1} \cap \widehat{K}_{\Omega_2}. \tag{2.8.21}$$

Since Ω_1 and Ω_2 are pseudoconvex $\widehat{K}_{\Omega_1}$ and $\widehat{K}_{\Omega_2}$ are compact subsets. This implies that each avoids some open neighborhood of $\partial\Omega_1$ or $\partial\Omega_2$ respectively. Since

$$\partial\Omega_1 \cap \Omega_2 = \Omega_1 \cap \partial\Omega_2 \cup \Omega_2 \cap \partial\Omega_1$$

it follows from (2.8.21) that $\widehat{K}_{\Omega_1 \cap \Omega_2}$ is a compact subset of $\Omega_1 \cap \Omega_2$.

Note that the same argument shows that the intersection of two domains of holomorphy is a domain of holomorphy.

To study unions we need a result about sequences of subharmonic functions.

Lemma 2.8.22. *Suppose that $u_i(z)$ is a decreasing sequence of subharmonic functions in a domain $D \subset \mathbb{C}$. If $\lim u_i(z_0) \neq -\infty$ for some $z_0 \in D$ then*

$$u(z) = \lim_{i \to \infty} u_i(z)$$

if finite almost everywhere and subharmonic.

Proof. To see that $u(z) > -\infty$ for almost every z we use that fact that if $B(z_0, r) \subset\subset D$ then for every i

$$u_i(z_0) \leq \frac{1}{\pi r^2} \iint u_i(z) dx dy. \tag{2.8.23}$$

Since the sequence is decreasing we can apply Lebesgue's monotone convergence theorem to conclude that

$$-\infty < u(z_0) \leq \frac{1}{\pi r^2} \iint u(z) dx dy. \tag{2.8.24}$$

From this it follows that $u(z) > -\infty$ for almost every $z \in B(z_0, r)$. We simply repeat the argument with a new point z_1 near to $\partial B(z_0, r)$ in this way we can show that $u(z) > -\infty$ for almost all $z \in D$. We can also use the Lebesgue theorem to conclude that for every $z \in D$ and $r > 0$ such that $B(z, r) \subset D$

$$u(z) \leq \frac{1}{\pi r^2} \iint\limits_{B(z,r)} u(z) dx dy. \tag{2.8.24}$$

This suffices to conclude that $u(z)$ is subharmonic.

To see this we observe that a function that satisfies (2.8.24) must satisfy the maximum principle. Thus if we take any harmonic function h then $u - h$ also satisfies the integral inequality and therefore the maximum principle. This implies that if $u - h \leq 0$ on $\partial B(w, r)$ then it is also negative in $B(w, r)$. This however was our definition of a subharmonic function.

Exercise 2.8.25. Prove that (2.8.24) and the upper semicontinuity of u imply that u satisfies the maximum principle.

Using the lemma we can study unions of pseudoconvex sets.

Theorem 2.8.26. *Suppose that Ω_i, $i = 1, \dots$ is an increasing sequence of pseudoconvex domains then*

$$\Omega = \bigcup_{i=1}^{\infty} \Omega_i$$

is pseudoconvex.

Proof. We use the characterization given by (2.8.14). Let R be a fixed polyradius. The sequence of functions $-\log \delta_{\Omega_i,R}(z)$ is decreasing for each fixed z. Evidently the limit is $-\log \delta_{\Omega,r}(z)$. In virtue of (2.8.14) each function in the sequence is plurisubharmonic and therefore by (2.8.22) the limit is as well.

We will now consider an important special case. We suppose that Ω has a C^2 boundary. This means that there is a function ρ twice differentiable in some neighborhood of Ω such that

$$\Omega = \{z;\, \rho(z) < 0\}$$

and furthermore

$$d\rho(z) \neq 0 \text{ for } z \in \partial\Omega.$$

Such a function is called a defining function for Ω. We have the following description of a pseudoconvex domain in terms of a defining function. We will only prove an important special case.

Theorem 2.8.27. *A domain with C^2–boundary is pseudoconvex if and only if there is a defining function, ρ such that*

$$\partial\overline{\partial}\rho(z)(X, \overline{X}) \geq 0, \text{ for all } z \in \partial\Omega, \text{ for which } \partial\rho(X) = 0. \tag{2.8.28}$$

Proof. One direction is obvious. Let

$$\rho(z) = -d(z, \Omega^c), z \in \Omega, \rho(z) = d(z, \Omega), z \in \Omega^c.$$

Using the implicit function theorem one can show that ρ is C^2 in some neighborhood of $\partial\Omega$. At such point we can compute $-\partial\overline{\partial}\log\rho$:

$$-\partial_{z_i}\partial_{\bar{z}_j}\log\rho = \sum_{i,j}[-\frac{\partial_{z_i}\partial_{\bar{z}_j}\rho}{\rho} + \frac{\partial_{z_i}\rho}{\rho}\frac{\partial_{\bar{z}_j}\rho}{\rho}]. \tag{2.8.29}$$

If $\partial\rho(X) = 0$ then the second term in (2.8.29) vanishes and the condition that $-\log\rho$ be plurisubharmonic reduces to the positivity of the first term. This evidently persists as we approach the boundary.

We will not prove this result in full generality but instead use a somewhat stronger hypotheses: we assume that $\partial\overline{\partial}\rho > 0$ on $\ker\partial\rho$ at $\partial\Omega$. We also assume that Ω is bounded. To show that Ω is pseudoconvex we apply (2.8.15) and produce a plurisubharmonic defining function. Clearly $-\log\rho$ is plurisubharmonic in a neighborhood of the boundary and blows up as we approach the boundary. To correct it in the interior we simply add a multiple of $|z|^2$. Thus we have a plurisubharmonic exhaustion function of the form

$$-\log r + M|z|^2,$$

which proves the theorem in this case.

A C^2–domain which has a defining function ρ which satisfies

$$\partial\overline{\partial}\rho(z) > 0 \text{ on } \ker\partial\rho \text{ at } \partial\Omega$$

is called strictly pseudoconvex. Consider the defining functions for Ω given by

$$\phi_t = e^{t\rho} - 1, t > 0.$$

If we compute $\partial\overline{\partial}\phi_t$ we see that

$$\partial\overline{\partial}\phi_t = te^{t\rho}(\partial\overline{\partial}\rho + t\partial\rho\overline{\partial}\rho). \tag{2.8.30}$$

Evidently if t is sufficiently large then $\partial\overline{\partial}\phi_t > 0$ at $\partial\Omega$ with no further restriction. Using techniques similar to those we have been using one can show that any pseudoconvex domain can be exhausted by smooth strictly pseudoconvex subdomains. These domains have much simpler analytic properties than general pseudoconvex domains and also have a very rich geometric structure which we will discuss. The canonical example is the unit ball

$$\mathbb{CB}^n = \{z \in \mathbb{C}^n;\, |z|^2 - 1 < 0\}.$$

It serves as the model domain for the study of strictly pseudoconvex domains in much the same way as the polydisk served as a model for a general pseudoconvex domain.

2.9 CR-structures and the Lewy extension theorem

Let $\Omega \subset \mathbb{C}$ be an open set with a smooth boundary. The question as to whether a function $f \in C^0(\Omega)$ is the boundary value of a function $u \in H(\Omega)$ is studied using a non–local, pseudodifferential operator defined on $\partial\Omega$. If we donte this operator by P_Ω then f is the boundary value of a holomorphic function if and only if $P_\Omega f = 0$. In more dimensions the situation is quite different. The condition that a function defined on $\partial\Omega$ be the boundary value of a holomorphic function defined in Ω is given by local, i.e. differential operators.

Recall that a complex structure on Ω is determined by a subbundle of the complexified tangent space, $T\Omega \otimes \mathbb{C}$, denoted by $T^{1,0}\Omega$. It satisfies two conditions; let $T^{0,1}\Omega = \overline{T^{1,0}\Omega}$ then

$$\begin{gathered} T^{1,0}\Omega \cap T^{0,1}\Omega = \{0\}, \\ \text{If } X, Y \text{ are sections of } T^{1,0}\Omega \text{ then so is } [X, Y]. \end{gathered} \tag{2.9.1}$$

In this formulation a function $u \in C^1(\Omega)$ is holomorphic if

$$\overline{Z}u = 0 \text{ for all sections,} \overline{Z} \text{ of } T^{0,1}\Omega. \tag{2.9.2}$$

From (2.9.2) we can easily derive necessary conditions for a function $f \in C^1(\Omega)$ to be the boundary value of a holomorphic function. In some cases these turn out to also be sufficient. Let M denote $\partial\Omega$ and J denote the almost complex structure underlying the complex structure. Suppose that $\overline{Z}$ is a section of $T^{0,1}\Omega$ with the property that $\operatorname{Re}\overline{Z}$ and $\operatorname{Im}\overline{Z}$ are tangent to M. Under this assumption it is immediately clear that the if $u \in C^1(\overline{\Omega})$ then $\overline{Z}u \restriction_M$ is determined by $u \restriction_M$. Thus we see that in order for $f = u \restriction_\Omega$ to be the boundary value of a holomorphic function it is necessary that

$$\overline{Z}f = 0 \text{ for all sections of } T^{0,1}\Omega \text{ which are tangent to } M. \tag{2.9.3}$$

These are called the tangential Cauchy–Riemann equations. If u is defined in a neighborhood of $\partial\Omega$ then an equivalent condition is given by

$$\overline{\partial}u \wedge \overline{\partial}\rho = 0 \text{ at } \partial\Omega \tag{2.9.3'}$$

here ρ is a defining function for Ω.

Exercise. Prove that (2.9.3) and (2.9.3') are equivalent

The next order of business is therefore to understand $T^{0,1}\Omega \cap TM \otimes \Omega$. This is a simple exercise in linear algebra. Since any invariant subspace of J is even dimensional it follows easily that JTM is not contained in TM and therefore

$$T\Omega \restriction_M = TM + JTM.$$

From this formula and the fact that $\dim(V + W) = \dim V + \dim W - \dim(V \cap W)$ it follows that

$$\dim TM \cap JTM = 2n - 2. \tag{2.9.4}$$

From (2.9.4) we immediately deduce that

$$\dim_{\mathbb{C}} TM \otimes \mathbb{C} \cap T^{0,1}\Omega \restriction_M = n - 1. \tag{2.9.5}$$

We denote the subspace

$$T^{0,1}M = TM \otimes \mathbb{C} \cap T^{0,1}\Omega \restriction_M .$$

Its conjugate is $T^{1,0}M$, clearly

$$T^{1,0}M \cap T^{0,1}M = \{0\}. \tag{2.9.6}$$

If X, Y are two sections of $T^{0,1}M$ then it is easy to see that they can be extended to a neighborhood of M in Ω as sections of $T^{0,1}\Omega$. From this observation and (2.9.1) it follows that

$$\text{If } X, Y \text{ are sections of } T^{0,1}M \text{ then so is } [X, Y]. \tag{2.9.7}$$

Thus we see that the complex structure on Ω induces a structure on a codimension 1, real submanifold. This structure is called a CR–structure. It is clear from the construction above, of an induced CR–structure, that a CR–structure can be defined intrinsically on an odd dimensional manifold. Abstractly we have

Definition 2.9.8**.** Let M be a $2n-1$–dimensional manifold and let $T^{0,1}M$ be a $n-1$–dimensional subbundle of $TM\otimes\mathbb{C}$ which satisfies the non–degeneracy condition (2.9.6) and the formal integrability condition (2.9.7) then we say that $T^{0,1}M$ defines a CR–structure on M.

Underlying the CR–structure is a real hyperplane field spanned by $T^{0,1}M+T^{1,0}M$. Locally a hyperplane field is defined as the kernel of a one form. Denote such a one form by θ. This one form allows us to study the degree to which the formal integrability condition is true integrability in the sense of Frobenius. Put differently, the condition (2.9.7) does not imply that $\ker\theta$ is tangent to a foliation of M. This would require knowing that $\theta[X,\overline{Y}]=0$ for X,Y sections of $T^{0,1}M$. The formula of Cartan states that

$$\theta[X,\overline{Y}] = X\theta(\overline{Y}) - \overline{Y}\theta(X) - d\theta(X,\overline{Y}). \tag{2.9.9}$$

Thus we see that true integrability properties of $\ker\theta$ are determined by $d\theta$.

For analytic purposes it is essential to know whether or not the manifold M contains any holomorphic submanifolds. If this were the case then we could find a vector field $\overline{Z}\in T^{0,1}M$ such that $[Z,\overline{Z}]=\alpha Z+\beta\overline{Z}$ and therefore

$$\theta([Z,\overline{Z}]) = d\theta(Z,\overline{Z}) = 0. \tag{2.9.10}$$

If on the other hand the hermitian pairing defined on $T^{0,1}M$ by

$$L(\overline{Z},\overline{Z}) = id\theta(Z,\overline{Z}). \tag{2.9.11}$$

is definite then M has no holomorphic submanifolds. The hermitian form defined in (2.9.11) is called the Levi form. It is defined up to a conformal factor. We can replace θ with a non–vanishing multiple $f\theta$; the Levi forms are related by:

$$d(f\theta)\restriction_{T^{1,0}M+T^{0,1}M} = fd\theta\restriction_{T^{1,0}M+T^{0,1}M}. \tag{2.9.12}$$

The complex structure defines an orientation on $\ker\theta$ thus if M is oriented then we can pick a definite sign for the conformal class of θ and therefore the signature of Levi form is well defined. We denote the signature by (m,n,p) if L is positive on a m–dimensional subspace, negative on an n–dimensional subspace and degenerate on a p–dimensional subspace. When we can choose a definite sign for the conformal class of θ we will say that M has a CR–orientation.

Definition 2.9.13**.** Suppose that M is an CR–oriented manifold such that the Levi form is positive definite then we say the structure is strictly pseudoconvex.

Now we return to the case of a domain $\Omega\subset\mathbb{C}^n$. Let ρ be a defining function for $\partial\Omega$, this implies that

$$\Omega = \{z;\rho(z)<0\}$$

with $d\rho(z)\neq 0$, $z\in\partial\Omega$. Since $d\rho=\partial\rho+\overline{\partial}\rho$ it follows that

$$-\partial\rho\restriction_{\partial\Omega} = \overline{\partial}\rho\restriction_{\partial\Omega}. \tag{2.9.14}$$

On the other hand a vector, $X\in T_z\Omega, z\in\partial\Omega$ is tangent to $\partial\Omega$ if and only if $d\rho(z)(X)=0$. Thus if $\overline{Z}\in T^{0,1}_z\Omega$, $z\in\partial\Omega$ then it is tangent to $\partial\Omega$ if anf only if $\overline{\partial}\rho(z)(\overline{Z})=0$. This proves the following geometric lemma.

Lemma 2.9.15**.** *Suppose that $\Omega\subset\mathbb{C}^n$ has a C^1–defining function ρ then*

$$T^{0,1}_z\partial\Omega = \ker\overline{\partial}\rho\restriction_{T^{0,1}_z\Omega}.$$

We let

$$\theta = -ij^*\overline{\partial}\rho, \tag{2.9.16}$$

here $j : \partial\Omega \hookrightarrow \Omega$ is the inclusion of the via a biholomorphic map. Evidently a CR–structure defined in this way has a CR–orientation: it is fixed by the hypothesis that a defining function is negative in Ω. Suppose that F is a biholomorphic map defined in a neighborhood, $U \subset \overline{\Omega}$ of $q \in \partial\Omega$. Suppose that it carries the interior of Ω locally onto the exterior. We suppose that ρ is defined in a neighborhood of $\partial\Omega$.

Since F is biholomorphic it follows that

$$F^*\partial\overline{\partial}\rho = \partial\overline{\partial}F^*\rho. \tag{2.9.17}$$

Since $-F^*\rho$ is also a defining function for $\partial\Omega$ near to q it follows from (2.9.17) that the signature of the Levi form satisfies

$$(m(F(q)), n(F(q)), p(F(q))) = (n(q), m(q), p(q)).$$

From this we deduce the proposition

Proposition 2.9.18. *If Ω is pseudoconvex and the Levi form has at least one positive eigenvalue at each point then there is no biholomorphic map carrying the $\partial\Omega$ to itself and mapping the interior of Ω onto the exterior.*

This already shows that there is a significant difference between the biholomorphic equivalence problem for one and several variables.

Now we return to the problem of holomorphic extension. As we shall see it is essentially a local problem in $\mathbb{C}^n$. Let ρ be a C^4 function defined in a neighborhood of $0 \in \mathbb{C}^n$ suppose that $\rho(0) = 0$ and that $d\rho(0) \neq 0$. Thus there is open set $U \subset \mathbb{C}^n$ such that $M = U \cap \{z;\, \rho(z) = 0\}$ is a smooth hypersurface. The next result shows the interaction between the problem of holomorphic extension and the signature of the Levi form.

Lewy Extension Theorem 2.9.19. *Let ρ be as above, assume that there exists a vector $Z \in T^{1,0}\mathbb{C}^n$ such that*

$$\partial\rho(0)(Z) = 0 \text{ and } \partial\overline{\partial}\rho(0)(Z, \overline{Z}) < 0.$$

Then there exists a neighborhood $\omega \subset U$ of 0 such that for every function $v \in C^4(\omega)$ which satisfies the tangential Cauchy Riemann equations,

$$\overline{\partial}v \wedge \overline{\partial}\rho \restriction_{\partial\Omega} = 0,$$

there is a function $V \in C^1(\omega)$ such that

$$v = V \text{ along } \rho = 0 \text{ and } \overline{\partial}V = 0 \text{ in } \omega_+, \tag{2.9.20}$$

where

$$\omega_+ = \{z \in \omega;\, \rho(z) \geq 0\}. \tag{2.9.21}$$

Proof. We will reduce this to the case considered in Theorem (2.4.8). First we change variables to obtain a somewhat simpler form for the defining function. The hypotheses imply that after a linear change of coordinates we can assume that

$$\begin{aligned} \rho(z, \bar{z}) = x_n + &\sum_{i,j=1}^{n} \partial_{z_i}\partial_{\bar{z}_j}\rho(0) z_i \bar{z}_j + \\ &\operatorname{Re}\sum_{i,j=1}^{n} \partial_{z_i}\partial_{z_j}\rho(0) z_i z_j + O(|z|^3). \end{aligned} \tag{2.9.22}$$

We make the holomorphic change of variables

$$z_j' = z_j, j = 1, \ldots, n-1,\; z_n' = z_n + i\sum_{i,j=1}^{n} \partial_{z_i}\partial_{z_j}\rho(0) z_i z_j.$$

In terms of the new coordinates the defining function assumes the simpler form:

$$\rho = \operatorname{Im} z_n' + \sum_{i,j=1}^{n} \partial_{z_i}\partial_{\bar{z}_j}\rho(0) z_i' \bar{z}_j' + O(|z|^3). \tag{2.9.23}$$

To simplify notation we drop the primes and use A_{ij} to denote the matrix of the hermitian form in (2.9.23).

The hypothesis implies that the form

$$\sum_{i,j=1}^{n-1} A_{ij} z_i \bar{z}_j$$

is not positive definite. By a linear change of coordinates we may achieve that $A_{11} < 0$. Therefore, since

$$\rho(z_1, 0, \dots, 0) = A_1 1 |z_1|^2 + O(|z_1|^3),$$

we can choose a $\delta > 0$ and then an $\epsilon > 0$ so that

$$\partial_{z_1}\partial_{\bar{z}_1}\rho(z) < 0 \text{ if } z \in \omega = \{z \in V; |z_1| < \delta, |z_2| + \cdots + |z_n| < \epsilon\}$$

and $\rho(z) < 0$ on the part of $\partial\omega$ where $|z_1| = \delta$. If we fix $z_2, \dots, z_n$ with $|z_2| + \cdots + |z_n| < \epsilon$ then the set of z_1 with $|z_1| \le \delta$ where $\rho(z) < 0$ is a connected set. This is so because ρ is negative where $|z_1| = \delta$ thus if there were two components then one would necessarily be compact and therefore ρ would have a local minimum in that component. This violates the hypothesis on $\partial_{z_1}\partial_{\bar{z}_1}\rho$ in ω. The point to this construction is that the boundaries of the disks contained in ω defined by $z_2, \dots, z_n$ constant do not intersect the hypersurface $\rho = 0$. This will allow us to use the Cauchy integral to solve the $\overline{\partial}$–equation.

Now we will construct a function V which agrees with v where $\rho = 0$ and is holomorphic in ω_+. First we construct a function V_0 in $C^2(\omega)$ such that

$$\overline{\partial} V_0 = O(\rho^2) \text{ along } \rho = 0. \tag{2.9.24}$$

By assumption

$$\overline{\partial} v = h_0 \overline{\partial}\rho + \rho h_1,$$

where $h_0 \in C^3(\omega)$ and $h_1 \in C^2(\omega; \Lambda^{0,1})$. Hence

$$\overline{\partial}(v - h_0\rho) = \rho(h_1 - \overline{\partial} h_0) = \rho h_2 \text{ where } h_2 \in C^2(\omega; \Lambda^{0,1}).$$

Since $\overline{\partial}(\rho h_2) = \overline{\partial}\rho \wedge h_2 + \rho\overline{\partial} h_2 = 0$ we have that $\overline{\partial}\rho \wedge h_2 = 0$ where $\rho = 0$. Thus we can write

$$h_2 = h_3 \overline{\partial}\rho + \rho h_4$$

where $h_3 \in C^2(\omega)$ and $h_4 \in C^1(\omega; \Lambda^{0,1})$. We set

$$V_0 = v - h_0\rho - \frac{1}{2} h_3 \rho^2,$$

to obtain that

$$\overline{\partial} V_0 = \rho^2 (h_4 - \frac{1}{2}\overline{\partial} h_3). \tag{2.9.25}$$

Note that at this point we could take $h_5 = (h_4 - \frac{1}{2}\overline{\partial} h_3)$ the condition $\overline{\partial}\rho^2(h_4 - \frac{1}{2}\overline{\partial} h_3) = 0$ implies that

$$h_5 = h_6 \overline{\partial}\rho + \rho h_7.$$

If we set $V_1 = V_0 - \frac{1}{3}\rho^3 h_6$ then $\overline{\partial} V_1 = O(\rho^3)$. Evidently if the data is sufficiently differentiable we can continue this indefinitely and obtain a function V' such that $\overline{\partial} V'$ vanishes to infinite order along $\rho = 0$.

To complete this argument V_0 suffices. Define a $(0,1)$–from in ω by

$$\alpha = \begin{cases} \overline{\partial} V_0 & \text{in } \omega_+ \\ 0 & \text{in } \omega \setminus \omega_+. \end{cases}$$

In virtue of (2.9.25) it is clear that $\alpha \in C^1(\omega; \Lambda^{0,1})$and $\overline{\partial}\alpha = 0$. By the construction of ω, $\alpha = 0$ in a neighborhood of $\omega \cap \{|z_1| = \delta\}$. If

$$\alpha = \sum_{j=1}^{n} \alpha_j d\bar{z}_j,$$

then we set

$$f(z_1, z') = \frac{1}{2\pi\imath} \int\limits_{|z_1|=\delta} \frac{\alpha_1(\tau, z')d\tau \wedge d\overline{\tau}}{(\tau - z_1).} \tag{2.9.26}$$

Clearly $f \in C^1(\omega)$. Since the integrand in (2.9.26) is compactly supported the integrability conditions and the complex version of Stokes' theorem imply that

$$\overline{\partial} f = \alpha \text{ in } \omega$$

and furthermore f vanishes in $\omega \setminus \omega_+$. This is because f is holomorphic in $\omega \setminus \omega_+$ and vanishes in some neighborhood of $\{|z_1| = \delta\} \cap \omega$. The argument given above showed that $\omega \setminus \omega_+$ is connected therefore f must vanish in the whole set. Since f is differentiable it follows that $f = 0$ along $\rho = 0$ consequently if we set

$$V = V_0 - f$$

then

$$\overline{\partial} V = 0 \text{ in } \omega_+ \text{ and } V \restriction_{\rho=0} = v \restriction_{\rho=0} .$$

This completes the proof of the theorem.

This theorem has several important corollaries

Corollary 2.9.27. *If ρ is plurisubharmonic along $\rho = 0$ and the Levi form has at least one positive eigenvalue then any function u defined on $\rho = 0$ which satisfies the tangential Cauchy Riemann equations has an extension to a neighborhood of $\rho < 0$ as a holomorphic function. Furthermore the extension is local.*

This shows quite clearly that the property of being a boundary value of a holomorphic function is very different in one and several variables.

As another corollary we see that if the Levi form of $\{\rho = 0\}$ has eigenvalues of both signs then the theorem implies that any function which satisfies the tangential Cauchy Riemann equations extends to be holomorphic in a full neighborhood of $\rho = 0$. This is in sharp contrast to the case of a pseudoconvex domain where there exist holomorphic functions which are smooth up to the boundary but do not extend.

Another case to consider is when the Levi form is identically zero. In this case the boundary is foliated by complex manifolds. For example the real hyperplane in $\mathbb{C}^n$ given by $y_n = 0$. The tangential CR–equations are the

$$\partial_{\bar{z}_j} u = 0, j = 1, \ldots, n-1.$$

Clearly any function of x_n satisfies these equations but in general has no extension as a holomorphic function.

We close this section with two more corollaries.

Corollary 2.9.28. *Suppose that $\Omega \subset \mathbb{C}^n$ is bounded, C^2 and the Levi form of $\partial\Omega$ has at least one positive eigenvalue at every point. Then any solution $f \in C^1(\Omega)$ of the tangential Cauchy Riemann equations has an extension to Ω as a holomorphic function.*

Proof. Let u be a function on $\partial\Omega$ satisfying the tangential CR–equations. The Lewy extension theorem implies that we can extend u to some neighborhood of $\partial\Omega$ as a holomorphic function. With out loss of generality we can assume that the neighborhood is connected. The Hartogs extension theorem then gives an extension to all of Ω.

Actually the convexity assumption is not necessary in this corollary. The technique for constructing a formal solution to $\overline{\partial} u = 0$ along $\rho = 0$ presented in the proof of (2.9.19) applies without change to the boundary of a C^2–domain.

Corollary 2.9.29. *Suppose that ρ satisfies the hypotheses of* (2.9.19) *and v satisfies the tangential Cauchy Riemann equations, then the extension of v as a holomorphic function to ω_+ is unique.*

Proof. The Lewy extension result provides a local procedure for extending v to ω_+. In the normal form given for ρ it is clear that $\partial_{z_1}\rho(z_1, z')$ does not vanish along the locus $\rho = 0$ away from $z_1 = 0$. Thus for an open set of z' the boundary of the set of z_1 such that $\rho(z_1, z') > 0$ is a single smooth circle. Thus we can apply the Cauchy integral formula to obtain a representation of the extension of v to ω_+ in terms of its values on $\rho = 0$. This formula is valid in open subset of ω_+ and thus establishes the uniqueness of the continuation.

This result has as a corollary the fact that if a holomorphic function vanishes on an open subset of a hypersurface with nondegenerate Levi form then it is, in fact, identically zero.

2.10 The Weierstraß preparation theorem

This is the last theorem in the local theory of holomorphic functions which we will consider. It provides a local description of the zero set of a holomorphic function. This result is essential in the local study of the intersections of analytic varieties and the study of the local ring structure of germs of holomorphic functions. It is the higher dimensional analogue of the one dimensional result to the effect that the behavior of a holomorphic function near z_0 is determined by the order of vanishing of $f(z) - f(z_0)$. That is there is a unique integer n and a holomorphic function $v(z)$ such that

$$f(z) = (z - z_0)^n v(z), \quad v(z_0) \neq 0.$$

The function, v is a 'unit' in the ring of germs of holomorphic functions at z_0, thus we have a unique factorization in this ring.

The simplest holomorphic functions are polynomials, next simpler might be functions of the form

$$P(w, z) = \sum_{i=0}^{k} a_i(z) w^i \tag{2.10.1}$$

where $a_i(z), i = 0, \ldots, k$ are holomorphic near to zero. If we suppose that $a_k(0) \neq 0$ but $a_i(0) = 0, i = 1, \ldots, k$ then the zero set of P has a simple local description as a branched cover of $\mathbb{C}^n$. More generally if $F(w, z) = q(w, z)P(w, z)$ with $q(0, 0) \neq 0$ then the zero set of F is also a branched cover. Moreover q is a unit and we could try proving a unique factorization theorem by studying polynomials with analytic coefficients. In fact every holomorphic function has a local representation of this sort.

Suppose that F is holomorphic in some neighborhood of $0 \in \mathbb{C}^{n+1}$ and not identically zero. From this we conclude that there is some direction $Z \in \mathbb{C}^{n+1}$ such that the function of one variable $f(w) = F(wZ)$ is not identically zero. Let k be the non–negative integer such that

$$f(w) = w^k g(w), \quad g(0) \neq 0. \tag{2.10.2}$$

We introduce coordinates (z, w) into $\mathbb{C}^{n+1}$ so that wZ corresponds to $z = 0$.

Weierstraß Preparation Theorem 2.10.3. *Suppose that $F(w, z)$ is holomorphic in a neighborhood of $(0, 0) \in \mathbb{C}^{n+1}$ and satisfies* (2.10.2) *then there exists a analytic polynomial*

$$P(w, z) = w^k + \sum_{i=1}^{k-1} a_i(z) w^i,$$

and a holomorphic function $q(w, z)$ defined in a neighborhood of $0 \in \mathbb{C}^{n+1}$ such that $q(0, 0) \neq 0$ and

$$F(w, z) = q(w, z)P(w, z). \tag{2.10.4}$$

This theorem is a special case of another result called the Weierstraß division theorem. This provides a generalization of the 'Euclidean' algorithm for dividing polynomials to the context of holomorphic function in several variables.

The Weierstraß Division Theorem 2.10.5. *Let $F(w,z)$ and k be as above and $G(w,z)$ an arbitrary holomorphic function defined in a neighborhood of $(0,0)$ then there exist unique holomorphic functions $q(w,z)$ and $r(w,z)$ such that*

$$G(w,z) = q(w,z)F(w,z) + r(w,z)$$

where

$$r(w,z) = \sum_{i=0}^{k-1} a_i(z)w^i.$$

(2.10.5)⟹(2.10.3). To deduce (2.10.3) from (2.10.5) we simply let $G(w,z) = w^k$, then

$$w^k = q(w,0)F(w,0) + r(w,0). \tag{2.10.6}$$

From the hypotheses on F and the form of r it is clear that $q(0,0) \neq 0$ thus

$$F(w,z) = \frac{1}{q(w,z)}(w^k - r(w,z)).$$

This proves (2.10.3).

Uniqueness in (2.10.5). This follows from a simple application of Rouchè's theorem in one complex variable. Suppose that

$$G(w,z) = q_1F + r_1 = q_2F + r_2, \tag{2.10.7}$$

then

$$r_1 - r_2 = F(q_2 - q_1). \tag{2.10.8}$$

Since $F(w,0) = w^k g(w)$ it follows that for z sufficiently near to 0 that $F(w,z)$ has at least k zeros in some small disk about 0. On the other hand $r_1 - r_2$ is a polynomial of degree at most $k-1$ in w. Therefore (2.10.8) implies that $r_1 - r_2$ is identically zero.

The construction of q, r uses a slightly indirect argument. We define

$$P_k(w,\lambda) = w^k + \sum_{i=1}^{k-1} \lambda_i w^i.$$

This is a holomorphic function in $\mathbb{C} \times \mathbb{C}^k$. We have a second division theorem from which we can deduce the first:

Polynomial Division Theorem 2.10.9. *Let $G(w,z)$ be a holomorphic function defined in a neighborhood of 0 then there exist holomorphic functions $q(w,z,\lambda)$ and $r(w,z,\lambda)$ in a neighborhood of 0 in $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^k$ such that*

$$G(w,z) = q(w,z,\lambda)P_k(w,\lambda) + r(w,z,\lambda),$$

where

$$r(w,z,\lambda) = \sum_{i=0}^{k-1} r^i(z,\lambda)w^i.$$

(2.10.9)⟹(2.10.5). We apply the polynomial division theorem to F, G to obtain

$$F = q_F P_k + r_F \quad G = q_G P_k + r_G. \tag{2.10.10}$$

We can deduce a few facts about q_F and r_F:

$$q_F(0) \neq 0 \text{ and } r_F^i(0,0) = 0, i = 0\dots, k-1. \tag{2.10.11}$$

To prove these statements we let $z = 0, \lambda = 0$ in (2.10.10),

$$w^k g(w) = F(w,0) = q_F(w,0,0)w^k + \sum_{i=0}^{k-1} r_F^i(0,0)w^i. \tag{2.10.12}$$

The statements follows by comparing coefficients of powers of w in (2.10.12). Now we set $f_i(\lambda) = r_F(0,\lambda)$, we claim that

$$\det \partial_{\lambda_j} f_i(0) \neq 0.$$

To prove this we set $z = 0$ in (2.10.10) and differentiate with respect to λ_j, evaluating at $\lambda = 0$ we obtain

$$0 = \partial_{\lambda_j} q_F(w,0,0)w^k + w^j + \sum_{i=0}^{k-1} \partial_{\lambda_j} f_i(0)w^i.$$

By comparing coefficients of w^i we deduce that

$$\partial_{\lambda_j} f_i(0) = 0, \text{ if } i > j \text{ and } \partial_{\lambda_i} f_i(0) = -q_F(0,0). \tag{2.10.13}$$

From (2.10.13) it follows that

$$\det \partial_{\lambda_j} f_i(0) = (-q_F(0,0))^k. \tag{2.10.14}$$

In light of (2.10.11) and (2.10.14) we can apply the holomorphic implicit function theorem to find holomorphic functions $\theta_i(z), i = 0, \ldots, k-1$, defined in some neighborhood of zero, such that

$$\theta_i(0) = 0, \quad r_F^i(z, \theta(z)) = 0, i = 0, \ldots, k-1. \tag{2.10.15}$$

Substituting from (2.10.15) into (2.10.8) we derive that

$$F(w,z) = q_F(w,z,\theta(z))P_k(w,\theta(z)). \tag{2.10.16}$$

Since $\theta(0) = 0$ and $q_F(0,0,0) \neq 0$ we can write

$$G(w,z) = q(w,z)F(w,z) + r(w,z)$$

where

$$q(w,z) = \frac{q_G(w,z,\theta(z))}{q_F(w,z,\theta(Z))}, \quad r(w,z) = \sum_{i=0}^{k-1} r_G^i(z,\theta(z))w^i. \tag{2.10.17}$$

This completes the deduction of (2.10.5) from (2.10.8).

All that remains is to prove (2.10.8).

Proof of (2.10.8). This follows from the Cauchy integral formula. We can write

$$G(w,z) = \frac{1}{2\pi\imath} \int_\gamma \frac{G(\eta,z)d\eta}{(\eta - w)} \tag{2.10.18}$$

where γ is some contour enclosing w. By comparing coefficients it follows that

$$P_k(w,\lambda) - P_k(\eta,\lambda) = (\eta - w)\sum_{i=0}^{k-1} s_i(\eta,\lambda)w^i$$

where $s_i(\eta, \lambda)$ are analytic functions. Dividing we obtain

$$\frac{P_k(w,\lambda)}{\eta - w} - \frac{P_k(\eta,\lambda)}{\eta - w} = \sum_{i=0}^{k-1} s_i(\eta,\lambda) w^i. \tag{2.10.19}$$

As $P_k(\eta, 0) = \eta^k$, we can find a contour, γ along which $P_k(\eta, \lambda)$ does not vanish for sufficiently small λ. Integrating along this contour we obtain

$$G(w,z) = \frac{1}{2\pi\imath} \int_\gamma \frac{G(\eta,z)d\eta}{(\eta - w)} \frac{P_k(\eta,\lambda)}{P_k(\eta,\lambda)}. \tag{2.10.20}$$

Using (2.10.19) in (2.10.20) we get

$$\begin{aligned} G(w,z) =& \frac{1}{2\pi\imath} \int_\gamma \frac{G(\eta,z)d\eta}{(\eta - w)} \frac{P_k(w,\lambda)}{P_k(\eta,\lambda)} - \\ & \sum_{i=0}^{k-1} \left(\frac{1}{2\pi\imath} \int_\gamma G(\eta,z) s_i(\eta,\lambda) d\eta \right) w^i. \end{aligned} \tag{2.10.21}$$

The theorem follows by setting

$$\begin{aligned} q(w,z,\lambda) =& \frac{1}{2\pi\imath} \int_\gamma \frac{G(\eta,z)d\eta}{(\eta - w)} \frac{1}{P_k(\eta,\lambda)} \quad \text{and} \\ r(w,z,\lambda) =& - \sum_{i=0}^{k-1} \left(\frac{1}{2\pi\imath} \int_\gamma G(\eta,z) s_i(\eta,\lambda) d\eta \right) w^i. \end{aligned}$$

Exercise 2.10.22**.**

(1) Show that if F and G are polynomials as in (2.10.1) then the function q in (2.10.5) is as well.
(2) Show that if F is a polynomial as in (2.10.1) and has a factorization of the form $F = g_1 g_2$ where g_1, g_2 are holomorphic, then they can be taken to be polynomials.
(3) Using the previous part and the theorem from commutative algebra that if a ring R is a unique factorization domain, then so is $R[z]$, show that the holomorphic functions defined in a neighborhood of $0 \in \mathbb{C}^{n+1}$ is a unique factorization domain. Note that the units are functions which do not vanish at 0.

References

The bulk of the material in §2.1–§2.9 comes form *An Introduction to Complex Analysis in Several Variables* by Lars Hörmander. Additional material was taken from *Introduction to Holomorphic Functions of Several Variables, I,II* by Robert C. Gunning. Section 2.10 follows *Stable Mapping and Their Singularities* by M. Golubitsky and V. Guillemin.

Version: 1.2; Revised:3-21-91; Run: February 5, 1998