NOTES ON ELLIPTIC BOUNDARY VALUE PROBLEMS FOR THE LAPLACE OPERATOR

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In these notes we present the pseudodifferential approach to elliptic boundary value problems for the Laplace operator acting on functions on a smoothly bounded compact domain in a compact manifold. This is an elaboration of the classical method of multiple layer potentials. After a short discussion of this method we consider the theory of homogeneous distributions on $\mathbb{R}^n$. This is useful in our subsequent discussion of boundary value problems and provides an interesting concrete complement to the rather abstract general theory developed earlier in the course. We then turn to boundary value problems. We analyze the smoothness and boundary regularity of multiple layer potentials for the Laplace equation. This allows the reduction of a boundary value problem to the solvability of a system of pseudodifferential equations on the boundary itself. After considering several different boundary value problems for smooth data we establish the Sobolev regularity properties of the single and double layer potentials. The estimates allow us to extend the existence results to data with finite differentiability and also establish the standard "elliptic estimates" for the solutions of elliptic boundary value problems for the Laplacian. This treatment is culled from material in L. Hörmander, *The analysis of Linear Partial Differential Operators, III*, M. Taylor, *Partial Differential Equations, II* and *Introduction to the theory of Linear Partial Differential Equations* by J. Chazarain and A. Piriou. I would finally like to thank Dara Cosgrove for the remarkable typing job.

0. Preface

A fundamental class of problems in partial differential equations is elliptic boundary value problems. The classical problems of this type are the Dirichlet and Neumann problems for the Laplace equation on a smoothly bounded domain, $\Omega \subseteq \mathbb{R}^n$:

Dirichlet Problem

Find a function $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that (D) $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u\,|_{\,b\Omega} = f \end{cases}$

Neumann Problem

Find a function $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that (N) $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = g \end{cases}$

Here $\frac{\partial u}{\partial \nu}$ is the outward normal derivative along $b\Omega$. We would like conditions under which (D) and (N) are solvable and relate the regularity of the solution u to the data f or g respectively. A starting point for the analysis of this problem is Green's formula: We let $G(x, y)$ denote the fundamental solutions of the Laplace equation in $\mathbb{R}^n$,

$$G(x, y) = \begin{cases} c_2 \log |x - y| & n = 2 \\ c_n |x - y|^{2-n} & n > 2 \end{cases}$$

A simple integration by parts shows that

$$\int_\Omega G(x,y)\Delta u\, dy = \begin{cases} \int_{b\Omega} (G(x,y)\frac{\partial u}{\partial \nu}(y) - \frac{\partial G}{\partial \nu_y}(x,y)u(y))\, d\sigma(y) \ + u(x) & x \in \Omega \\ \int_{b\Omega} (G(x,y)\frac{\partial u}{\partial \nu}(y) - \frac{\partial G}{\partial \nu_y}(x,y)u(y))\, d\sigma(y) & x \in \overline{\Omega}^c \end{cases} \tag{0.1}$$

From this formula several things are apparent

1) We can always reduce the inhomogeneous problem, $\Delta u = f$ to a homogeneous problem by letting $u = v + w$ where

$$\Delta v = 0$$
$$w = \int_\Omega G(x,y) f(y) dy$$

2) Suppose that we can show that for $f, g \in C^\infty(b\Omega)$ the functions which are defind, a priori in $b\Omega^c$ by

$$\mathcal{S}f(x) = \int_{b\Omega} G(x,y) f(y) d\sigma(y)$$
$$\mathcal{D}g(x) = \int_{b\Omega} \frac{\partial G}{\partial \nu_y}(x,y) g(y) d\sigma(y)$$

have extensions as elements of $C^\infty(\overline{\Omega})$ and $C^\infty(\Omega^C)$, respectively. Then $\mathcal{S}f\,|_{\,b\Omega^\pm}$, $\mathcal{D}g\,|_{\,b\Omega^\pm}$ are given by linear operators, $S^\pm f$, $D^\pm g$. Let u_0 and u_1 denote $u\,|_{\,b\Omega}$ and $\frac{\partial u}{\partial \nu}\,|_{\,b\Omega}$. If u is a harmonic function in Ω then Green's formula implies that

$$u_0 = -S^+ u_1 + D^+ u_0$$

So

$$S^+ u_1 = (D^+ - I) u_0 \tag{0.2}$$

If we could show that S^+ were an invertible operator then (0.2) would imply that

$$u_1 = (S^+)^{-1} (D^+ - I) u_0 \tag{0.3}$$

Using (0.3) we can solve the Dirichlet problem.

We let $g = (S^+)^{-1}(D^+ - I) f,$ and
$u = \mathcal{D}f - \mathcal{S}g$

So That $u_0 = D^+ f - S^+ (S^+)^{-1} (D^+ - I) f$
$= f$

as desired.

If $\mathcal{S}f$ and $\mathcal{D}g$ have smooth extensions it would also follow that

$$\partial_\nu^\pm \mathcal{S}f = S_1^\pm f$$

$$\partial_\nu^\pm \mathcal{D}g = D_1^\pm g$$

so that

$$u_1 = D_1^+ u_0 - S_1^+ u_1$$

so if D_1^+ is invertible then

$$u_0 = (D_1^+)^{-1} (I + S_1^+) u_1$$

and we could therefore solve the Neumann problem.

In the next few sections we will show that $\mathcal{S}f$ and $\mathcal{D}f$ have the smooth extension property posited above. This is established by using the Fourier representation of the Green's function and techniques from the theory of pseudodifferential operators. In fact we show that, for each $k \in \mathbb{N}_0$ there are ψDOs $D_k^{\pm}$, $S_k^{\pm} \in \Psi^*(b\Omega)$ such that

$$(\partial_\nu^{\pm})^k \mathcal{S}f = S_k^{\pm} f$$
$$(\partial_\nu^{\pm})^k \mathcal{D}f = D_k^{\pm} f\ .$$

With this information we can give sufficient conditions for a more general boundary value problem:

$$\begin{cases} \Delta u = 0 \\ b_0 u_0 + b_1 u_1 = g \quad b_i \in \Psi^*(b\Omega) \end{cases} \tag{0.4}$$

to be Fredholm problems. This will be the case if the pseudodifferential system

$$\begin{pmatrix} (D^+ - I)u_0 - S^+ u_1 \\ b_0 u_0 + b_1 u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

is elliptic. In this case (0.4) is solvable for g satisfying finitely many linear conditions. We also establish that the solution u depends on the boundary data in a specific way: for example in the Dirichlet problem

$$\|U\|_{H^s(\Omega)} \leq C_s \|f\|_{H^{s-\frac{1}{2}}(b\Omega)} \qquad s > \tfrac{1}{2}$$

and for the Neumann problem

$$\|U\|_{H^s(\Omega)} \leq C_s \|g\|_{H^{s-\frac{3}{2}}(b\Omega)} \qquad s > \tfrac{1}{2}\ .$$

In fact as it requires no additional effort we establish these results for the Δ_g defined by a metric g on a smooth, compact manifold with boundary.

1. Introduction

Let M be a smooth compact manifold with nonempty boundary. Choose a smooth Riemannian metric g on M. This defines a second order elliptic operator, Δ_g acting on $C^\infty(M)$. If the metric is given in local coordinates, $(x_1, \ldots, x_n)$ by

$$ds^2 = \sum g_{ij} dx_i dx_j$$

then

$$\Delta_g f(x) = \frac{1}{\sqrt{g}} \sum \partial x_i g^{ij} \sqrt{g} \partial x_j f(x)$$

where g^{ij} is the matrix inverse to g_{ij} and $g = \det g_{ij}$. The symbol of this operator is easily seen to be

$$\sigma_2(\Delta_g)(x, \xi) = |\xi|_g^2$$

and therefore the operator is elliptic. In this chapter we consider boundary value problems of the form:

$$\begin{cases} \Delta_g u = 0 \\ b_0 u\,|_{bM} + b_1 \frac{\partial u}{\partial \nu}\,|_{bM} = g\ . \end{cases} \tag{1.1}$$

Here $b_i \in \Psi^*(bM)$ are pseudodifferential operators and $\frac{\partial u}{\partial \nu}$ is the outer normal derivative of u along bM. Without loss of generality we can assume that M is a domain in a compact manifold, $\widehat{M}$, and that the metric g has a smooth extension to a metric on $\widehat{g}$.

In a previous section we analyzed $\Delta_{\widehat{g}}$ on $\widehat{M}$ and showed that there is an operator $G_0 \in \Psi^{-2}(\widehat{M})$ such that

$$\Delta_{\widehat{g}} G_0 = G_0 \Delta_{\widehat{g}} = I - \pi_0$$

where $\pi_0(f) = \int_{\widehat{M}} f dV_{\widehat{g}}$.

Choose a point $p \in \widehat{M} \backslash M$ and define: the function G on $\overline{M} \times \overline{M}$ by

$$G(x,y) = G_0(x,y) - [G_0(p,y) + G_0(x,p)].$$

A simple calculation shows that

$$\Delta_g^y G_x = \Delta_g^y G(x,y) = \delta_x - \frac{1}{V} - [\delta_p - \frac{1}{V}] = \delta_x - \delta_p$$

where $V = \mathrm{Vol}(\widehat{M})$. Thus if we select a function $u \in C_c^\infty(M)$ then

$$\langle G_x, \Delta_g u \rangle = u(x)\,.$$

For each $x \in M$, $G_x(y)$ is an integrable function in $C^\infty(\overline{M} \backslash \{x\})$. Choosing $\psi \in C_c^\infty(U_x)$ such that $\psi(x) = 1$ and U_x is a neighborhood of x with $U_x \subset\subset M$ we see that

$$\begin{aligned}\langle G_x, D_g u \rangle &= \langle G_x, \Delta_g \phi u \rangle + \langle G_x, \Delta_g (1-\phi) u \rangle \\ &= u(x) + \int_M G_x(y) \Delta_g (1-\phi) u \, dy\,.\end{aligned}$$

We can integrate by parts in the second term to obtain

$$= u(x) + \int_{bM} (G_x(y) \tfrac{\partial u(y)}{\partial \nu} - \partial_{\nu_y} G_x(y) u(y))\, d\sigma(y)\,.$$

If $\Delta_g f = 0$ then we obtain Green's formula:

$$u(x) = \int_{bM} (\partial_{\nu_y} G_x(y) u(y) - G_x(y) \tfrac{\partial u}{\partial \nu})\, d\sigma(y)\,. \tag{1.2}$$

For (x,y) in a neighborhood of $\overline{M} \times \overline{M}$, $G(x,y)$ is the Schwarz kernel of a pseudodifferential operator of order -2 which differs from $G_0(x,y)$ by a smooth kernel, thus $\sigma(G) = \sigma(\overline{G}_0)$.

We use formula (1.2) to reduce the solution of the boundary value problem, (1.1) to the solution of a pseudodifferential equation on bM. At the same time we obtain conditions on b_0, b_1 that imply that the solution to (1.1) satisfies standard "elliptic estimates": If $s \in \mathbb{R}$ greater than $\frac{1}{2}$ then

$$\|u\|_{H^s(M)} \leq c(\|u\,|_{bM}\|_{H^{s-\frac{1}{2}}(bM)} + \|\tfrac{\partial u}{\partial \nu}\|_{H^{s-\frac{3}{2}}(bM)})\,.$$

For $f \in C^\infty(bM)$ we define two operators:

$$\mathcal{S}f(x) = \int_{bM} G(x,y) f(y) d\sigma(y) \qquad x \in M$$

this is called a single layer potential;

$$\mathcal{D}f(x) = \int_{bM} \partial_{\nu_y} G(x,y) f(y) d\sigma(y)$$

this is called a double layer potential.

A harmonic function satisfies

$$u(x) = \mathcal{D}u_0(x) - \mathcal{S}u_1(x)$$

where $u_0 = u\,|_{bM}$ and $u_1 = \frac{\partial u}{\partial \nu}\,|_{bM}$. Suppose we show that

$$\begin{aligned}Su(x) &= \lim_{x \to bM} \mathcal{S}u(x) \\ Du(x) &= \lim_{x \to bM} \mathcal{D}u(x)\end{aligned}$$

exist. Then for a harmonic function

$$u_0 = Du_0 - Su_1\,.$$

If we could show, for example, that

$$(I-D)u_0 = -Su_1$$

can be solved, either for u_0 or u_1 then we could substitute into the boundary condition to obtain:

$$(b_1 - b_0(I-D)^{-1}S)u_1 = g\ . \tag{1.3}$$

Thus a reasonable condition to impose on $b_1,\ b_0$ is that $b_1 - b_0(I-D)^{-1}S$ is an elliptic operator. Then a solution to (1.3) exists for g satisfying finitely many linear conditions.

We will in fact show that for $f \in C^\infty(bM)$ $\mathcal{S}f$, $\mathcal{D}f$ have extensions as elements of $C^\infty(\overline{M})$ and show that

$$\begin{aligned} \partial_\nu^k \mathcal{S}f\,|_{b\Omega} &= A^k f, && A^k \in \Psi^{k-1}(bM) \\ \partial_\nu^k \mathcal{D}f\,|_{b\Omega} &= B^k f, && B^k \in \Psi^{k}(bM)\ . \end{aligned}$$

Computing the principal symbols of these operators will allow us to analyze (1.1). Finally we prove estimates

$$\begin{aligned} \|\mathcal{S}f\|_{H^s(M)} &\le C_s \|f\|_{H^{s-\frac{3}{2}}(bM)} \\ \|\mathcal{D}f\|_{H^s(M)} &\le \widetilde{C}_s \|f\|_{H^{s-\frac{1}{2}}(bM)} \end{aligned} \qquad s \in \mathbb{R}. \tag{1.4}$$

In addition to the general calculus of pseudodifferential operators on manifolds which we have already developed, we require some specialized tools to carry through this analysis: the theory of homogeneous distributors. In the next section we present such a theory. After that we analyze the single and double layer potentials and apply this analysis to study (1.1). In the final section we prove the estimates given in (1.4).

2. Homogeneous Distributions

Suppose that some $m \in \mathbb{R}$, $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfies:

$$\phi(\lambda x) = \lambda^{-m}\phi(x) \quad \lambda \in (0,\infty)\ ,$$

we say that ϕ is homogeneous of degree m. Such a function certainly defines an element of $(C_c^\infty(\mathbb{R}^n \backslash \{0\}))'$ by

$$\ell_\phi(\psi) = \langle \phi, \psi\rangle = \int_{\mathbb{R}^n \backslash \{U\}} \phi\psi dx.$$

If we define $\psi_\lambda(x) = \lambda^n \psi(\lambda x)$ then a simple change of variables shows that

$$\langle \phi, \psi_\lambda\rangle = \lambda^m \langle \phi, \psi\rangle\ . \tag{2.1}$$

This is the weak formula of homogeneity. If $\ell \in C^{-\infty}(\mathbb{R}^n \setminus \{0\})$ satisfies (2.1) then we say that ϕ is homogeneous of degree m. In this section we consider the problem of extending ℓ_ϕ as a distribution in $\mathcal{S}'(\mathbb{R}^n)$ and the extent to which the extended distribution can be made homogeneous.

We begin with the one dimensional case. For $s \in \mathbb{C}$ we define

$$x_+^s = \begin{cases} e^{s\log x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0\ . \end{cases} \tag{2.2}$$

Here $\log z$ is defined so that $\log x \in \mathbb{R}$ if $x > 0$. If $\Re(s) > -1$ and $\psi \in \mathcal{S}(\mathbb{R})$ then

$$I_+^s(\psi) = \int_{-\infty}^{\infty} x_+^s \psi(x)dx = \int_0^\infty x_+^s \psi(x)dx$$

converges absolutely and evidently defines a homogeneous distribution of degree s. In fact $I_+^s(\psi)$ is an analytic function of s in $\Re(s) > -1$. If $\Re(s) > 0$ we can integrate by parts k–times to obtain

$$I_+^s(\psi) = \int_0^\infty \frac{x_+^{s+k}}{(s+1)\cdots(s+k)}\psi^{[k]}(x)dx\ . \tag{2.3}$$

From (2.3) it follows that $I_+^s(\psi)$ has a meromorphic extension to the entire complex plane with simple poles at the negative integers, $-\mathbb{N}$.

$$\lim_{s\to -k}(s+k)I_+^{s+k}(\psi) = (-1)^{k-1}\int_0^\infty \frac{\psi^{[k]}(x)dx}{(k-1)!} = \frac{(-1)^k\psi^{[k-1]}(0)}{(k-1)!}$$

For $s \notin -\mathbb{N}$ it is easy to verify that

$$\psi \mapsto I_+^s(\psi)$$

defines a distribution which is homogeneous of degree s. Note that for $\Re(s) > 0$

$$\begin{aligned}(\tfrac{d}{dx}I_+^s)(\psi) &= I_+^s(-\psi')\\ &= -\int_0^\infty s_+^s\psi'(x)dx\\ &= s\int_0^\infty x_+^{s-1}\psi(x)dx\\ &= sI_+^{s-1}(\psi)\ .\end{aligned}$$

Thus

$$(\tfrac{d}{dx}I_+^s)(\psi) = sI_+^{s-1}(\psi) \qquad \text{for } \Re(s) > 0\ , \tag{2.4}$$

as both sides of (2.4) are meromorphic the equation extends to $s \notin -\mathbb{N}_0$. Note also that for $\Re(s) > -1$:

$$\begin{aligned}(xI_+^s)(\psi) &= I_+^s(x\psi)\\ &= \int_0^\infty x_+^s x\psi dx\\ &= \int_0^\infty x_+^{s+1}\psi dx\\ &= I_+^{s+1}(\psi)\ .\end{aligned}$$

Therefore as above

$$xI_+^s = I_+^{s+1} \qquad s \notin -\mathbb{N}\ .$$

Finally observe that for $\psi \in C_c^\infty(\mathbb{R}\backslash\{0\})$

$$I_+^s(\psi) = \int_0^\infty x^s\psi(x)dx\ , \qquad s \notin -\mathbb{N}$$

so I_+^s is an extension of x^s as a homogeneous distribution in $\mathcal{S}'(\mathbb{R})$. We are left with the cases: $s \in -\mathbb{N}$. We define an extension by subtracting the pole:

$$\begin{aligned}I_+^{-k}(\psi) &\overset{\text{def}}{=} \lim_{s\to -k}\left(I_+^s(\psi) - \frac{(-1)^k}{s+k}\frac{\psi^{(k-1)}(0)}{(k-1)!}\right)\\ &= \lim_{s\to -k}\int_0^\infty \left(\frac{x_+^{s+k}\psi^{[k]}(x)}{(s+1)\cdots(s+k)} + \frac{(-1)^k\psi^{[k]}(x)}{(s+k)(k-1)!}\right)dx\\ &= \frac{(-1)^{k-1}}{(k-1)!}\left[\int_0^\infty (\log x\psi^{[k]}(x))dx + \left(\sum_{j=1}^{k-1}\frac{1}{j}\right)\psi^{(k-1)}(0)\right]\ , \qquad (2.5)\end{aligned}$$

the sum is absent if $k = 1$.

Let us investigate the homogeneity of I_+^{-k}. A calculation using the previous formula shows that:

$$I_+^{-k}(\psi_\lambda) = \lambda^k \left[I_+^{-k}(\psi) - \frac{(-1)^{k-1}}{(k-1)!} \log \lambda \psi^{[k-1]}(0) \right] .$$

In other words, I_+^{-k} is not homogeneous of degree $(-k)$ as

$$I_+^{-k}(\psi_\lambda) - \lambda^k I_+^{-k}(\psi) = -\lambda^k \log \lambda \frac{(-1)^{k-1}}{(k-1)!} \delta_0^{(k-1)}(\psi) . \tag{2.6}$$

In fact there is <u>no</u> homogeneous extension of x_+^{-k}, any other extension, ℓ would differ from I_+^{-k} by a distribution supported at zero, hence

$$\ell = I_+^{-k} + \sum_j c_j \delta_0^{(j)} .$$

But $\delta^{(j)}$ is homogeneous of degree $-(i+j)$ and therefore no such sum can ever remove the log term in (2.6). Observe that

$$(x\psi)^{(j)} = x\psi^{(j)} = x\psi^{[j]} + j\psi^{[j-1]}$$

Using this relation in the definition of I_+^{-k} we easily show that

$$xI^{-k} = I_+^{1-k}$$

and by induction

$$x^j I_+^{-k} = I_+^{j-k} . \tag{2.7}$$

Finally we obtain that

$$\begin{aligned} \tfrac{d}{dx} I_+^{-k} &= -I_+^{-k}(\psi') \\ &= -kI_+^{-1-k}(\psi) - \frac{(-1)^{k-1}\psi^{(k)}(0)}{k} . \end{aligned} \tag{2.8}$$

The failure of I_+^{-k} to be homogeneous is reflected in the failure of the Euler equation

$$x\partial_x I_+^{-k} \neq -kI_+^{-k}.$$

We define another homogeneous distribution on $\mathbb{R}$ by setting

$$x_-^s = \begin{cases} 0 & x \geq 0 \\ |x|^s & x < 0 \end{cases} .$$

As $\langle x_-^s, \psi \rangle = \langle x_+^s, \check{\psi} \rangle$ where $\check{\psi}(x) = \psi(-x)$ the extension of this family follows in a straight forward way from our analysis of x_+^s. Every homogeneous function of degree s on $\mathbb{R}\backslash\{0\}$ is of the form

$$\phi = a_- x_-^s + a_+ x_+^s, a_\pm \in \mathbb{C} .$$

So we see that if $s \notin -\mathbb{N}$ then ϕ has a unique extension as a homogeneous distribution in $\mathcal{S}'(\mathbb{R})$. If $s = -k$ then ϕ may or may not have a homogeneous extension:

$$\langle \psi, \psi_\lambda \rangle - \lambda^{-k} \langle \phi, \psi \rangle = \frac{(-1)^{k-1} - \lambda^k \log \lambda \psi^{[k-1]}(0)}{(k-1)!} (a_+ + (-1)^{k-1} a_-) .$$

So ϕ has an extension as a homogeneous distribution iff $a_+ = (-1)^{k-1} a_- = 0$. For example the functions x^{-k} have extensions as homogeneous distributions. We denote these by $\underline{x}^{-k}$. If

$k = 1$ then

$$\begin{aligned}
\underline{x}^{-1}(\psi) &= x_+^{-1}(\psi) - x_-^{-1}(\psi) \\
&= -\int_0^\infty \log x\psi'(x)dx - \int_{-\infty}^0 \log(-x)\psi'(x)dx \\
&= -\lim_{\epsilon\downarrow 0}\left[\int_\epsilon^\infty \log x\psi'(x)dx + \int_{-\infty}^{-\epsilon} \log(-x)\psi'(x)dx\right] \\
&= -\lim_{\epsilon\downarrow 0}\left[\log x\psi(x)\mid_\epsilon^\infty - \int_\epsilon^\infty \frac{\psi(x)}{x}dx + \log(-x)\psi(x)\mid_{-\infty}^{-\epsilon} - \int_{-\infty}^{-\epsilon}\frac{\psi(x)}{x}dx\right] \\
&= P.V.\int_{-\infty}^\infty \frac{\psi(x)}{x}dx + \lim_{\epsilon\downarrow 0}(\psi(\epsilon)-\psi(-\epsilon))\log\epsilon \\
&= P.V.\int_{-\infty}^\infty \frac{\psi(x)}{x}dx
\end{aligned}$$

So

$$\underline{x}^{-1}(\psi) = P.V.\int_{-\infty}^\infty \frac{\psi(x)}{x}dx = -\int_{-\infty}^\infty \log|x|\psi'(x)dx\ ,$$

which implies that

$$\partial_x \log|x| = \underline{x}^{-1}\ .$$

Another interesting and important family of homogeneous distributions arise from a slightly different regularization of x^s:

$$\langle (x\pm i0)^s, \psi\rangle = \lim_{\epsilon\downarrow 0}\int (x\pm i\epsilon)^s\psi(x)dx\ .$$

It turns out that the functions $\Psi_s^\pm = \langle (x\pm i\epsilon)^s, \psi\rangle$ are entire functions of s if $\epsilon > 0$ with uniform limits as $\epsilon\downarrow 0$. $\Psi_x^\pm$ is clearly uniformly analytic in $Re\ s > -1$. We use Taylor's formula to obtain:

$$\Psi_x^\pm(\epsilon) = \int_{-\infty}^{-1} + \int_1^\infty (x\pm i\epsilon)^s\psi(x)dx + \int_{-1}^1 (x\pm i\epsilon)^s\sum_{j=0}^k \frac{\psi^{(j)}(0)x^j}{j!} + \int_{-1}^1 (x\pm i\epsilon)^s r_k(x)dx$$

where $r_k(x) = \psi(x) - \sum_{j=0}^k \frac{\psi(0)x^j}{j!}$. Clearly $|r_k(x)| = o(|x|^k)$ and therefore all terms but the third integral are analytic in $Re\ s > -k$. Let $C^\pm = \{|z| = 1;\ \pm Imz > 0\}$. By Cauchy's theorem:

$$\int_{-1}^1 (x\pm i\epsilon)^s x^j dx = \int_{C^\pm} (z\pm i\epsilon)^s z^j dz$$

These are uniformly entire functions of s as $\epsilon\downarrow 0$. This shows that $(x\pm i0)^s$ is well defined for all s. For $Re\ s > -1$ $\quad\langle (x\pm i0)^s, \psi_\lambda\rangle = \lambda^{-s}\langle (x\pm i0)^s, \psi\rangle$. As both sides are entire, this identity persists for all s and therefore $(x\pm i0)^s$ is a homogeneous distribution of degree s. Using Cauchy's theorem we see that

$$\begin{aligned}
\langle [(x+i0)^{-k} - (x-i0)^{-k}], \psi\rangle &= \int_{C^+-C^-} (\sum_{j=0}^{k+1}\frac{\psi^{(j)}(0)}{j!}z^j)z^{-k}dz \\
&= (2\pi i)\frac{\psi^{(k-1)}(0)}{(k-1)!}
\end{aligned}$$

Before proceeding to the n-dimensional case, we compute the Fourier transforms of these distributions. First we make a general observation about the Fourier transform of a homogeneous distribution:

Suppose $\phi \in \mathcal{S}'(\mathbb{R}^n)$ satisfies:

$$\langle \phi, \psi_\lambda \rangle = \lambda^{-s} \langle \phi, \psi \rangle$$

then

$$\langle \widehat{\phi}, \psi_\lambda \rangle = \langle \phi, \widehat{\psi}_\lambda \rangle.$$

A simple calculation shows that $\widehat{\psi}_\lambda(\xi) = \widehat{\psi}(\xi/\lambda) = \lambda^n \widehat{\psi}_\lambda(\xi)$. Thus

$$\langle \widehat{\phi}, \psi_\lambda \rangle = \lambda^n \langle \phi, \widehat{\psi}_{1/\lambda} \rangle = \lambda^{n+s} \langle \phi, \widehat{\psi} \rangle$$

Hence the Fourier transform is homogeneous of order $-(n+s)$.

We now compute the Fourier transforms of x_+^s, $s \notin -\mathbb{N}$. For $Re\ s > -1$ we see that

$$\begin{aligned} x_+^s(\xi) &= \lim_{\epsilon \downarrow 0} \int_0^\infty e^{-ix\cdot\xi} e^{-\epsilon x} x^s dx \\ &= \lim_{\epsilon \downarrow 0} \int_0^\infty e^{-x(\epsilon + i\xi)} x^s dx. \end{aligned}$$

We compute this using the Cauchy integral formula

$$arg\ z(\epsilon + i\xi) = 0, \text{defines the contour}, \ \Gamma^R_{\epsilon,\xi}$$

$$\int_{\Gamma^R_{\epsilon,\xi}} e^{-z(\epsilon+i\xi)} z^s dz = 0$$

It is elementary to show that along $\{|z| = R\} \bigcap \Gamma^R_{\epsilon,\xi} \quad | e^{-z(\epsilon+i\xi)} | \leq e^{-\epsilon R}$ and therefore as $R + \infty$ we obtain:

$$\begin{aligned} \int_0^\infty e^{-x(\epsilon+i\xi)} x^s dx &= \int_0^\infty e^{-x} (\frac{y}{\epsilon + i\xi})^s \frac{dy}{(\epsilon + i\xi)} \\ &= \frac{\Gamma(s+1)}{(\epsilon + i\xi)^{s+1}} \\ &= \frac{\Gamma(s+1) e^{+\frac{\pi i}{2}(s+1)}}{(\xi - i\epsilon)^{s+1}} \end{aligned}$$

Thus $\widehat{x_+^s}(\xi) = \dfrac{\Gamma(s+1) e^{\frac{-\pi i(s+1)}{2}}}{(\xi - i0)^{s+1}}$ for $Re\ s > -1$. These two functions are meromorphic in $\mathbb{C}$ and so must coincide for $s \notin -\mathbb{N}$. We compute

$$\begin{aligned} \widehat{x_+^{-1}}(\xi) &= \lim_{\epsilon \downarrow 0} \widehat{x_+^{\epsilon - 1}}(\xi) - \frac{1}{\epsilon} \widehat{\delta}_0(\xi) \\ &= \lim_{\epsilon \downarrow 0} \Gamma(\epsilon) e^{\frac{-\pi i \epsilon}{2}} (\xi - i0)^\epsilon - \frac{1}{\epsilon} \\ &= \begin{cases} \lim\limits_{\epsilon \downarrow 0} \dfrac{\Gamma(\epsilon+1) e^{+\frac{\pi i \epsilon}{2}} \xi^\epsilon - 1}{\epsilon} & \text{for } \xi > 0 \\ \lim\limits_{\epsilon \downarrow 0} \dfrac{\Gamma(\epsilon+1) e^{-\frac{\pi i \epsilon}{2}} |\xi|^\epsilon - 1}{\epsilon} & \text{for } \xi < 0 \end{cases} \\ &= \log|\xi| + \frac{\pi i}{2} \operatorname{sgn}\ \xi + \Gamma'(1). \end{aligned} \tag{2.9}$$

To compute $\widehat{x_+^{-k}}$ one simply uses (2.8). From the definition of $\underline{x}^{-1}$ and (2.9) we obtain

$$\widehat{\underline{x}^{-1}}(\xi) = \pi i \ \operatorname{sgn}\ \xi$$

As $\quad \partial_x \underline{x}^{-k} = -k\underline{x}^{-(k+1)} \quad$ we see that

$$\begin{aligned}\widehat{\underline{x}^{-k}}(\xi) &= k!(-1)^{k-1}(i\xi)^{k-1}\pi i \text{ sgn } \xi \\ &= \pi i k!(-i)^{k-1}\xi^{k-1}\text{sgn } \xi.\end{aligned}$$

We now consider this extension problem in dimensions greater than 1. Suppose that $\phi \in C^\infty(\mathbb{R}^n\backslash\{0\})$ and for some $m \in C$ satisfies:

$$\phi(\lambda x) = \lambda^m \phi(x) \qquad \text{for } \lambda \in \mathbb{R}_+ \tag{2.10}$$

We want to extend ϕ to $\mathbb{R}^n$ as a distribution which satisfies (2.10) in so far as this is possible. For $\psi \in C_c^\infty(\mathbb{R}^n\backslash\{0\})$ we have the relation:

$$\begin{aligned}\langle \phi, \psi\rangle &= \int \phi(x)\psi(x)dx \\ &= \int_{S^{n-1}} \phi(\omega)\int_0^\infty \psi(r\omega)r^{m+n-1}dr\, d\sigma(\omega)\end{aligned}$$

If we interpret

$$\begin{aligned}\int_0^\infty \psi(r\omega)r^{m+n-1} &= \langle r_+^{m+n-1}, \psi(r\omega)\rangle \\ &= R_m(\psi)(\omega)\end{aligned}$$

then we can use the one dimensional results to obtain the desired extension:

$$\langle \widetilde{\phi}, \psi\rangle = \int_{S^{n-1}} \phi(\omega)R_m(\psi)(\omega)d\sigma(\omega) \ , \qquad \psi \in \mathcal{S}(\mathbb{R}^n) \tag{2.11}$$

As $r_+^{m+n-1} \in \mathcal{S}'(\mathbb{R})$ and $\omega \mapsto \psi(r\omega)$ is a smooth mapping of $\mathbb{S}^{n-1}$ into $\mathcal{S}(\mathbb{R})$ it is clear that $R_m(\psi)(\omega) \in C^\infty(\mathbb{S}^{n-1})$ so that $\langle \widetilde{\phi}, \psi\rangle$ is well defined even if $\phi\Big|_{\mathbb{S}^{n-1}} \in C^{-\infty}(\mathbb{S}^{n-1})$. It defines a distribution which is homogeneous so long as $m \in \{-n, -n-1, \ldots\}$.

This extension process has several nice properties:

1) If $P(x)$ is a homogeneous polynomial then

$$(\widetilde{P(x)\phi}) = P(x)\widetilde{\phi} \quad \text{for all } m \in \mathbb{C} \ .$$

2) If $m \notin \{1-n, -n, \ldots\}$ then

$$(\widetilde{\partial x_i \phi}) = \partial x_i \widetilde{\phi} \quad i = 1, \ldots, n$$

Both statements are immediate consequences of (2.11).

If $m = -(n+k)$ then $\widetilde{\phi}$ is not always a homogeneous distribution:

with $\psi(\lambda x) = \lambda^n \psi(\lambda x)$

$$\begin{aligned}\langle \widetilde{\phi}, \psi_\lambda\rangle &= \int \phi(\omega)R_{-(n+k)}(\psi)(\omega)d\sigma(\omega) \\ &= \int \phi(\omega)\lambda^{k+n}\Big[R_{-(n+k)}(\psi)(\omega) - \log\lambda\frac{(-1)^k}{k!}\partial_r^k\psi(r\omega)\Big|_{r=0}\Big]d\sigma(\omega) \ .\end{aligned} \tag{2.12}$$

Thus we see that

$$\langle \widetilde{\phi}, \psi_\lambda\rangle - \lambda^{k+n}\langle \widetilde{\phi}, \psi\rangle = \frac{-\lambda^{k+n}\log\lambda(-1)^k}{k!}\sum_{|\alpha|=k}\partial_x^\alpha\psi(0)\mathcal{M}(\phi\omega^\alpha)$$

where

$$\mathcal{M}(\phi\omega^\alpha) = \int_{|\omega|=1}\phi(\omega)\omega^\alpha d\sigma(\omega) \ .$$

The log-term vanishes if and only if

$$\mathcal{M}(\phi\omega^\alpha) = 0 \qquad \forall\, \alpha \text{ with } |\alpha| = k \ .$$

As in the one dimensional case, if the extension $\widetilde{\phi}$ is not homogeneous then it is not possible to find a homogeneous extension. This is because any other extension differs from $\widetilde{\phi}$ by a distribution supported at 0 and therefore a finite sum of the form $\sum c_\alpha \partial_x^\alpha \delta_0$. As this is itself a sum of homogeneous distributions no such sum can suffice to remove the log-term in (2.12). Note that if $\phi(-x) = -(-1)^k \phi(x)$ then $\mathcal{S}(\omega^\alpha \phi) = 0 \;\forall\; \alpha$ with $|\alpha| = k$. This condition is satisfied by ratios of homogeneous polynomials in odd dimensions.

The Fourier transforms of homogeneous distributions are again homogeneous distributions:

$$\begin{aligned}
\langle \widehat{\widetilde{\phi}}, \psi_\lambda \rangle &= \langle \widetilde{\phi}, \widehat{\psi_\lambda} \rangle \\
&= \lambda^{-n} \langle \widetilde{\phi}, \widehat{\psi}_{\frac{1}{\lambda}} \rangle \\
&= \lambda^{-(m+n)} \langle \widetilde{\phi}, \widehat{\psi} \rangle \\
&= \lambda^{-(m+n)} \langle \widehat{\widetilde{\phi}}, \psi \rangle \ .
\end{aligned}$$

If $\widetilde{\phi}$ is homogeneous of degree m then $\widehat{\widetilde{\phi}}$ is homogeneous of degree $-(m+n)$. If $m = -(k+n)$ and $\widetilde{\phi}$ is not homogeneous then $\widehat{\widetilde{\phi}}$ also transforms with a log-term:

$$\langle \widehat{\widetilde{\phi}}, \psi_\lambda \rangle = \lambda^{-k} \langle \widehat{\widetilde{\phi}}, \psi \rangle - \frac{\lambda^{-k} \log \lambda (-1)^k}{k!} \sum \mathcal{S}(\phi \omega^\alpha) \langle x^\alpha, \psi \rangle \tag{2.13}$$

From (2.13) we easily obtain that if ϕ is homogeneous of degree $-(k+n)$ then

$$\widehat{\widetilde{\phi}} = \widehat{\phi}_1 + \log|x| p(x)$$

where $\widehat{\phi}_1$ is homogeneous of degree k and $p(x)$ is a homogeneous polynomial of degree k.

In all cases it is easy to show that if $\phi \in C^\infty(\mathbb{R}^n \backslash \{0\})$ is homogeneous of degree m then $\widehat{\widetilde{\phi}}\Big|_{\mathbb{R}^n \backslash \{0\}}$ is smooth as well. To prove this we select a function $\chi \in C_c^\infty(\mathbb{R}^n)$ so that $\chi \equiv 1$ in a neighborhood of 0. Then

$$\widehat{\widetilde{\phi}} = \widehat{\chi \widetilde{\phi}} + [(1-\chi)\widetilde{\phi}]^\wedge \ .$$

The first term is analytic as $\chi\widetilde{\phi}$ is a compactly supported distribution. Using the oscillatory integral definition of $[(1-\chi)\widetilde{\phi}]^\wedge$ we obtain that

$$[(1-\chi)\widetilde{\phi}]^\wedge(\xi) = \int \Delta^k [(1-\chi)\phi(x)] \frac{e^{-ix\cdot\xi}}{|\xi|^{2k}} dx \ . \tag{2.14}$$

Any derivatives applied to $1-\chi$ again leads to the Fourier transform of a function in $C_c^\infty(\mathbb{R}^n)$, the only term which is therefore not obviously smooth in $\mathbb{R}^n \backslash \{0\}$ is:

$$\frac{1}{|\xi|^{2k}} \int (1-\chi)(\Delta^k \phi) e^{-ix\cdot\xi} dx \ .$$

Since $\Delta^k \phi$ is homogeneous of order $m-2k$. For any fixed j there is a k so that this expression is absolutely convergent along with all derivatives of order j. This completes the proof.

We close with a simple application of these ideas:

Let $P(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$ be an elliptic operator, that is

$$P(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

is nonvanishing in $\mathbb{R}^n \backslash \{0\}$. There exists a fundamental solution E of the form

$$E(x,y) = e_0(x-y) + \log|x-y| p(x-y)$$

where e_0 is homogeneous of degree $m-n$ and p is a polynomial of degree $m-n$ (identically zero if $m < n$.) To see this we observe that if $\widehat{E} = \widetilde{\frac{1}{P(\xi)}}$
then

$$\begin{aligned} P(\xi)\widehat{E} &= \left(P(\xi) \cdot \frac{1}{P(\xi)}\right)^{\sim} \\ &= \widetilde{1} \\ &= 1 \end{aligned}$$

This implies that $P(D)E = \delta_0$.

3. Elliptic boundary value problem for the Laplacian

3.1. **Multiple layer potentials.** As it introduces no additional complexity we consider the somewhat more general problem of a boundary value problem for the Laplace, Δ_g, of a metric, g on a compact domain, Ω with smooth boundary in a compact manifold, X. In section 2 we showed that there is a fundamental solution defined on a neighborhood, Ω_1, of Ω. This is represented by a function $Q \in C^\infty(\Omega_1 \times \Omega_1 \backslash \Delta)$ which satisfies:

$$\Delta_g^y Q(x, y) = \delta_x(y) \qquad x \in \Omega_1 \tag{3.1}$$

in the distribution sense, that is

$$\int Q(x, y)\Delta\phi(y)dVol(y) = \phi(x) \qquad \forall \phi \in C_c^\infty(\Omega_1) .$$

Q is the Schwartz kernel of a ψDO in $\Psi^{-2}(\Omega_1)$, its principal symbol, $\sigma_{-2}(q)$ satisfies:

$$\sigma_{-2}(Q)(x, \xi) = |\xi|_{g(x)}^{-2} . \tag{3.2}$$

For $f \in C^\infty(b\Omega)$ we define:

$$\mathcal{S}f(x) = \int_{b\Omega} Q(x, y)f(y)d\sigma(y)$$

and

$$\mathcal{D}f(x) = \int_{b\Omega} \frac{\partial Q}{\partial \nu_y}(x, y)f(y)d\sigma(y)$$

Here $d\sigma$ is the surface measure on $b\Omega$ and ∂_ν is the outer normal derivative along $b\Omega$. It is immediate from (3.1) that $\mathcal{S}f, \mathcal{D}f \in C^\infty(\Omega)$ and:

$$\Delta_g \mathcal{S}f(x) = \Delta_g \mathcal{D}f(x) = 0 \qquad \text{for } x \in \Omega_1 \backslash b\Omega .$$

If $\Delta_g u = 0$ in Ω then Green's formula states that

$$u(x) = \mathcal{D}u_0(x) - \mathcal{S}u_1(x) \tag{3.3}$$

Here $u_0 = u\Big|_{b\Omega}$ and $u_1 = \frac{\partial u}{\partial \nu}\Big|_{b\Omega}$. We now consider the operators $\mathcal{D}$, $\mathcal{S}$ and prove the following basic result:

Theorem 3.1. *If $f \in C^\infty(b\Omega)$ then $\mathcal{S}f$, $\mathcal{D}f$ have extensions as elements of $C^\infty(\overline{\Omega})$. For each $k \in \mathbb{N}_0$ there are ψDO S_k, D_k in $\Psi^{k-1}(b\Omega)$ and $\Psi^k(b\Omega)$ respectively such that:*

$$\begin{aligned} \partial_\nu^k \mathcal{S}f\Big|_{b\Omega} &= S_k f \\ \partial_\nu^k \mathcal{D}f\Big|_{b\Omega} &= D_k f \end{aligned} \qquad k = 0, 1, \ldots \tag{3.4}$$

Moreover

$$\begin{aligned} \sigma_{-1}(S_0)(x, \xi) &= \tfrac{1}{2}|\xi|_g^{-1} \\ \sigma_{-1}(D_0)(x, \xi) &= \tfrac{1}{2} \end{aligned} \qquad (x, \xi) \in T^*b\Omega . \tag{3.5}$$

In the theorem we use the metric induced on $b\Omega$ from the metric defined on X.

Proof: A more invariant description of the operators, $\mathcal{S}$ and $\mathcal{D}$ takes advantage of the fact that pseudodifferential operators act on distributions. Let $\delta_{b\Omega}$ denote the distribution:

$$\langle \phi, \delta_{b\Omega} \rangle = \int_{b\Omega} \phi d\sigma \ .$$

Then the operators $\mathcal{S}$ and $\mathcal{D}$ are defined for $f \in C^\infty(b\Omega)$ by:

$$\begin{aligned} \mathcal{S}f &= Q(f\delta_{b\Omega}) \\ \mathcal{D}f &= -Q(\partial_\nu(f\delta_b\Omega)) \ . \end{aligned} \tag{3.6}$$

From these formulae it is apparent that, as distributions

$$\begin{aligned} \text{singsupp } \mathcal{S}f &\subseteq \text{supp } f \\ \text{singsupp } \mathcal{D}f &\subseteq \text{supp } f \ . \end{aligned} \tag{3.7}$$

From (3.7) it is clear that to prove the theorem it suffices to work in a single local coordinate patch. We choose coordinates to simplify the form of the metric, $|\xi|_g^2$: Let $\widetilde{U}$ be a small disk centered on a point $p \in b\Omega$ and let $(x_1, \ldots, x_n)$ be coordinates defined on $\widetilde{U} \cap b\Omega$. We define x_0 to be the signed geodesic distance from a position $\widetilde{U}$ to a point on $b\Omega$. We take x_0 to be positive for points in Ω and negative for points in $X\backslash\Omega$, this function is smooth in some neighborhood of $b\Omega$. In a possibly smaller neighborhood, U of p, we can use $(x_0, x_1, \ldots, x_n)$ as coordinates. In such a coordinate patch the metric takes the form:

$$ds^2 = dx_0^2 + \sum_{i,j=1}^n g_{ij}(x_0, x) dx_i dx_j \ ,$$

the principal symbol of Δ_g is then $\sigma_2(\Delta_g) = \xi_0^2 + |\xi|_g^2$ where we use the notation:

$$|\xi|_g^2 = \sum_{i,j=1}^n g^{ij} \xi_i \xi_j .$$

Since it entails no further effort and is useful for subsequent applications we will consider the limiting behavior as $x_0 \to 0$ from above and below. We use $(+)$ to denote $x_0 \searrow 0$ and $(-)$ to denote $x_0 \nearrow 0$. Recall that in our domain, Ω, the variable $x_0 > 0$. Relative to Ω, $\partial_{x_0}\Big|_{x_0=0}$ is the inward pointing unit normal vector.

The complete symbol of Q, in this local coordinate system takes the form:

$$\sigma(Q)(x_0, x; \xi_0, \xi) = q(x_0, x; \xi_0, \xi) \sim (\xi_0^2 + |\xi|_g^2)^{-2} + \sum_{j=-\infty}^{-3} a_j(x_0, x; \xi_0, \xi) \ ,$$

here a_j is homogeneous in (ξ_0, ξ) of order j in (ξ_0, ξ). In the sequel, the statement that a function of the form $k(x_0, x; \xi_0, \xi)$ 'is homogeneous' means that it is homogeneous as a function of (ξ_0, ξ). For our applications we require a more precise statement than this: for each j there is a smooth family of homogeneous polynomials, $p_j(x_0, x; \xi_0, \xi)$ and an integer k_j, such that:

$$a_j(x_0, x; \xi_0, \xi) = \frac{p_j(x_0, x; \xi_0, \xi)}{|\xi|_g^{2k_j}} \ .$$

In other words, the terms in the asymptotic expansion of $\sigma(Q)$ are rational functions in (ξ_0, ξ). This is a simple consequence of the parametrix construction. For $f \in C_c^\infty(U \bigcap b\Omega)$ we see that

$$\mathcal{S}f(x_0, x) = \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} e^{ix\cdot\xi} e^{ix_0\cdot\xi_0} \widehat{f}(\xi) q(x_0, x; \xi_0, \xi) \frac{d\xi_0 d\xi}{(2\pi)^{n+1}}$$

In the case at hand the ξ_0-integral converges absolutely, in other cases one uses (3.6) to give such expression meaning as an oscillatory integral. If $\psi(a,b)$ is a function in $C^\infty(\mathbb{R}^2)$ such that

$$\psi_1(a,b) = \begin{cases} 1 & |a|^2 + |b|^2 > 1 \\ 0 & |a|^2 + |b|^2 < \frac{1}{2} \end{cases}$$

then

$$r_N(x_0, x; \xi_0, \xi) = q(x_0, x; \xi_0, \xi) - \psi_1(\xi_0, |\xi|_g)\big((\xi_0^2 + |\xi|_g^2)^{-2} + \sum_{j=-N}^{-3} a_j\big) \tag{3.8}$$

belongs to $S^{-(N+1)}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$.

We have the following simple lemma:

Lemma 3.1. *: If $N \geq 1$ then for $f \in C_c^\infty(\mathbb{R}^n)$*

$$R_N f(x_0, x) = \int_{\mathbb{R}^n} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi_0 \cdot x_0} e^{i\xi \cdot x} r_N(x_0, x; \xi_0, \xi) \frac{d\xi_0 d\xi}{(2\pi)^{n+1}} \tag{3.9}$$

belongs to $C^{N-1}(\mathbb{R}^{n+1})$ moreover $R_N f(0,x) = \rho_N(f)(x)$, where $\rho_N \in \Psi^{-N}(\mathbb{R}^n)$.

Proof: Proving $R_N f(x_0, x)$ has $(N-1)$-derivatives is simply a matter of differentiation under the integral sign. The only degradation to the convergence of the integral arises from differentiating $e^{i\xi_0 x_0}$, but this term can be differentiated $(N-1)$-times leading to an integrand which is $0(\frac{1}{\xi_0^2})$. Restricting to $\xi_0 = 0$ gives:

$$R_N f(0,x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{ix\cdot\xi} \Big(\int r_N(0, x; \xi_0, \xi) \frac{d\xi_0}{2\pi}\Big) \frac{d\xi}{(2\pi)^n}$$

Thus we see that

$$\sigma(\rho_N) = \int r_N(0, x; \xi_0, \xi) \frac{d\xi_0}{2\pi}$$

$$|D_x^\alpha D_\xi^\beta r_N(0, x; \xi_0, \xi)| \leq \frac{C_{\alpha\beta}}{(1 + |\xi_0| + |\xi|_g)^{N+1+|\beta|}} \ .$$

Integrating this estimate shows that

$$|D_x^\alpha D_\xi^\beta \sigma(\rho_N)(x, \xi)| \leq \frac{\widetilde{C}_{\alpha\beta}}{(1 + |\xi|_g)^{N+|\beta|}} \ .$$

This completes the proof of the lemma.

As a simple corollary we have:

Corollary 3.1. *If $k < N-1$ then $\partial_{x_0}^k R_N f(x_0, x)$ is continuous and $\partial_{x_0}^k R_N f(0,x) = \rho_N^k(f)(x)$ where $\rho_N^k \in \Psi^{k-N}(\mathbb{R}^n)$.*

Proof: We can simply differentiate (3.9) to obtain:

$$\begin{aligned} \partial_{x_0}^k R_N f(x_0, x) &= \int \widehat{f}(\xi) e^{ix\cdot\xi} \int \partial_{x_0}^k \big(r_N(x_0, x; \xi_0, \xi) e^{ix_0\cdot\xi_0}\big) \frac{d\xi_0}{2\pi} \frac{d\xi}{(2\pi)^n} \\ &= \int \widehat{f}(\xi) e^{ix\cdot\xi} \int r_N(x_0, x; \xi_0, \xi)(i\xi_0)^k e^{ix_0\cdot\xi_0} \frac{d\xi_0}{2\pi} \frac{d\xi}{(2\pi)^n} + \text{l.o.t.} \end{aligned}$$

Here l.o.t. is a sum of terms of order strictly less that $k - N$.

The lemma, its corollary and (3.8) show that to prove the theorem it suffices to consider each term in the asymptotic expansion separately. First we analyze the principal term. We define:

$$\mathcal{S}_0 f(x_0,x) = \int\int \frac{\widehat{f}(\xi)e^{ix\cdot\xi}}{(\xi_0^2+|\xi|_g^2)}\psi_2(|\xi|_g)\psi_1(\xi_0,|\xi|_g)e^{ix_0\cdot\xi_0}\frac{d\xi_0 d\xi}{(2\pi)^{n+1}} \ .$$

$$\text{Here} \quad \psi_2(a) = \begin{cases} 0 & |a|\le 1 \\ 1 & |a|\ge 2 \end{cases} \ .$$

We evaluate the ξ_0 integral, for $|\xi|_g > 1$, using the Cauchy residue formula we obtain:

$$\int \frac{e^{ix_0\cdot\xi_0}}{(\xi_0^2+|\xi|_g^2)}\frac{d\xi_0}{2\pi} = \frac{e^{-|x_0||\xi|_g}}{2|\xi|_g} \tag{3.10}$$

Thus we see that

$$\mathcal{S}f(x_0,x) = \int \widehat{f}(\xi)\psi_2(|\xi|_g)\frac{e^{-|x_0||\xi|_g}}{2|\xi|_g}e^{ix\cdot\xi}\frac{d\xi}{(2\pi)^n} \tag{3.11}$$

From (3.11) it is clear that $\mathcal{S}_0 f(x_0,x)$ has smooth extensions to $\mathbb{R}^n\times(-\infty,0]$ and $\mathbb{R}^n\times[0,\infty)$ which in general do $\underline{\underline{\text{not}}}$ agree across $x_0=0$. If g is independent of x_0 then

$$\begin{aligned}\lim_{x_0\to 0^\pm}\partial_{x_0}^k\mathcal{S}_0 f(x_0,x) &= \begin{cases} \frac{1}{2}\displaystyle\int \widehat{f}(\xi)\psi_2(|\xi|_g)|\xi|_g^{k-1}e^{ix\cdot\xi}\tfrac{d\xi}{(2\pi)^n} & (-) \\ \frac{(-1)^k}{2}\displaystyle\int \widehat{f}(\xi)\psi_2(|\xi|_g)|\xi|_g^{k-1}e^{ix\cdot\xi}\tfrac{d\xi}{(2\pi)^n} & (+)\end{cases} \\ &= \begin{cases} S_{k_0}^- f(x) & (-) \\ S_{k_0}^+ f(x) & (+)\end{cases}\end{aligned}$$

Note that $S_{k_0}^\pm \in \Psi^{k-1}(\mathbb{R}^n)$ and that

$$\sigma_{k-1}(S_{k_0}^\pm) = \begin{cases} \frac{1}{2}|\xi|_g^{k-1} & (-) \\ \frac{(-1)^k}{2}|\xi|_g^{k-1} & (+)\end{cases}$$

More generally if g does depend on x_0 then it follows easily from (3.11) and the fact that $\psi_2(|\xi|_g)|\xi|_g$ is a symbol that

$$\lim_{x_0\to 0^\pm}\partial_{x_0}^k\mathcal{S}_0 f(x_0,x) = S_{k_0}^\pm f + E_k^\pm f \qquad \text{here } E_k^\pm \in \Psi^{k-2}(\mathbb{R}^n).$$

This completes the analysis of leading part.

Now we need to consider the lower order terms in the asymptotic expansion for $\sigma(Q)$. These are expressions of the form:

$$\mathcal{A}_j f(x_0,x) = \int\int \widehat{f}(\xi)e^{ix\cdot\xi}a_j(x_0,x_i;\xi_0,\xi)e^{i\xi_0\cdot x_0}\psi_1(\xi_0,|\xi|_g)\frac{d\xi_0 d\xi}{(2\pi)^{n+1}}$$

where a_j is homogeneous in (ξ_0,ξ) of degree j. In fact a somewhat more precise statement is true:

$$a_j(x_0,x;\lambda\xi_0,\lambda\xi) = \lambda^j a_j(x_0,x_i;\xi_0,\xi) \qquad \lambda\in\mathbb{R}\backslash\{0\} \ . \tag{3.12}$$

This follows because a_j is a rational function in (ξ_0,ξ). From this we can conclude that taking the Fourier transform of a_j in the ξ_0-variable will not lead to terms of the form $\log|x_0|$. Up to a smoothing term:

$$\mathcal{A}_j f(0,x) = \int \widehat{f}(\xi)e^{ix\cdot\xi}\psi_2(|\xi|_g)\Big[\int a_j(x_0,x;\xi_0,\xi)\frac{e^{i\xi_0\cdot x_0}}{2\pi}d\xi_0\Big]d\xi \ . \tag{3.13}$$

There are two approaches to evaluating the ξ_0-integral. We can use the fact that

$$a_j(x_0,x;\xi_0,\xi) = \frac{p_j(x_0,x;\xi_0,\xi)}{(\xi_0^2+|\xi|_g^2)^{k_j}}$$

and the Cauchy integral formula to obtain that for $|\xi|_g > 1$:
for $x_0 \lessgtr 0$:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{p_j(x_0, x; \xi_0, \xi)e^{ix_0\xi_0}}{(\xi_0^2 + |\xi|_g^2)^{k_j}} \frac{d\xi_0}{2\pi} &= \frac{i}{(k_j-1)!}\partial_{\xi_0}^{k_j-1}\Big[\frac{e^{ix_0\xi_0}p_j(x_0, x; \xi_0, \xi)}{(\xi_0 \pm i|\xi|_g)^{k_j}}\Big]\Big|_{\xi_0 = i \text{ sgn } x_0|\xi|_g} \\ &= \frac{e^{-|x_0||\xi|_g}q_j(x_0, x; \pm i|\xi|_g, \xi)}{(\pm i|\xi|_g)^{2k_j-1}}.\end{aligned} \tag{3.14}$$

Recall that deg $p_j - 2k_j = -j$, while q_j is no longer a homogeneous polynomial, a moment's reflection shows that:

$$\deg\, q_j - (2k_j - 1) = 1 - j\ . \tag{3.15}$$

From (3.13) and the second line in (3.14) it is evident that $\mathcal{A}_j f(x_0, x)$ again has smooth extensions to $\mathbb{R}^n \times (-\infty, 0]$ and $\mathbb{R}^n \times [0, \infty)$. From (3.14) and (3.15) we conclude that

$$\lim_{x_0 \to 0^{\pm}} \partial_{x_0}^k \mathcal{A}_j f(x_0, x) = A_{jk}^{\pm} f(x) \qquad \text{where } A_{jk}^{\pm} \in \Psi^{k+1-j}(\mathbb{R}^n).$$

This analysis used only that $j \leq -2$ and the general form of the symbols, $\{a_j\}$.

There is a second approach to this computation that works in greater generality. To compute

$$\alpha_j(x_0, x; \xi) = \int a_j(x_0, x; \xi_0, \xi)e^{i\xi_0 \cdot x_0}\frac{d\xi_0}{2\pi} \tag{3.16}$$

we rewrite $a_j(x_0, x; \xi_0, \xi) = |\xi|_g^{1+j} a_j(x_0, x; \frac{\xi_0}{|\xi|_g}, \omega)$ where $\omega = \frac{\xi}{|\xi|_g}$. We let $s = \frac{\xi_0}{|\xi|_g}$ in (3.16) to obtain:

$$\alpha_j(x_0, x; \xi) = |\xi|_g^{1+j}\int_{-\infty}^{\infty} a_j(x_0, x; s, \omega)e^{isx_0|\xi|_g}ds\ .$$

The fact that we can allow the integral to run from $-\infty$ to ∞ is a consequence of (3.12).

To evaluate this integral we make further usage of the homogeneity of a_j:

$$a_j(x_0, x; s, \omega) = s^j a_j(x_0, x; 1, \frac{\omega}{s})\ ,$$

where again we use (3.12). We use the Taylor expansion for a_j:

$$a_j(x_0, x; s, \omega) = s^j\Big[\sum_{|\alpha| \leq N}\frac{\partial_\xi^\alpha}{\alpha!}a_j(x_0, x; 1, 0)(\frac{\omega}{s})^\alpha + r_N(x_0, x; 1, \frac{\omega}{s})\Big] \tag{3.17}$$

where $r_N(x_0, x; 1, y) = 0(|y|^{N+1})$ at $y = 0$.

Let $\phi(s) = \begin{cases} 1 & |s| < 1 \\ 0 & |s| > 2 \end{cases}$ then

$$\begin{aligned}\alpha_j(x_0, x; \xi) = |\xi|_g^j &\int a_j(x_0, x; s, \omega)\phi(s)e^{is\cdot x_0|\xi|_g}\frac{ds}{2\pi} \\ &+ |\xi|_g^j\int\Big(\sum_{|\alpha| \leq N}\frac{\partial_\xi^\alpha a_j(x_0, x; 1, 0)}{\alpha!}(x_0, x; 1, 0)(\frac{\omega}{s})^\alpha + \\ &r_N(x_0, x; 1, \frac{\omega}{s})s^j e^{isx_0|\xi|_g}(1 - \phi(s))ds\end{aligned}$$

The first term is obviously a smooth function of x_0 taking values in $\Psi^j(\mathbb{R}^n)$, its k-fold x_0-derivative takes values in $\Psi^{j+k}(\mathbb{R}^n)$.

To evaluate the second term we observe that

$$s^{j-|\alpha|}\big(1 - \phi(s)\big) = \underline{s}^{j-|\alpha|} - \phi(s)\underline{s}^{j-|\alpha|}$$

where $\underline{s}^m$ denotes the homogeneous distribution on $\mathbb{R}$ defined in §2. As $\phi(s)\underline{s}^{j-|\alpha|}$ is a compactly supported distribution, its Fourier transform is a smooth function of x_0. On the other hand, we have computed the Fourier transforms of $\underline{s}^{j-|\alpha|}$ and it is:

$$\widehat{\underline{s}^{j-|\alpha|}}(\xi_0) = i\pi(|\alpha| - j)!(i\xi_0)^{|\alpha|-j-1}\operatorname{sgn}\ \xi_0 \ .$$

Thus the sum in (3.17) contributes:

$$|\xi|_g^j \sum_{|\alpha|\leq N} \frac{\partial_\xi^\alpha a_j(x_0, x; 1, 0)}{\alpha!}\omega^\alpha i\pi(|\alpha| - j)!(ix_0|\xi|_g)^{|\alpha|-j-1}\operatorname{sgn}(x_0) \ .$$

Recalling that $j \leq -3$ we see that these terms have smooth extensions to $\mathbb{R}^n \times (-\infty, 0]$ and $\mathbb{R}^n \times [0, \infty)$. The error term $r_N(x_0, x; 1, \frac{\omega}{s})s^j(1 - \phi(s))$ is $N - (j + 1)$-times differentiable. As N is arbitrary we once again conclude that $\mathcal{A}_j f(x_0, x)$ has smooth extensions to $\mathbb{R}^n \times (-\infty, 0]$ and $\mathbb{R}^n \times [0, \infty)$. Using the various expressions used to evaluate (3.16) we again show that $A^{\pm}_{jk} \in \Psi^{k+1-j}(\mathbb{R}^n)$, we leave the details of this argument to the reader. This completes the proof of the statement in Theorem 3.1 regarding $\mathcal{S}f$. A similar argument could be used to study $\mathcal{D}f$, however this is not really necessary as the properties of $\mathcal{D}f$ are easily deduced from those of $\mathcal{S}f$.

Observe that

$$\mathcal{D}f(x_0, x) = \int\int e^{ix\cdot\xi}e^{ix_0\cdot\xi_0}(-i\xi_0)q(x_0, x; \xi_0, \xi)\frac{\widehat{f}(\xi)d\xi}{(2\pi)^{n+1}} \tag{3.18}$$

On the other hand:

$$\begin{aligned} \partial_{x_0}\mathcal{S}f(x_0, x) &= \int\int e^{ix\cdot\xi}e^{ix_0\cdot\xi_0}(i\xi_0)q(x_0, x; \xi_0, \xi)\frac{\widehat{f}(\xi)d\xi_0 d\xi}{(2\pi)^{n+1}} \\ &\quad + \int\int e^{ix\cdot\xi}e^{ix_0\cdot\xi_0}\frac{\partial q}{\partial x_0}(x_0, x; \xi_0, \xi)\frac{\widehat{f}(\xi)d\xi_0 d\xi}{(2\pi)^{n+1}} \\ &= \mathcal{S}_1 f + \mathcal{S}_2 f \ . \end{aligned} \tag{3.19}$$

The symbol in the second term in (3.19), $\frac{\partial q}{\partial x_0}$ defines an operator with exactly the same properties as $\mathcal{S}$ itself. Of course, the symbols are different but the orders of the operators which appear in $\partial^k_{x_0}\mathcal{S}_2 f(0^\pm, x)$ are identical to those which arose in $\partial^k_{x_0}\mathcal{S}f(0^\pm, x)$. So we see that

$$\mathcal{D}f = -\partial_{x_0}\mathcal{S}f + \mathcal{S}_2 f \ . \tag{3.20}$$

Thus all the required properties of $\mathcal{D}f$ follow in a straightforward manner from the analysis we have just presented. For example, we see that

$$\begin{aligned} \sigma_0(D_0^\pm) &= \begin{cases} \frac{(-1)}{2} & (-) \\ \frac{1}{2} & (+) \end{cases} \\ \sigma_1(D_1^\pm) &= \begin{cases} \frac{-|\xi|_g}{2} & (-) \\ \frac{+|\xi|_g}{2} & (+) \end{cases} \end{aligned} \tag{3.21}$$

using these calculations we can easily prove the following proposition:

Proposition 3.1. *If $f \in C^\infty(b\Omega)$ then $\mathcal{S}f(x_0, x)$ is continuous as $x_0 \to 0$ whereas*

$$\mathcal{D}f(0^+, x) - \mathcal{D}f(0^-, x) = f \ . \tag{3.22}$$

Similarly $\partial_{x_0}\mathcal{D}f(x_0, x)$ is continuous as $x_0 \to 0$ whereas

$$\partial_{x_0}\mathcal{S}f(0^+, x) - \partial_{x_0}\mathcal{S}f(0^-, x) = -f \ . \tag{3.23}$$

3.2. **The Calderon projector and Dirichlet to Neumann operator.** If u is a harmonic function in Ω then

$$u(x) = \mathcal{D}u_0 - \mathcal{S}u_1,$$

where $u_0 = u\big|_{b\Omega}$ and $u_1 = \partial_\nu u$. So in particular

$$\begin{gathered} u_0 = D_0^+ u_0 - S_0^+ u_1 \\ \text{and} \\ u_1 = D_1^+ u_0 - S_1^+ u_1 \ . \end{gathered} \tag{3.24}$$

If we let f, g be smooth functions on $b\Omega$ then

$$U = \mathcal{D}f - \mathcal{S}g$$

is a harmonic function defined in Ω. From Theorem 3.1 it follows that

$$\begin{aligned} U_0 &= D_0^+ f - S_0^+ g \\ U_1 &= D_1^+ f - S_1^+ g \end{aligned} \tag{3.25}$$

We define the operator:

$$C\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} D_0^+ & -S_0^+ \\ D_1^+ & -S_1^+ \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

From (3.24) it follows that C is a projection operator, that is $C^2 = C$. From Theorem 3.1 it follows that C is a pseudodifferential operator with principal symbol:

$$\sigma_{\text{prin}}(C) = \frac{1}{2}\begin{pmatrix} 1 & \frac{-1}{|\xi|_g} \\ -|\xi|_g & 1 \end{pmatrix} .$$

As expected $\sigma_{\text{prin}}(C)^2 = \sigma_{\text{prin}}(C)$. This operator is called the Calderon projector.

Another operator of considerable interest is the Dirichlet to Neumann operator. As we shall soon see, the classical Dirichlet problem,

$$\text{(D)} \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u\big|_{b\Omega} = f \end{cases}$$

can always be solved. We defined an operator

$$\mathcal{N}f = \frac{\partial u}{\partial \nu}\Big|_{b\Omega}$$

where u is the solution to (D). Observe that S_0^+ is an elliptic ψDO on $b\Omega$ and let P_0^+ denote a parametrix:

$$P_0^+ S_0^+ = I + e_0^+$$

where $E_0^+ \in \Psi^{-\infty}(b\Omega)$. Using (3.24) we deduce that

$$u_1 = (P_0^+ D_0^+ - P_0^+)u_0 - E_0^+ u_1 \ .$$

So again $\mathcal{N}$ is a ψDO of order 1 with

$$\sigma_1(\mathcal{N}) = \frac{|\xi|_g}{2} \ .$$

The Dirichlet to Neumann operator plays an important role in geophysical inverse scattering problems.

3.3. **Elliptic boundary value problems.** Now we use the properties of the multilayer potentials to show that certain boundary value problems for the Laplace operator are solvable for data satisfying finitely many conditions. Let's consider the simplest such problem, the Dirichlet problem. With $u = \mathcal{D}u_0 - \mathcal{S}u_1$ we see that

$$(I - D_0^+)u_0 = -S_0^+ u_1 \ .$$

If we can solve for u_1, then setting

$$u_1 = -(S_0^+)^{-1}(I - D_0^+)f$$

we obtain the Cauchy data for the solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u\Big|_{b\Omega} = f & , \end{cases}$$

i.e. if we set

$$u = \mathcal{D}f + \mathcal{S}(S_0^+)^{-1}(I - D_0^+)f \tag{3.26}$$

$$\begin{aligned} \text{then} \quad \Delta u &= 0 \quad \text{and} \\ u\Big|_{b\Omega} &= D_0^+ f + S_0^+(S_0^+)^{-1}(I - D_0^+)f \\ &= f \ . \end{aligned}$$

We need to show that S_0^+ is invertible. As its symbol is $\frac{1}{2}|\xi|_g^{-1}$ it is clear that for every $s \in \mathbb{R}$:

$$S_0^+ : H^s(b\Omega) \to H^{s+1}(b\Omega)$$

is a Fredholm operator of index 0 and $\ker S_0^+ \subseteq C^\infty(b\Omega)$. To avoid technical difficulties we now consider only the case of $\Omega \subseteq \mathbb{R}^n$. If there is a function f_0 such that $S_0^+ f_0 = 0$ then we define

$$u = \mathcal{S}f_0 \ .$$

This function is harmonic in $\mathbb{R}^n \backslash \Omega$. As its boundary values from inside Ω equal $S_0^+ f = 0$ it follows from the maximum principle that $U^+\Big|_{\Omega} \equiv 0$. Let U^- denote $U\Big|_{\mathbb{R}^n\backslash\Omega}$. As $\mathcal{S}f_0$ is continuous, it follows that $U^-\Big|_{b\Omega} = 0$ as well. On the other hand we know from Proposition 3.1 that $\partial_\nu U^+ - \partial_\nu U^- = -f$, and therefore

$$\partial_\nu U^- = f \ .$$

As

$$U(x) = c_n \int_{b\Omega} \frac{f(x)d\sigma(y)}{|x-y|^{n-2}}$$

it follows that $\lim\limits_{|x|\to\infty} U^-(x) = 0$. We can therefore apply the maximum principle to conclude that $U^-|_\Omega \equiv 0$. This implies that $f = 0$, hence S_0^+ is an invertible operator. This shows that the Dirichlet problem always has a smooth solution for a smooth bounded domain, $\Omega \subseteq \mathbb{R}^n$ and $f \in C^\infty(b\Omega)$. Using the formula (3.26) and estimates for $\mathcal{S}$ and $\mathcal{D}$ proved in the last section, we will extend this result to $f \in H^s(b\Omega)$ for $s > \frac{1}{2}$ as well as obtaining a precise statement of the regularity of the solution, u.

Without restricting to $\Omega \subseteq \mathbb{R}^n$ we could have concluded that the range of S_0^+ is of finite codimension. Thus there are finitely many linear functionals $\ell_1, \ldots, \ell_m$ so that if

$$\ell_j((I - D_0^+)f) = 0 \quad j = 1, \ldots, m$$

then equation

$$(I - D_0^+)f = -S_0^+ u_1$$

has a solution, g. Again setting

$$u = \mathcal{D}f - \mathcal{S}g$$

we obtain that

$$\begin{aligned} u\big|_{b\Omega} &= D_0^+ f - S_0^+ g \\ &= D_0^+ f + (I - D_0^+) f \\ &= f \quad , \end{aligned}$$

as desired. As before, we will be able to apply the estimates proved in the next section to extend this existance result to boundary data with finite differentiability and also obtain precise regularity results. In fact, the Dirichlet problem is always solvable on a smooth compact manifold with boundary.

Now we turn our attention to more general boundary value problems:

$$(P_b) \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ b_0 u_0 + b_1 u_1 = f & \text{on } b\Omega \end{cases}$$

Here $b_0 \in \Psi^k(b\Omega)$ and $b_1 \in \Psi^{k-1}(b\Omega)$. We want conditions which imply that this is an "elliptic problem". For us this will be simply the condition that the system of equations:

$$\begin{pmatrix} I - D_0^+ & S_0^+ \\ b_0 & b_1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{3.27}$$

is elliptic. This is a principal symbol calculation:

Definition: The boundary value problem, P_b is elliptic if

$$|\xi|_g \sigma_{k-1}(b_1) - \sigma_k(b_0)$$

is elliptic. This is the determinant of the principal symbol of the operator in (3.27). In this case (3.27) is a Fredholm system, the principal symbol of the parametrix is given by:

$$\frac{1}{D} \begin{pmatrix} \sigma_{k-1}(b_g) & \frac{-1}{2|\xi|_g} \\ -\sigma_k(b_0) & \frac{1}{2} \end{pmatrix} \quad , \qquad \xi \neq 0,$$

where $D = \frac{1}{2}\left(\sigma k - 1(b_1) - \frac{\sigma_k(b_0)}{|\xi|_g}\right)$. Using the symbol calculus for $\Psi^*(b\Omega)$ we easily show that there exists a matrix of operators,

$$\begin{pmatrix} R & S \\ T & U \end{pmatrix}$$

such that

$$\sigma \begin{pmatrix} R & S \\ T & U \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \sigma_{k-1}(b_1) & \frac{-1}{2|\xi|_g} \\ -\sigma_k(b_0) & \frac{1}{2} \end{pmatrix}$$

$$\text{and} \qquad \begin{pmatrix} R & S \\ T & U \end{pmatrix} \begin{pmatrix} I - D_0^+ & S_0^+ \\ b_0 & b_1 \end{pmatrix} = I + E \ ,$$

$$\begin{pmatrix} I - D_0^+ & S_0^+ \\ b_0 & b_1 \end{pmatrix} \begin{pmatrix} R & S \\ T & U \end{pmatrix} = I + E'$$

where $E, E' \in \Psi^{-\infty}(b\Omega; \mathbb{C}^2)$. This shows that $\begin{pmatrix} I - D_0^+ & S_0^+ \\ b_0 & b_1 \end{pmatrix}$ is a Fredholm map whenever the boundary problem is elliptic. From this we easily deduce:

Theorem 3.2. *If (P_b) is an elliptic boundary value problem then it has a smooth solutions for all $f \in C^\infty(b\Omega)$ which satisfy a finite number of linear conditions of the form:*

$$\int_{b\Omega} f g_j d\sigma = 0 \qquad j = 1, \ldots, m$$

where $g_j \in C^\infty(b\Omega)$.

Corollary 3.2. *If (P_b) is elliptic then the set of solutions to P_b with $f = 0$ is finite dimensional.*

The classical Neumann problem

$$\text{(N)} \qquad \begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = f & \text{on } b\Omega \end{cases}$$

corresponds to $b_0 = 0$, $b_1 = 1$, this is obviously an elliptic problem which therefore has a solution for data which satisfies finitely many conditions. For a domain in $\Omega \subseteq X$ it is a consequence of Green's formula,

$$\int_\Omega u\Delta_g v - v\Delta_g u = \int u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}$$

applied to u, a solution of (N) and $v = 1$ that

$$\int_{b\Omega} f = 0$$

is a necessary condition for (N) to be solvable. In fact, for this case, this is the only condition though the proof again requires non-pseudodifferential techniques.

We can consider a more general problem of this type; the oblique derivative problem:

$$(O_Y) \qquad \begin{cases} \Delta u = 0 \\ \frac{\partial u}{\partial \nu} + Y(u\big|_{b\Omega}) = f\ . \end{cases}$$

Here $Y \in C^\infty(b\Omega; Tb\Omega)$ is a smooth vector field tangent to $b\Omega$. In this case

$$\sigma_0(b_1) = 1\ ,\ \sigma_1(b_0) = i\langle Y, \xi\rangle\ .$$

Recall that $(x,\xi) \in T^*b\Omega$. This problem is elliptic if

$$\Im\frac{\langle Y, \xi\rangle}{|\xi|} < 1 \qquad \forall\, \xi \neq 0\ .$$

This is the case if Y is a real vector field. There is a very important special case of this problem which is not elliptic. If $\Omega \subseteq \mathbb{C}^n$ is a domain with a smooth boundary then we choose a unit vector field, N transverse to $b\Omega$ such that $JN = T$ is tangent to $b\Omega$. Here J is the endomorphism of $T\mathbb{C}^n$ which defines the standard complex structure. The complex vector field $\overline{Z} = N + iJN$ is of type $(0,1)$, so if u is a holomorphic function in $\overline{\Omega}$ then $\overline{Z}u\big|_{b\Omega} \equiv 0$. There is an infinite dimensional space of such functions; these functions are also harmonic. This implies that the oblique derivative problem:

$$(\sigma_{\overline{Z}}) \qquad \begin{cases} \Delta n = 0 & \text{in } \Omega \\ \overline{Z}u\big|_{b\Omega} = f \end{cases}$$

is not Fredholm: it has an infinite dimensional null space. The endomorphism, J, is orthogonal and therefore

$$I_m\frac{\langle \overline{Z}, \xi\rangle}{\xi} = \frac{\langle JN, \xi\rangle}{|\xi|}$$

attains the value 1 in exactly one codirection. This shows that $\sigma_{\overline{Z}}$ is not an elliptic problem in the sense defined above. This problem is called the $\overline{\partial}$-Neumann problem and is of fundamental importance in the theory of holomorphic functions of several variables.

What does the condition

$$(I - D_0^+)u_0 + S_0^+ u_1 = 0 \tag{3.28}$$

mean? To understand this we consider a simple model situation, the upper half space $\mathbb{R}^{n+1}_+ = [0,\infty)\times\mathbb{R}^n$. We want to solve the equation

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ b_0 u\Big|_{b\Omega} + b_1\frac{\partial u}{\partial\nu} = f \ . \end{cases}$$

Take the Fourier transform of the equation in the $\mathbb{R}^n$-factor. We obtain

$$\frac{\partial^2\widehat{u}}{\partial x_0^2}(x_0,\xi) - |\xi|^2\widehat{u}(x_0,\xi) = 0 \ . \tag{3.29}$$

The general solution to (3.28) is of the form:

$$\widehat{u}(x_0,\xi) = a_-|\xi|e^{-x_0|\xi|} + a_+(\xi)e^{x_0|\xi|} \ .$$

The condition (3.28) is

$$\frac{1}{2}\big(a_-(\xi) + a_+(\xi)\big) + \frac{|\xi|}{2|\xi|}\big(a_+(\xi) - a_-(\xi)\big) = 0,$$

that is $a_+(\xi) = 0$. In other words, the exponentially growing part is identically zero. In P.D.E. the existence of exponentially growing solutions is intimately tied to non-well posedness. If b_0 and b_1 are convolution operators, then the boundary condition becomes:

$$\big(\widehat{b_0}(\xi) - |\xi|\widehat{b_1}(\xi)\big)a_-(\xi) = \widehat{f}(\xi) \ .$$

Elliptically is then the condition that we can solve for the remaining coefficient, $a_-(\xi)$ in terms of the data $\widehat{f}(\xi)$.

A similar interpretation for these conditions exists in the general case: one simply replaces the operator Δ_g at a boundary point $(0,x)$ with the model problem that comes form freezing the coefficients of the principal symbol at $(0,x)$:

$$\frac{\partial^2}{\partial x_0^2}\widehat{u}(x_0,\xi) - |\xi|_g^2\widehat{u}(x_0,\xi) = 0 \ .$$

We leave this to the interested reader, see also Hörmander, vol. 3.

In the next and final section we prove that the operator $\mathcal{S}$ and $\mathcal{D}$ extend to define bounded maps between the L^2–Sobolev spaces on $b\Omega$ and those of Ω.

4. Basic elliptic estimates

In the previous sections we showed that the multiple later potentials $\mathcal{S}$, $\mathcal{D}$ define maps from $C^\infty(b\Omega)$ to $C^\infty(\overline{\Omega})$. This shows that when an elliptic boundary value problem with C^∞ boundary data is solvable, the solution is smooth up to $b\Omega$. In this section we prove the following estimates: for each $s\in\mathbb{R}$

$$\begin{aligned} &\mathcal{S} : H^s(b\Omega) \to H^{s+\frac{3}{2}}(\Omega) \\ &\mathcal{D} : H^s(b\Omega) \to H^{s+\frac{1}{2}}(\Omega) \ . \end{aligned} \tag{4.1}$$

Using these estimates and the density of $C^\infty(b\Omega)$ in $H^s(b\Omega)$ we can use our previous analysis to solve elliptic boundary value problems with boundary data of finite differentiability.

To prove (4.1) we introduce local coordinates (x_0,x) to flatten out the boundary where, as before, the lines $x =$const are arclength parametrized geodesics orthogonal to $b\Omega$. Using partitions of unity it evidently suffices to prove that

$$\begin{aligned} &\widetilde{\mathcal{S}} : H^s_{\text{loc}}(\mathbb{R}^n) \to H^{s+\frac{3}{2}}_{\text{loc}}\big(\mathbb{R}^n\times[0,\infty)\big) \quad \text{and} \\ &\widetilde{\mathcal{D}} : H^s_{\text{loc}}(\mathbb{R}^n) \to H^{s+\frac{1}{2}}_{\text{loc}}\big(\mathbb{R}^n\times[0,\infty)\big) \end{aligned} \tag{4.1$'$}$$

where $\widetilde{\mathcal{S}}$,$\widetilde{\mathcal{D}}$ are local coordinate representations of $\mathcal{S}$ and $\mathcal{D}$. We start with a lemma:

Lemma 4.1. *Let $a(x_0, x; \xi_0, \xi)$ be a homogeneous function in (ξ_0, ξ) of the form:*

$$a(x_0, x; \xi_0, \xi) = \frac{p(x_0, x; \xi_0, \xi)}{q(x_0, x; \xi_0, \xi)}$$

where p and q are homogeneous polynomials in (ξ_0, ξ), with deg $p -$ deg $q = j$. Suppose that for $\xi \neq 0$ $q(x_0, x; \xi_0, \xi) = 0$ has no real roots. Let

$$k(x_0, x; \xi) = \int a(x_0, x; \xi_0, \xi) e^{ix_0 \xi_0} d\xi_0$$

for $x_0 > 0$ and $\xi \neq 0$ be defined as an oscillatory integral. Then for each pair of multiindices, (α_0, α), (β_0, β) we have the following estimates:

$$(x^\beta x_0^{\beta_0} \partial_x^\alpha \partial_{x_0}^{\alpha_0} k) \leq c(1 + |\xi|)^{j+1+\alpha_0-\beta_0} \ . \tag{4.2}$$

Proof: Using the oscillatory integral definition means simply that for $x_0 > 0$ and sufficiently large ℓ,

$$k(x_0, x; \xi) = (\tfrac{-1}{ix_0})^\ell \int_{-\infty}^{\infty} \partial_{\xi_0}^\ell a(x_0, x; \xi_0, \xi) e^{ix_0 \cdot \xi_0} d\xi_0 \ . \tag{4.3}$$

If ℓ is large enough, then (4.3) is an absolutely convergent expression. Since a is a ratio of polynomials, we can define the contour of integration to a simple closed curve, Γ, contained in the upper half plane which encloses the zeros of $q(x_0, x; \xi)$. Thus:

$$\begin{aligned} k(x_0, x; \xi) &= (\tfrac{-1}{ix_0})^\ell \int_{\Gamma_\xi} \partial_{\xi_0}^\ell a(x_0, x; \xi_0, \xi) e^{ix_0 \cdot \xi_0} d\xi_0 \\ &= \int_{\Gamma_\xi} a(x_0, x; \xi_0, \xi) e^{ix_0 \cdot \xi_0} d\xi_0 \ . \end{aligned} \tag{4.4}$$

The last statement follows: if $f(\xi_0)$, $g(\xi_0)$ are holomorphic functions in the neighborhood of a simple closed contour, γ, then

$$\begin{aligned} \int_\gamma \partial_{\xi_0} f(\xi_0) g(\xi_0) d\xi_0 &= \int_\gamma df g \\ &= -\int_\gamma f dg \end{aligned}$$

by Stokes' theorem. The estimates now follow easily:

$$x_0^{\beta_0} x^\beta \partial_x^\alpha \partial_{x_0}^{\alpha_0} k = (-i)^{\beta_0} x^\beta \int_\Gamma \partial_{x_0}^{\alpha_0} \big(\partial_x^\alpha a(x_0, x; \xi_0, \xi) \partial_{\xi_0}^{\beta_0} e^{ix_0 \cdot \xi_0}\big) d\xi_0$$

Again we can integrate by parts in ξ_0 and apply the Leibniz formula to compute $\partial_{x_0}^{\alpha_0}$. We obtain:

$$(-i)^{\beta_0} x^\beta \sum_{j=0}^{\alpha_0} c_j \int_{\Gamma_\xi} \partial_{x_0}^j \partial_x^\alpha \partial_{\xi_0}^{\beta_0} a(i\xi_0)^{\beta_0 - j} e^{ix_0 \cdot \xi_0} d\xi_0 \ . \tag{4.5}$$

As q is homogeneous the length of Γ_ξ can be taken proportional to $\xi|$ the estimates in (4.2) follow easily from this, (4.5) and the symbolic estimates satisfied by a. To prove the estimates (4.1′) it evidently suffices to prove analogous results for the homogeneous terms in the asymptotic expansion of the symbols of Q and $\frac{\partial Q}{\partial \nu_y}$. Handling the remainder terms is an easy exercise which we leave to the reader.

We would like to use the simpler form of the Sobolev norms available for $\mathbb{R}^{n+1}$ (as opposed to $\big(\mathbb{R}^n \times [0, \infty)\big)$. To that end we extend $k(x_0, x; \xi)$ to all of $\mathbb{R}^{n+1}$ in such a way that the estimates, (4.2), continue to hold. This can be accomplished using the Seeley extension theorem, (see Melroses notes). Choosing constants $\{\lambda_p\}$ we set:

$$\widetilde{k}(x_0, x; \xi) = \sum_{p=1}^{\infty} \lambda_p k(-2^p x_0, x; \xi)$$

where we can suppose that k is compactly supported in (x_0, x). There exist constants, $\{C_s\}$ such that for any s

$$\|\widetilde{k}\|_{H^s(\mathbb{R}^{n+1})} \leq C_s \|k\|_{H^s(\mathbb{R}^{n+1})}.$$

Let

$$\widehat{k}(\zeta, \xi) = \int_{\mathbb{R}^{n+1}} e^{i(x_0 \cdot \zeta_0 + x \cdot \zeta)} \widetilde{k}(x_0, x; \xi) dx_0 dx \; .$$

We use Lemma 4.1 to obtain estimates for $\widehat{k}$:

Lemma 4.2. *For all $a, b \in M$ there are constants, $C_{a,b}$ so that:*

$$\left|\widehat{k}(\zeta, \xi)\right| \leq C(1+|\xi|)^2(1+|\zeta|)^{-a}\Big(1+\frac{|\zeta_0|}{1+|\xi|}\Big)^{-b} . \tag{4.6}$$

Proof: We use the previous lemma and take $q > a$ and $p > 1$ to conclude that

$$|\partial_{x_0}^{\alpha} D_x^{\alpha} \widetilde{k}(x_0, x; \xi) \leq C(1+|\xi|)^{j+1+\alpha_0}(1+|x|)^{-q}(1+|x_0|(1+|\xi|)^{-p} \; .$$

Therefore

$$|\xi_0^{\alpha_0} \zeta^{\alpha} \widehat{k}(\zeta, \xi)| \leq C(1+|\xi|)^{j+1+\alpha_0} \int_{-\infty}^{\infty} \big(1+|x_0|(1+|\xi|)\big)^{-p} dx_0 \qquad \leq C(1+|\xi|)^{j+\alpha_0} \; .$$

This easily implies (4.6). Let

$$u(x) = \int e^{ix \cdot \xi} \widetilde{k}(x_0, x; \xi) \widehat{v}(\xi) d\xi \; .$$

Because

$$\|u\Big|_{\mathbb{R}\times[0,\infty)}\|_{H^s(\mathbb{R}^n \times [0,\infty))} \leq C\|u\|_{H^s(\mathbb{R}^{n+1})}$$

is suffices to show that

$$\|u\|_{H^{s-j+\frac{1}{2}}(\mathbb{R}^{n+1})} \leq C\|v\|_{H^s(\mathbb{R}^n)} \; . \tag{4.7}$$

Using the weak formulation of the norm we see that it suffices to show that

$$|\langle u, \phi\rangle| \leq C\|v\|_s \|\phi\|_{j+\frac{1}{2}-s} \; ,$$

for all $\phi \in C_c^{\infty}(\mathbb{R}^{n+1})$. Note that

$$\widehat{u}(y) = \int \widehat{k}(y_0, y-\xi, \xi)\widehat{v}(\xi) d\xi$$

and so by the Plancherel theorem:

$$\langle u, \phi\rangle = \int \widehat{k}(y_0, y-\xi, \xi)\widehat{v}(\xi)\widehat{\phi}(-y_0, -y) dy_0 dy d\xi$$

and therefore:

$$|\langle u, \phi\rangle| \leq C\|\phi\|_{-s+j+\frac{1}{2}} \|V\|_0 \quad \text{where}$$

$$V(y_0, y) = \int |\widehat{k}(y_0, y-\xi; \xi)|(1+|y_0|+|y|)^{s-j-\frac{1}{2}} |\widehat{v}(\xi)| d\xi \; .$$

Thus we need to prove that

$$\|V\|_0 \leq C\|v\|_s \; .$$

We replace $\widehat{k}$ by its estimate, (4.6), and define

$$W(y_0, y) = \int (1+|\xi|)^j (1+|y_0|+|y|)^{s-j-\frac{1}{2}} (1+|y-\xi|)^{-r} (1+\tfrac{|y_0|}{1+|\xi|})^{-q} |\widehat{v}(\xi)| d\xi \; .$$

Evidently $|v(y_0, y)| \leq |w(y_0, y)|$ and so it suffices to show that

$$\|W\|_0 \leq C\|V\|_s \; .$$

We let $\frac{y_0}{1+|y|}=\widetilde{y}_0$ and obtain that

$$\|W\|_0^2=\int\left[(1+|y|^2)^{\frac{1}{2}}W(y,(1+|y|)y_0)\right]^2dydy_0\ .$$

We use Peetre's inequality:

$$(1+|\xi|^2)_s\leq 2^{|s|}(1+|y|^2)^s(1+|\xi-y|^2)^{|s|}$$

to obtain that:

$$\begin{aligned}(1+|y|)^{\frac{1}{2}}(1+|\xi|)^{j-s}(1+(1+|y|)y_0+|y|)^{s-j-\frac{1}{2}}(1+|y-\xi|)^{-r}\leq\\ C(1+|y_0|)^{s-j+\frac{1}{2}}(1+|y-\xi|)^{-r+|j-s|}(1+\frac{|y_0|}{1+|y-\xi|})^{-q}\end{aligned}$$

so that

$$\|W\|_0^2\leq C\int(1+|y_0|^2)^{s-j-\frac{1}{2}}\|\gamma(\cdot,y_0)\|_0^2dy_0\quad\text{where}$$

$$\gamma(y,y_0)=\int(1+|\xi-y|)^{q-r+|j-s|}(1+|\xi-y|+|y_0|)^{-q}(1+|\xi|)^s|\widehat{v}(\xi)|d\xi\ .$$

In fact by the Cauchy Schwartz inequality:

$$\|\gamma(\cdot,y_0)\|_0^2\leq C\|V\|_g^2S^2(y_0)$$

where

$$S^2(y_0)=\int(1+|y|)^{q-r+|j-s|}(1+|y_0|+|y|)^{-q}dy\ .$$

We choose $q>n-1$, $q>s-j+n-\frac{1}{2}$ and $r>|j-s|+q$ to obtain that

$$S(y_0)\leq C(1+|y_0|)^{-q+n-1}$$

and therefore:

$$\begin{aligned}\|W\|_0^2&\leq C\|V\|_s^2\int(1+\|y_0\|^2)^{s-j-\frac{1}{2}-q+n-1}dy_0\\ &\leq\widetilde{C}\|V\|_s^2\ .\end{aligned}$$

This completes the proof.

The proof given here comes essentially verbatim from the book of Chazarain and Piriou. We apply the estimate (4.7) with $j=-2$ for $\widetilde{\mathcal{S}}$ and $j=-1$ for $\widetilde{\mathcal{D}}$ to obtain the estimates (4.1′). We apply these results to prove estimates for the solution of the boundary value problems considered in the previous section.

If $u=\mathcal{D}f-\mathcal{S}g$ for $f,g\in C^\infty(b\Omega)$ then (4.1) implies that

$$\|u\|_{H^s(\Omega)}\leq C(\|f\|_{H^{s-\frac{1}{2}}(b\Omega)}+\|g\|_{H^{s-\frac{3}{2}}(b\Omega)})\ . \tag{4.8}$$

If $\{f_n,g_n\}\subset C^\infty(b\Omega)$ are sequences which converge in $H^{s-\frac{1}{2}}(b\Omega)$ and $H^{s-\frac{3}{2}}(b\Omega)$ to (f,g) respectively, then $u_n=\mathcal{D}f_n-\mathcal{S}g_n$ converges in $H^s(\Omega)$ to $u=\mathcal{D}f-\mathcal{S}g$.

If $s>\frac{1}{2}$ then u has a well defined restriction to $b\Omega$ and if $s>\frac{3}{2}$ then $\frac{\partial u}{\partial\nu}$ is also well defined.

From (2.26) we see that the solution of the Dirichlet problem: ($\Delta u=0$ in Ω, $u\Big|_{b\Omega}=f$) is given by

$$u=\mathcal{D}f+\mathcal{S}(s_0^+)^{-1}(I-D_0^+)f\ . \tag{4.9}$$

If $f\in H^s(b\Omega)$, for $s>0$ then $u\in H^{s+\frac{1}{2}}(\Omega)$ satisfies

$$\begin{aligned}\Delta u&=0\qquad\text{in }\Omega\\ u\Big|_{b\Omega}&=f\ .\end{aligned}$$

The boundary condition has to be interpreted in the sense of traces, i.e. in terms of the bounded linear map

$$H^s(\Omega) \to H^{s-\frac{1}{2}}(b\Omega) \quad s > \frac{1}{2}\,.$$

Note that (4.8) and (4.9) imply that

$$\|u\|_{H^s(\Omega)} \leq C\|f\|_{H^{s-\frac{1}{2}}(b\Omega)}\,.$$

For the more general elliptic boundary value problems, (P_b) where we need to solve

$$(P_b) \quad \begin{cases} \Delta_g u = 0 & \text{in } \Omega \\ b_0(u\Big|_{b\Omega}) + b_1\frac{\partial u}{\partial \nu} = f \end{cases}$$

where $b_0 \in \Psi^k(b\Omega), b_1 \in \Psi^{k-1}(b\Omega)$. If for $f \in H^s(b\Omega)$ there is a solution u then $u \in H^{s+\frac{1}{2}+k}(\Omega)$. In general this type of boundary value problem makes sense for $s + k > 0$.

Theorem 4.1. *If (P_b) is an elliptic boundary value problem then there is a finite number of functions $\{\psi_1, \ldots, \psi_m\} \subseteq C^\infty(b\Omega)$ such that for $s > -k$ the problem (P_b) has a solution u (in the weak sense) for all $f \in H^s(b\Omega)$ which satisfy*

$$\langle \psi_i, f\rangle = 0 \qquad i = 1, \ldots, m\,.$$

The solution u satisfies the estimates:

$$\|u\|_{H^{s+\frac{1}{2}+k}(\Omega)} \leq C\|f\|_{H^s(b\Omega)}\,. \tag{4.10}$$

We have proven this theorem for a compact domain with smooth boundary in a manifold. The operator, Δ_g is the Laplacian defined by a smooth metric. This is just an example of the sort of results which can be proven by this method for systems of elliptic differential and pseudodifferential equations on domains in manifolds. Beyond this there is a theory of subelliptic boundary value problems. In this theory the $\frac{1}{2}$ appearing on the left hand side of (4.10) is replaced by a number in the interval $[-\frac{1}{2}, \frac{1}{2}]$. These results can also be proven using pseudodifferential methods, see M. Taylor, *Pseudodifferential Operators.*

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