Chapter 2

Elementary properties of holomorphic functions in several variables

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2.1 Holomorphy for functions of several variables

In this chapter we introduce holomorphic functions of several variables and deduce their simpler properties. Much is routine generalization from the one–variable case via the Cauchy integral formula. Though even the elementary theory of the $\overline{\partial}$–equation is more involved. The extension theorems in several variables are quite different from the single variable case; there is a straightforward analogue of Riemann's removable singularities but Hartogs' theorem is truly a multi–variable result.

We consider the theory of power series in many variables, their convergence properties are quite different from the single variable case. This follows in part from the Hartogs' result mentioned above. To begin to understand these new phenomena we consider various notions of 'convexity'.

We begin with the real vector space $\mathbb{R}^m$. Let J denote a map from $\mathbb{R}^m$ into $\mathrm{Gl}_m(\mathbb{R})$. We think of J as an automorphism of the vector bundle $T\mathbb{R}^m$. If J satisfies the identity

$$J^2 = -\,\mathrm{Id} \tag{2.1.1}$$

at every point then it is defines an almost complex structure. Note that (2.1.1) implies that the minimal polynomial for J is t^2+1. Since this has simple roots it follows that J is diagonalizable

Using J we can split $\mathbb{R}^m \otimes \mathbb{C}$ into the two eigenspaces of J corresponding to eigenvalues $i, -i$. We denote these by $T_J^{1,0}\mathbb{R}^m$ and $T_J^{0,1}\mathbb{R}^m$. The complex vector space $\mathbb{R}^m \otimes \mathbb{C}$ has a natural conjugation defined by

$$\overline{v \otimes \alpha} = v \otimes \bar{\alpha}.$$

Since J is a real transformation (i. e. it commutes with the conjugation defined above) it follows that

$$T_J^{0,1}\mathbb{R}^m = \overline{T_J^{1,0}\mathbb{R}^m}. \tag{2.1.2}$$

From (2.1.2) it follows that m is even, we'll denote it by $m = 2n$. An almost complex structure defines a complex structure provided the following 'integrability' condition is satisfied

$$\text{If } X, Y \text{ are sections of } T_J^{1,0}, \text{ then so is } [X, Y]. \tag{2.1.3}$$

If $U \subset \mathbb{R}^m$ then we say that a function $f \in \mathcal{C}^1(U)$ is J-holomorphic if for every smooth section X of $T_J^{1,0}\mathbb{R}^m$ we have that

$$\bar{X}f = 0.$$

It is by no means obvious that this system of equations will have any solutions.

If J is real analytic and satisfies (2.1.3) then the Frobenius theorem can be applied to find local coordinates $x_i, y_i, i = 1, \ldots, n$ such that

$$T_J^{1,0}\mathbb{R}^{2n} = \mathrm{sp}\{\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})\}. \tag{2.1.4}$$

In this case we can show that the functions $z_j = x_j + iy_j$ are holomorphic functions. Even if J is only finitely differentiable then a deep theorem of Newlander and Nirenberg states that (2.1.3) is necessary and sufficient for the existence of local coordinates which satisfy (2.1.4).

Exercises 2.1.5**.** A complex structure on an open set $U \subset \mathbb{R}^2$ is determined by choosing a smooth, complex vector field $\bar{Z}$.

(1) Show that any vector field such that $\overline{\bar{Z}_p} \neq \lambda \bar{Z}_p$ for any point $p \in U$ defines an integrable complex structure on U.

(2) Show that if $\bar{Z}_1$ and $\bar{Z}_2$ are two such vector fields and there is a function $f \in \mathcal{C}^\infty(U)$ so that $\bar{Z}_1 = f\bar{Z}_2$ then a function is holomorphic with respect to $\bar{Z}_1$ if and only if it is holomorphic with respect to $\bar{Z}_2$. For this reason we say that $\bar{Z}_1$ and $\bar{Z}_2$ define equivalent complex structures.

(3) Show that every almost structure on an open set in U is equivalent to one of the form $\partial_{\bar{z}} + \mu(z,\bar{z})\partial_z$ or $\partial_z + \mu(z,\bar{z})\partial_{\bar{z}}$, where μ is a smooth function that satisfies

$$|\mu(z,\bar{z})| < 1 \text{ for all } z \in U.$$

(4) Let U and V be open subsets and $\psi : U \to V$ a diffeomorphism, if $\bar{Z}$ defines an almost complex structure on U then $\bar{W} = \psi_*(\bar{Z})$ defines an almost complex structure on V. Show that a function $g \in \mathcal{C}^\infty(V)$ is holomorphic with respect to $\bar{W}$ if and only if $\psi^*(g) \in \mathcal{C}^\infty(U)$ is holomorphic with respect to $\bar{Z}$.

(5) Explain what the Newlander-Nirenberg tells us for the case of an almost complex structure defined in an open subset of $\mathbb{R}^2$. Hint: it allows you to find a representation for all solutions of $\bar{Z}u = 0$ in a neighborhood of a point $p \in U$

Exercises 2.1.6**.**

(1) Show that if J is a linear transformation of $\mathbb{R}^{2n}$ satisfying (2.1) then the almost complex structure it defines is integrable.

(2) Prove that the space

$$\mathcal{J} = Gl(2n,\mathbb{R})/Gl(n,\mathbb{C})$$

parametrizes the almost complex structures defined by linear transformations. Note that if $A = X + iY \in Gl(n,\mathbb{C})$ then

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \in Gl(2n,\mathbb{R}).$$

(3) Show that the following condition on $J : U \to \mathrm{Gl}(2n,\mathbb{R})$ is equivalent to (2.1.3): for every pair of vector fields, X, Y the quadratic form

$$N(X,Y) = [JX,Y] + [X,JY] - J[X,Y] + J[JX,JY]$$

vanishes.

(4) Show that for $f, g \in \mathcal{C}^\infty$, $N(fX, gY) = fgN(X,Y)$. This shows that N is a tensor; it is called the Nijenhaus tensor.

For the time being we will only consider almost complex structures defined by linear transformations. The 'canonical' almost complex structure is defined by the matrix

$$J_0 = \begin{pmatrix} 0 & -\operatorname{Id} \\ \operatorname{Id} & 0 \end{pmatrix}.$$

We use $\mathbb{C}^n$ to denote $\mathbb{R}^{2n}$ with its canonical complex structure. It is a consequence of exercise (2.1.5)b that every linear complex structure is linearly equivalent to this one. The eigenspaces are given by

$$\begin{aligned} T^{1,0}\mathbb{C}^n &= \operatorname{sp}\{\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i}), i = 1, \dots, n\} \\ T^{0,1}\mathbb{C}^n &= \operatorname{sp}\{\partial_{\bar{z}_i} = \frac{1}{2}(\partial_{x_i} + i\partial_{y_i}), i = 1, \dots, n\} \end{aligned}$$

The dual spaces are denoted by

$$\begin{aligned} \Lambda^{1,0}\mathbb{C}^n &= \operatorname{sp}\{dz_i, i = 1, \dots, n\} \\ \Lambda^{0,1}\mathbb{C}^n &= \operatorname{sp}\{d\bar{z}_i, i = 1, \dots, n\}. \end{aligned} \tag{2.1.7}$$

As in the case of one variable we can express any derivative of a function in terms of these vector fields. Thus

$$df = \partial f + \overline{\partial} f, \text{ where } \partial f = \sum_{i=1}^{n} \partial_{z_i} f dz_i, \quad \overline{\partial} f = \sum_{i=1}^{n} \partial_{\bar{z}_i} f d\bar{z}_i. \tag{2.1.8}$$

As before we call the $1,0$–part of df ∂f and the $0,1$–part $\overline{\partial} f$.

Definition 2.1.9. If $\Omega \subset \mathbb{C}^n$ is open and $u \in C^1(\Omega)$ then u is holomorphic provided that

$$\overline{\partial} u = 0. \tag{2.1.10}$$

We denote the set of such functions by $H(\Omega)$.

Note that (2.1.10) is a system of $2n$ equations for 2 unknown functions. If $n > 1$ it is overdetermined, so in some sense it is surprising that it has solutions at all. On the other hand note that the $\overline{\partial}$-operator satisfies the Leibniz formula

$$\overline{\partial}(fg) = f\overline{\partial} g + g\overline{\partial} f$$

This implies that the set of solutions to $\overline{\partial} f = 0$ is a ring under ordinary pointwise multiplication of functions. This means that if there is a single non-constant function f such that $\overline{\partial} f = 0$ then there is an infinite dimensional vector space of solutions spanned by $\{f^k : k \in \mathbb{N}\}$.

It is a very important fact in one dimension that the composition of two holomorphic functions is holomorphic. To generalize this to several variables we need to define a 'holomorphic mapping'

Definition 2.1.12. A mapping from $\Omega \subset \mathbb{C}^n$ to $\mathbb{C}^m$ is holomorphic provided the coordinate functions are. That is if $U = (u_1, \dots, u_m)$ then $\overline{\partial} u_i = 0, i = 1, \dots, m$. We denote such mappings by $H(\Omega; \mathbb{C}^m)$. If a holomorphic mapping between two open subsets is invertible with holomorphic inverse then the mapping is said to be *biholomorphic.*

Proposition 2.1.13. *Suppose that* $f \in H(\Omega'), \Omega' \subset \mathbb{C}^m$ *and* $U \in H(\Omega; \mathbb{C}^m)$ *has range contained in* Ω' *then*

$$U^* f(z) = f(U(z)) \in H(\Omega). \tag{2.1.14}$$

Proof. We compute $dU^* f$:

$$dU^* f = \sum_{i=1}^{n} \sum_{j=1}^{m} \partial_{u_j} f \partial_{z_i} u_j dz_i.$$

All other terms are absent as df and $du_j, j = 1, \dots, m$ are of type $1,0$. This proves the claim.

The implicit and inverse function theorems also extend easily to holomorphic mappings

Theorem 2.1.15. *Let $f_j(w,z)$ be analytic functions in a neighborhood of the point $(w_0,z_0) \in \mathbb{C}^m \times \mathbb{C}^n$ and assume that $f_j(w_0,z_0)=0$, $j=1,\dots,m$. Finally suppose that*

$$\det \partial_{w_j} f_k(w_0,z_0) \neq 0, \tag{2.1.16}$$

then there is a neighborhood V of z_0 and a holomorphic mapping $w(z) \in H(V,\mathbb{C}^m)$ with

$$w(z_0)=w_0 \text{ and } f_j(w(z),z)=0. \tag{2.1.17}$$

Proof. Identifying $Gl(n,\mathbb{C})$ with a subgroup of $Gl(2n,\mathbb{R})$ as in (2.1.5), reinterpreting (2.1.16) allows us to apply the standard implicit function theorem to the system of $2m$ real equations $\operatorname{Re} f_j = 0, \operatorname{Im} f_j = 0$ with respect to the variables $\operatorname{Re} w_j, \operatorname{Im} w_j, \operatorname{Re} z_i, \operatorname{Im} z_i$. This uniquely determines differentiable functions $\operatorname{Re} w_j(\operatorname{Re} z_i, \operatorname{Im} z_i), \operatorname{Im} w_j(\operatorname{Re} z_i, \operatorname{Im} z_i)$ which satisfy (2.1.17).

At this point we can reexpress everything in terms of $z_i, \bar{z}_i$ and differentiate the equations in (2.1.17) to obtain

$$\sum_{k=1}^{m} \partial_{w_k} f_j dw_j + \sum_{i=1}^{n} \partial_{z_i} f_j dz_i = 0, j=1,\dots m. \tag{2.1.18}$$

In light of (2.1.16) we can use (2.1.18) to solve for the dw_k in some neighborhood of (w_0,z_0). It is clear that these one forms are of type $1,0$ and therefore $w_k(z)$ are holomorphic functions.

The exterior algebra $\Lambda^*\mathbb{R}^{2n} \otimes \mathbb{C}$ can be split into *p,q–types* using the complex co-vectors $dz_i, d\bar{z}_i$ as a basis. Let

$$I = \{1 \leq i_1 < \dots < i_p \leq n\}; J = \{1 \leq j_1 < \dots < j_q \leq n\},$$

be multi-indices, then we define

$$dz^I = dz_{i_1} \wedge \dots \wedge dz_{i_p}, d\bar{z}^J = d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}.$$

As usual we define $|I|=p, |J|=q$.

Definition 2.1.19. A differential form, ω is said to be of type p,q if it has a local representation as

$$\omega = \sum_{I,J;|I|=p,|J|=q} f_{IJ} dz^I \wedge d\bar{z}^J. \tag{2.1.20}$$

The set of such forms defined on $U \subset \mathbb{C}^n$ is denoted by $\Lambda^{p,q}(U)$.

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Exercise 2.1.21. Prove that the notion of p,q–type is invariant under biholomorphic mappings.

The operators $\partial, \overline{\partial}$ extend to define differential operators on $\Lambda^{p,q}(U)$. In fact

$$\begin{aligned} \partial &: \Lambda^{p,q}(U) \longrightarrow \Lambda^{p+1,q}(U) \text{ is defined by } \partial\omega = \sum_{I,J} \partial f_{IJ} \wedge dz^I \wedge d\bar{z}^J, \\ \overline{\partial} &: \Lambda^{p,q}(U) \longrightarrow \Lambda^{p,q+1}(U) \text{ is defined by } \overline{\partial}\omega = \sum_{I,J} \overline{\partial} f_{IJ} \wedge dz^I \wedge d\bar{z}^J. \end{aligned} \tag{2.1.22}$$

It is not dificult to show that that $\partial^2 = \overline{\partial}^2 = 0$; since $d^2=0$ as well, this implies that

$$\partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Exercise 2.1.23.

(1) Show that $\overline{\partial}^2 : \Lambda^{p,q}(U) \to \Lambda^{p,q+2}$ is zero for all (p,q).
(2) Show that one can define the $1,0$ and $0,1$–parts of *df* with respect to any almost complex structure,
(3) Use the previous part to define ∂_J and $\overline{\partial}_J$,
(4) Show that the almost complex structure defined by J is integrable if and only if $\overline{\partial}_J^2 = 0$.

The main topic of this course is the inhomogeneous Cauchy-Riemann equation

$$\overline{\partial} u = \alpha,\ \alpha \in \Lambda^{0,1}(U).$$

This is a system of n-equations for 1 unknown function. Because $\overline{\partial}^2 = 0$ a necessary condition for this equation to have a solution is that

$$\overline{\partial}\alpha = 0.$$

This condition is vacuous in one complex dimension. We see that consideration of $0,1$–forms leads inevitably to $0,2$–forms and so to consideration of $0,3$–forms, etc. Though the case of $0,1$–forms is most important for applications, it presents no essential difficulty to consider p,q–forms from the start.

2.2 The Cauchy formula for polydiscs and its elementary consequences

The unit disk in $\mathbb{C}$ is a model domain for the study of the local properties of holomorphic functions. In several variables there are two, quite different analogues of the unit disk, the ball and the polydisk. The ball of center w and radiius R is

$$\mathbb{B}(w;R) = \{z \in \mathbb{C}^n;\ |z - w| < R\}.$$

A ball has a smooth boundary. A polydisk is defined by an n–tuple of positive numbers $r_1, \dots, r_n$ and by a point $w \in \mathbb{C}^n$

$$D(w;r) = D(w;r_1,\dots,r_n) = \{(z_1,\dots,z_n); |z_i - w_i| < r_i, i = 1,\dots,n\}.$$

As we shall soon see the local analysis of holomorphic functions on polydisks is very similar to the one variable case. Note however that a polydisk does not have a smooth boundary. It's boundary has a distinguished subset defined by

$$\partial_0 D(w;r) = \{(z_1,\dots,z_n) : |z_i - w_i| = r_i\}.$$

This is the lowest dimensional "stratum"of the boundary but, as we shall see, it is also the most important. Later on we will show that the complex analytic geometry of the unit ball has more in common with the unit disk, though the analysis on the ball is quite a bit more involved.

For functions that are holomorphic in a polydisc we have a direct generalization of the Cauchy integral formula.

Cauchy Integral Formula 2.2.1. *Let $u(z)$ be continuous is $\overline{D}(w;r)$ and holomorphic separately in each variable then*

$$u(z) = \left(\frac{1}{2\pi\imath}\right)^n \int_{\partial_0 D}\cdots\int \frac{u(w)dw_1\dots dw_n}{(w_1 - z_1)\dots(w_n - z_n)}. \tag{2.2.2}$$

Proof. We can prove this inductively. This is simply (1.3.16) if $n = 1$. Suppose that it is proved for $n-1$. The polydisk $D(w;r)$ can be written as a product

$$D(w;r) = D(w_1;r_1) \times D'.$$

Here D' is a polydisk in $\mathbb{C}^{n-1}$. The inductive hypothesis implies that for each fixed $z \in D(w_1;r_1)$ we have the representation

$$u(z, z_2, \dots, z_n) = \left(\frac{1}{2\pi\imath}\right)^{n-1} \int_{\partial_0 D'}\cdots\int \frac{u(z,\zeta')d\zeta_2\dots d\zeta_n}{(\zeta_2 - z_2)\dots(\zeta_n - z_n)}. \tag{2.2.3}$$

On the other hand the continuity hypothesis and (1.3.16) imply that for each $\zeta' \in \partial_0 D'$

$$u(z,\zeta') = \frac{1}{2\pi\imath}\int_{|\zeta_1 - w_1| = r_1} \frac{u(\zeta,\zeta')d\zeta}{\zeta - z}. \tag{2.2.4}$$

We can put the integral in (2.2.4) into (2.2.3), for $(z,z') \in D(w;r)$ the combined integrand is continuous and therefore we can apply the Fubini theorem to identify the iterated integral with the (2.2.2).

Note that in the hypothesis of (2.2.1) we did not assume that u was C^1 in $D(w;r)$ but rather continuous and separately holomorphic. The Cauchy formula has the following corollary

Corollary 2.2.5. *If u satisfies the hypotheses of* (2.2.1) *then $u \in \mathcal{C}^\infty(D(w;r))$ and therefore $u \in H(D(w;r))$.*

The assertions follow by differentiating the integral representation. The maximum modulus principle follows from

Corollary 2.2.6. *If $u \in C^0(D(w;r)) \cap H(D(w;r))$ then*

$$|u(w)| \leq \max_{z \in \partial_0 D(w;r)} |u(z)| \tag{2.2.7}$$

with equality only if $u \restriction_{\partial_0 D(w;r)}$ is constant.

Proof. Exactly the same argument as in the one variable case.

In the case of equality we can use the Cauchy integral representation to deduce that u is actually constant in $\overline{D}(w;r)$.

The Maximum Modulus Principle 2.2.8. *If $\Omega \subset \mathbb{C}^n$ is an open set and $u \in C^0(\overline{\Omega}) \cap H(\Omega)$ then $|u|$ does not assume its maximum at an interior point unless u is constant.*

Proof. This follows from (2.2.6) and the fact that $\forall\, w \in \Omega$ we can find a positive n–tuple r such that $D(w;r) \subset\subset \Omega$.

The several variables result is actually a bit stronger than the one variable case. If two holomorphic functions on a disk in $\mathbb{C}$ agree on the boundary then they agree in the whole disk. In many variables we see that if two holomorphic functions in a polydisk agree on the distinguished boundary then they agree in the polydisk as well. The boundary of the polydisk is a manifold with corners of dimension $2n-1$ whereas the distinguished boundary has dimension n.

We also have a generalization of (1.3.7) to the case at hand. Estimates for the derivatives of a holomorphic function follow from this representation. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n–tuple of non–negative integers. Then we define the differential operator

$$\partial^\alpha = \partial_{z_1}^{\alpha_1} \ldots \partial_{z_n}^{\alpha_n}.$$

We also define

$$|\alpha| = \alpha_1 + \cdots + \alpha_n; \quad \alpha! = \alpha_1! \ldots \alpha_n!$$

The operator $\overline{\partial}^\alpha$ is defined analogously. Note that when ∂ or $\overline{\partial}$ has a multi-index superscript it means something different from the operator defined in (2.1.22).

Holomorphic functions of several variables have compactness properties with respect to locally uniform convergence identical to the one variable case.

Theorem 2.2.10. *If $u_n \in H(\Omega)$ converges locally uniformly to a function u then the limit is holomorphic. If $u_n \in H(\Omega)$ is locally uniformly bounded then u_n has a locally uniformly convergent subsequence.*

Proof. Since $H(\Omega) \subset \mathcal{C}^0(\Omega)$ it follows that the locally uniform limit, u of u_n is continuous. Applying the one variable result to each coordinate separately we conclude that the limit function is separately holomorphic is all variables as well. Thus we can apply Corollary 2.2.5 to conclude that the limit is actually holomorphic.

Using the Cauchy integral formula we can deduce the existence of convergent power series expansions for holomorphic functions. The notion of convergence that is appropriate in this context is that of *normal convergence.* If $\{a_\alpha(z) : \quad z \in \Omega\}$ is a collection of functions defined on a open set Ω then we say that

$$\sum_\alpha a_\alpha(z)$$

converges normally if

$$\sum_\alpha \sup_K |a_\alpha|$$

converges for every $K \subset\subset \Omega$. Evidently we can rearrange a normally convergent series and obtain the same limit. If the functions $a_\alpha(z)$ are holomorphic then clearly the limit is holomorphic as well.

Theorem 2.2.11**.** *If u is holomorphic in a polydisk, $D(0;r)$ then we have*

$$u(z) = \sum_{\alpha} \frac{\partial^\alpha u(0) z^\alpha}{\alpha!}. \tag{2.2.12}$$

With normal convergence for $z \in D(0;r)$.

Proof. We observe that

$$[(\zeta_1 - z_1) \dots (\zeta_n - z_n)]^{-1} = \sum_{\alpha} \frac{z^\alpha}{\zeta^\alpha \zeta_1 \dots \zeta_n}. \tag{2.2.13}$$

The series in (2.2.13) converges normally for $(z, \zeta) \in D \times \partial_0 D$. If $u \in C^0(D(0;r))$ then we can interchange the order of integration and summation in the Cauchy integral formula to obtain

$$u(z) = \sum_{\alpha} z^\alpha \left(\frac{1}{2\pi\imath}\right)^n \int\limits_{\partial_0 D} \frac{u(\zeta) d\zeta_1 \dots d\zeta_n}{\zeta^\alpha \zeta_1 \dots \zeta_n}. \tag{2.2.14}$$

This is again normally convergent in $D(0;r)$. A simple calculation using Cauchy's formula shows that

$$\partial^\alpha u(0) = \frac{\alpha!}{(2\pi i)^n} \int\limits_{\partial_0 D} \frac{u(\zeta) d\zeta}{\zeta^\alpha \zeta_1 \dots \zeta_n}. \tag{2.2.15}$$

Putting (2.2.15) into (2.2.14) implies (2.2.12) in this special case. Otherwise we simply apply this argument to polydisks $D(0;r')$ with $r'_i < r_i, i = 1, \dots, n$.

Once again (2.2.10) and (2.2.11) imply that a function is holomorphic if and only if it has a normally convergent power series expansion about every point. As in the one variable case we obtain estimates on the derivatives of u from the Cauchy formula.

Cauchy Estimates 2.2.16**.** *If $u \in H(D(0,r))$ and $|u| \leq M$ then*

$$|\partial^\alpha u(0)| \leq M \alpha! r^{-\alpha}.$$

Proof. These estimates follow by applying (2.2.15) to smaller polydisks.

In addition we also have an analogue of Schwarz's lemma for functions of several variables

Schwarz's Lemma 2.2.17**.** *If u is holomorphic in a neighborhood of a closed ball $B(0;R)$, $|u(z)| \leq M$ in the ball and*

$$\partial^\alpha u(0) = 0 \text{ if } |\alpha| < k$$

for some positive integer k then

$$|u(z)| \leq M R^{-k} |z|^k. \tag{2.2.18}$$

Proof. Let $Z \in \mathbb{C}^n$ with $|Z| = 1$. Then the function $f(t) = u(tZ)$ is holomorphic for $t \in B(0;R) \subset \mathbb{C}$ and satisfies $|f(t)| \leq M$ and $f^{[j]}(0) = 0, j = 0, \dots, k-1$. The classical Schwarz lemma implies that

$$|f(t)| \leq M R^{-k} |t|^k. \tag{2.2.19}$$

Since Z is an arbitrary point on the unit sphere (2.2.19) implies (2.2.18).

Exercise 2.2.20**.** What can you conclude if equality holds at some point in (2.2.18).

Though the generalization of Runge's theorem to several variables is in general quite complicated, there is one case which is essentially trivial. If $D_1 \subset\subset D_2 \subset\subset D$ are polydisks and $f \in H(D_2)$ then given $\epsilon > 0$ we can find $F \in H(D)$ which satisfies

$$\sup_{z \in D_1} |F(z) - f(z)| < \epsilon. \tag{2.2.31}$$

One simply expands f in a Taylor series about the center of D_2, a sufficiently large partial sum satisfies (2.2.31). For latter reference we formulate this as a proposition

Proposition 2.2.32**.** *If D is a polydisk then the uniform closure on D of functions holomorphic in a neighborhood of D equals the uniform closure on D of the holomorphic polynomials.*

Holomorphic functions in several variables have a remarkable property: if a function is separately holomorphic in each variable then it is holomorphic in the sense of definition (2.2.1). We proved a weak version of this above. Note that this sort of a property fails if we replace separately holomorphic with separately real analytic. For example

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

is real analytic when restricted to any vertical or horizontal line, however this function fails to be continuous at $(0,0)$.

Appendix A. The Bochner-Martinelli formula

There is an integral formula which allows the expression of a holomorphic function defined on a smooth domain in terms of an integral over the full boundary of the domain. This called the Bochner-Martinelli formula. In some ways it gives a higher dimensional analogue of the Cauchy-Pompieu integral formula, though in important ways it falls short. Our aim is to represent a $\mathcal{C}^1$-function f defined on a connected open set, with smooth boundary in terms of its boundary values, $f \restriction_{\partial\Omega}$ and $\overline{\partial} f$ in the interior of Ω. We define an $(n,0)$-form

$$\omega(z) = dz_1 \wedge \cdots \wedge dz_n$$

and an $(n-1,0)$-form

$$\eta(z) = \sum_{j=1}^{n} (-1)^{j+1} z_j \widehat{dz_j},$$

where

$$\widehat{dz_j} = dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n.$$

Observe that

$$d\eta(z) = n\omega(z).$$

The form η is sometimes called the *Leray* form. Notice that the volume form on $\mathbb{C}^n$ is given by

$$\mathrm{dVol} = c_n \omega(z) \wedge \overline{\omega(z)},$$

the unit radial vector is represented in complex notation by

$$R = |z|^{-1} \sum_{j=1}^{n} [z_j \partial_{z_j} + \bar{z}_j \partial_{\bar{z}_j}].$$

If $S \hookrightarrow \mathbb{C}^n$ is a smooth hypersurface with unit normal vector field ν then the volume form induced on S is given by

$$\mathrm{dV}_S = i_\nu \, \mathrm{dVol} \restriction_S .$$

It is not difficult to show that

$$i_R\omega(z) = -|z|^{-1}\eta(z), \quad i_R\overline{\omega(z)} = |z|^{-1}\overline{\eta(z)},$$

from which it follows easily that

$$\mathrm{dV}_{\partial B_r(0)} = c_n[\eta(z) \wedge \overline{\omega(z)} + (-1)^n \omega(z) \wedge \overline{\eta(z)}].$$

Note that $\overline{\eta(z)} \wedge \omega(z)$ is an $(n, n-1)$ form and therefore

$$d\overline{\eta(z)} \wedge \omega(z) = \overline{\partial}\overline{\eta(z)} \wedge \omega(z) n\overline{\omega(z)} \wedge \omega(z).$$

This formula implies that for any $z_0 \in \mathbb{C}^n$ and $r > 0$ we have

$$\int_{\partial B_r(z_0)} \overline{\eta(z)} \wedge \omega(z) = n \int_{B_r(z_0)} \overline{\omega(z)} \wedge \omega(z).$$

As $\overline{\omega(z)} \wedge \omega(z)$ is a constant multiple of the volume form on $\mathbb{C}^n$ we see that

(2.A.1)
$$\int_{\partial B_r(z_0)} \overline{\eta(z)} \wedge \omega(z) = nC_n r^{2n}.$$

With this motivation we can state the Bochner-Martinelli formula

Theorem [Bochner-Martinelli]**.** *Suppose that $\Omega \subset \mathbb{C}^n$ is a bounded domain with a smooth boundary and $f \in \mathcal{C}^1(\overline{\Omega})$ then for all $z \in \Omega$ we have the formula*

$$f(z) = \frac{1}{nC_n} \int\limits_{\partial\Omega} \frac{f(\zeta)\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}} - \frac{1}{nC_n} \int\limits_{\Omega} \frac{\overline{\partial} f}{|\zeta - z|^{2n}} \wedge \overline{\eta(\zeta - z)} \wedge \omega(z).$$

Here

$$\eta(\zeta - z) = \sum_{j=1}^{n} (-1)^{j+1} (\zeta_j - z_j) \widehat{d\zeta_j}.$$

Proof. The result is a simple consequence of Stokes' theorem. Fix $z \in \Omega$, for sufficiently small $\epsilon > 0$ we let

$$\Omega_{z,\epsilon} = \Omega \setminus B_\epsilon(z).$$

On $\Omega_{z,\epsilon}$ the form

$$\psi(\zeta) = \frac{f(\zeta)\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}}$$

is smooth. It is an $(n, n-1)$-form and therefore $d\psi = \overline{\partial}\psi$. The Leibniz formula tells us that

$$\overline{\partial}_\zeta \psi = \frac{\overline{\partial}_\zeta f(\zeta) \wedge \overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}} + f(\zeta)\overline{\partial}_\zeta \left[\frac{\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}} \right].$$

Exercise 2.A.2**.** Prove that in $\mathbb{C}^n \setminus \{z\}$ we have

(2.A.3)
$$\overline{\partial}_\zeta \left[\frac{\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}} \right] = 0.$$

Using (2.A.3) and Stokes' theorem we see that

$$\int_{\Omega_{z,\epsilon}} d\psi(\zeta) = \int_{\Omega_{z,\epsilon}} \frac{\overline{\partial} f}{|\zeta - z|^{2n}} \wedge \overline{\eta(\zeta - z)} \wedge \omega(z)$$

(2.A.4)

$$= \int_{\partial\Omega} \frac{f(\zeta)\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}} - \int_{\partial B_\epsilon(z)} \frac{f(\zeta)\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{\epsilon^{2n}}.$$

Because $\overline{\eta(\zeta - z)} \wedge \omega(\zeta) = O(|\zeta - z|)$ we can allow ϵ to go to zero in the integral over $\Omega_{z,\epsilon}$. To complete the proof we need to compute

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(z)} \frac{f(\zeta)\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}}.$$

Because f is $\mathcal{C}^1$ we know that $|f(z) - f(\zeta)| = O(|\zeta - z|)$ and therefore

$$\frac{(f(\zeta) - f(z))\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}} = O(|\zeta - z|^{2-2n}),$$

hence

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(z)} \frac{(f(\zeta) - f(z))\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{\epsilon^{2n}} = 0.$$

On the other hand (2.A.1) implies that for any $\epsilon > 0$

$$\int_{\partial B_\epsilon(z)} \frac{f(z)\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}} = nC_n f(z).$$

Combining these formulæ completes the proof of the Bochner-Martinelli formula.

As a corollary of the general formula we obtain a representation for a holomorphic function in terms of its boundary values

Corollary 2.A.5. *If $f \in \mathcal{C}^1(\overline{\Omega})$ and $\overline{\partial} f = 0$ in Ω then*

$$f(z) = \frac{1}{nC_n} \int_{\partial\Omega} \frac{f(\zeta)\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}}.$$

Estimates for the derivatives of a holomorphic function in terms of the L^1-norms follow from the Bochner-Martinelli formula.

Corollary 2.A.6. *Suppose that $u \in H(\Omega)$ and that K is a compact subset with $K \subset\subset \omega \subset\subset \Omega$ then for each multi-index α there is a constant C_α such that*

$$\sup_{z \in K} |\partial^\alpha u(z)| \leq C_\alpha \|u\|_{L^1(\omega)}.$$

Proof. We simply cover K by a finite collection of balls. Using the Bochner-Martinelli formula we can apply the argument used to prove (1.4.2).

Since the "kernel function" does not depend on the domain and it allows us to express a holomorphic function in terms of its boundary values, this formula is in some sense an analogue of the Cauchy integral formula. Note however that the Cauchy kernel, $(\zeta - z)^{-1}$ depends holomorphically on z and therefore if $\Omega \subset \mathbb{C}$ and $g \in \mathcal{C}^0(\partial\Omega)$ then

$$G(z) = \frac{1}{2\pi i} \int\limits_{\partial\Omega} \frac{g(\zeta)d\zeta}{\zeta - z}$$

if always holomorphic in Ω. That is, using the Cauchy kernel we can "manufacture" holomorphic functions in Ω. The Bochner-Martinelli kernel

$$\frac{\overline{\eta(\zeta - z)} \wedge \omega(\zeta)}{|\zeta - z|^{2n}}$$

does not depend holomorphically on z and therefore cannot be used to manufacture holomorphic functions on an arbitrary domain in $\mathbb{C}^n$. In fact this kernel has more in common with the Newtonian potential $|\zeta - z|^{2-2n}$ than with the Cauchy kernel. If Ω satisfies special geometric properties then it is possible to construct a kernel which is a true analogue of the Cauchy kernel, expect that it depends in a complicated way on the domain itself. This kernel is called the Henkin-Ramirez kernel. The Bochner-Martinelli kernel is the starting point for this subject.

Appendix B. Sub-harmonic functions

In order to study this problem we need to establish a few elementary facts about sub–harmonic functions.

Definition 2.2.21**.** A function $u(z)$ defined in Ω an open subset of $\mathbb{C}$ is subharmonic provided

$$\begin{aligned} &u(z) \text{ is bounded from above,}\\ &u(z) \text{ is upper semicontinuous,}\\ &\pi r^2 u(z) \leq \iint\limits_{B(z,r)} u dxdy, \text{ whenever } B(z,r) \subset\subset \Omega. \end{aligned} \tag{2.2.23}$$

The assumption that u is upper semicontinuous, implies that the integral in (2.2.23) is well defined.

Exercise 2.2.24**.**

(1) If $f(z) \in H(\Omega)$ show that $\log|f(z)|$ is subharmonic,

(2) If φ is a convex monotone increasing function defined on $\mathbb{R}$ and u is subharmonic then so is $\varphi(u)$.

(3) Suppose that u, defined in an open set $\Omega \subset \mathbb{C}$, is upper semi-continuous and bounded from above. Show that u is sub-harmonic in Ω if and only if it satisfies the following condition:
If $D \subset\subset \Omega$ is a disk and $h \in \mathcal{C}^0(\overline{D})$ is a harmonic function defined on the disk, such that $f(z) \leq h(z)$ for all $z \in \partial D$ then

$$f(z) \leq h(z), \text{ for all } z \in D.$$

Lemma 2.2.25**.** *Suppose that $v_n(z)$ is a sequence of subharmonic functions defined in Ω such that $v_n(z) \leq M$, $n = 1, 2, \ldots,$ and $\limsup_{n\to\infty} v_n(z) \leq c$. Then given a compact subset K of Ω and an $\epsilon > 0$ there exists an N so that*

$$v_n(z) < c + \epsilon \text{ provided } z \in K,\, n > N.$$

Proof. We prove this by contradiction. Suppose that conclusion is false then we can choose a subsequence n_k and a sequence $z_{n_k} \in K$ such that $v_{n_k}(z_{n_k}) > c + \epsilon$. Since K is compact there is no loss in generality in assuming that this subsequence converges to $z^* \in K$. Since $K \subset\subset \Omega$ we can choose an $r > 0$ such that $B(z_{n_k}; 2r) \subset\subset \Omega$. Using (2.2.24) we observe that $f_k(z) = e^{u_{n_k}(z)}$ is a sequence of bounded, non–negative subharmonic functions. Since these functions are non–negative and the sequence $\{z_{n_k}\}$ converges to z^* it is clear that for sufficiently large k and $\delta > 0$

$$\int\limits_{B(z_{n_k},r)} f_k(w)dxdy \leq \int\limits_{B(z^*,r+\delta)} f_k(w)dxdy. \tag{2.2.26}$$

Fixing a $\delta > 0$ a simple application of Fatou's Lemma implies that

$$(2.2.27) \qquad \limsup_{k\to\infty} \int\limits_{B(z^*;r+\delta)} f_k(w)dxdy \le \int\limits_{B(z^*;r+\delta)} \limsup_{k\to\infty} f_k(w)dxdy \le e^c\pi(r+\delta)^2.$$

On the other hand since f_k is subharmonic

$$(2.2.28) \qquad e^{c+\epsilon}\pi r^2 \le \pi r^2 f_k(z_{n_k}) \le \int\limits_{B(z_{n_k};r)} f_k(w)dxdy.$$

Letting $k \to \infty$ in (2.2.29) and applying (2.2.26)–(2.2.27) we obtain

$$(2.2.30) \qquad e^{c+\epsilon}r^2 \le e^c(r+\delta)^2.$$

Since $\delta > 0$ is arbitrary and $\epsilon > 0$ is fixed (2.2.30) leads to a contradiction.

2.3 Hartogs' Theorem on separately holomorphic functions

In the previous section we defined a holomorphic function as a function which is continuously differentiable and satisfies the Cauchy–Riemann equations in each variable separately. In the course of deriving the Cauchy integral formula we established that a function which is merely continuous and satisfies the Cauchy–Riemann equations in each variable, with the other variables regarded as constant is actually holomorphic in the previous sense. In this section we prove the Theorem of Hartogs' which states that if a function is separately holomorphic in each variable then it is holomorphic in the above sense. No additional assumption needs to be made about the regularity of the function as a function of several variables.

Hartogs' Theorem 2.3.1. *If u is a complex valued function defined in an open subset $\Omega \subset \mathbb{C}^n$ which is holomorphic in each variable z_j when the other variables are given fixed arbitrary values then $u \in H(\Omega)$.*

Proof. The theorem is proved inductively using several lemmas and the result on subharmonic functions proved in §2.2. As a preliminary step we prove

Lemma 2.3.2. *Suppose that u satisfies the hypotheses of the previous theorem and in addition u is locally uniformly bounded then $u \in H(\Omega)$.*

Remark. The subtlety is that u might fail to be measurable as a function of $2n$–real variables so we cannot simply apply the Cauchy formula and Fubini's theorem.

Proof. Since u is uniformly bounded on compact subsets and holomorphic in each variable separately, the one–variable Cauchy estimates imply that $\partial_{z_i} u$ are locally uniformly bounded for each i. Since $\partial_{\bar{z}_i} u = 0$ it follows that all first partial derivatives of u are locally uniformly bounded and thus the mean value theorem implies that u is continuous. The conclusion then follows from (2.2.1).

Now we will show that u is bounded on some open set. We make an inductive hypothesis that (2.3.1) is proved for $n-1$. It is trivial for $n = 1$.

Lemma 2.3.3. *Let u satisfy the hypotheses of* (2.3.1) *and let $D = D_1 \times \cdots \times D_n \subset\subset \Omega$ be a closed polydisk Then there exist disks $D'_i \subset D_i, i = 1, \ldots, n-1$, with non–empty interior, such that u is bounded in the polydisk $D'_1 \times \cdots \times D'_{n-1} \times D_n$.*

Proof. We prove this using the Baire category theorem and the inductive hypothesis . Let

$$E_M = \{z' : z' \in \prod_{j=1}^{n-1} D_j \text{ and } |f(z', z_n)| \le M \text{ when } z_n \in D_n\}.$$

Since $f(z', z_n)$ is holomorphic for $z_n \in D_n$ it follows that

$$\prod_{j=1}^{n-1} D_j = \bigcup_{M\in\mathbb{N}} E_M.$$

The inductive hypothesis implies that, for fixed z_n, $f(z', z_n)$ is holomorphic in the first $n-1$ coordinates. This implies that $f(z', z_n)$ is continuous as a function of these coordinates from which we conclude that E_M is a closed set for each M. Since $\prod_{j=1}^{n-1} D_j$ is a complete metric space it follows from the Baire category theorem that E_M must have nonempty interior for some M. If z_0' lies in the interior then we can find a polydisk D' centered at z_0' so that $D' \times D_n$ satisfies the conclusion of the lemma.

A final lemma is required to finish the proof.

Lemma 2.3.4. *Let u be a complex valued function in a polydisk $D = \{z;\, |z_j - z_j^0| < R, j = 1 \dots, n\}$, assume that u is analytic in $z' = (z_1, \dots, z_{n-1})$ if z_n is fixed and that u is analytic and bounded in*

$$D' = \{z;\, |z_j - z_j^0| < r, j = 1 \dots, n-1, |z_n - z_n^0| < R\}$$

for some $r > 0$. Thenu is analytic in D.

Proof. For simplicity we assume that $z^0 = 0$. Choose R_1, R_2 with $0 < R_1 < R_2 < R$. By (2.2.12) we have a power series expansion for $u(z)$

$$u(z) = \sum_\alpha a_\alpha(z_n) {z'}^\alpha, \quad z \in D, \tag{2.3.5}$$

where the sum extends over $(n-1)$–multiindices. The coefficients are given by

$$a_\alpha(z_n) = \frac{\partial_z^\alpha u(0, z_n)}{\alpha!}.$$

The coefficients are analytic in z_n since $u(z)$ is analytic in D'. Because $u(z', z_n)$ is holomorphic in z' for $|z_j - z_j^0| < R, j = 1, \dots, n-1$

$$\limsup_{|\alpha|\to\infty} |a_\alpha(z_n)| R_2^{|\alpha|} = 0, \text{ for fixed } z_n \text{ with } |z_n| < R.$$

On the other hand since $u(z)$ is holomorphic in D' the Cauchy inequalities apply to give

$$|a_\alpha(z_n)| r^{|\alpha|} \le M, \tag{2.3.6}$$

if M is a bound for $|u|$ in D'.

The functions $z_n \longrightarrow \frac{\log|a_\alpha(z_n)|}{|\alpha|}$ are subharmonic for $|z_n| < R$. From (2.3.6) we conclude that these functions are uniformly bounded when $|z_n| < R$ and the $\limsup_{|\alpha|\to\infty}$ is at most $-\log R_2$ for each fixed z_n. We can therefore apply (2.2.25) to conclude that, for sufficiently large $|\alpha|$,

$$\frac{\log|a_\alpha(z_n)|}{|\alpha|} \le -\log R_1 \text{ if } z_n < R_1,$$

or in other words

$$|a_\alpha(z_n)| R_1^{|\alpha|} \le 1 \text{ for large } \alpha \text{ if } |z_n| < R_1.$$

This proves that the series in (2.3.5) converges normally in D. Since the terms are analytic the sum must also be.

Now we can complete the proof of the theorem. Given $\zeta \in \Omega$ we choose an $R > 0$ so that the polydisk $\{z;\, |z_j - \zeta_j| \le 2R, j = 1, \dots, n\}$ is contained in Ω. By Lemma (2.3.3) applied to the polydisk

$$D = \{|z_j - \zeta_j| \le R, j = 1, \dots, n-1,\ |z_n - \zeta_n| \le 2R\},$$

we can find a point z^0 with $|z_j^0 - \zeta_j| < R$ so that the hypotheses of Lemma (2.3.4) are satisfied with this z^0 and R as above. The lemma implies that u is holomorphic in a neighborhood ζ. This completes the proof of the theorem.

2.4 Solving the $\overline{\partial}$–equation in a polydisc, extension theorems

In this section we consider the inhomogeneous Cauchy–Riemann equation:

$$\overline{\partial}u = f,\ f \in \mathcal{C}_c^\infty(D;\Lambda^{p,q}). \tag{2.4.1}$$

As noted above this equation has a nontrivial integrability condition

$$\overline{\partial}f = 0. \tag{2.4.2}$$

We first consider the fundamental case of $(0,1)$–forms.

Proposition 2.4.3. *Suppose that $D \subset \mathbb{C}^n$, $n > 1$ is a polydisk and $f \in \mathcal{C}_c^k(D;\Lambda^{0,1})$ satisfying* (2.4.2). *Then there is a function $u \in \mathcal{C}_c^k(D)$ such that*

$$\overline{\partial}u = f.$$

Remark. In several variables note the interesting difference in the support properties of solutions to (2.4.1). In general the solution to (2.4.1) does not have compact support in one dimension.

Exercise 2.4.4.

(1) Find necessary and sufficient conditions on f for the solution to (2.4.1) to be compactly supported when $n = 1$.
(2) If $n = 1$ and the solution to (2.4.1) is compactly supported, what is the support of u?

Proof. We simply apply the one variable formula to obtain

$$u(z) = \frac{1}{2\pi\imath}\int \frac{f_1(w,z')dw \wedge d\bar{w}}{w - z_1}. \tag{2.4.5}$$

The regularity follows as in the single variable case. Applying Proposition (1.3.10) we obtain

$$\partial_{\bar{z}_1}u(z) = f_1(z).$$

We can differentiate with respect to $\bar{z}_2, \dots, \bar{z}_n$ under the integral sign to obtain

$$\partial_{\bar{z}_j}u(z) = \frac{1}{2\pi\imath}\int \frac{\partial_{\bar{z}_j}f_1(w,z')dw \wedge d\bar{w}}{w - z_1}. \tag{2.4.6}$$

The integrability condition implies that

$$\partial_{\bar{z}_j}f_1 = \partial_{\bar{z}_1}f_j. \tag{2.4.7}$$

Using this in (2.4.6) and taking account of the compact support of f we can apply (1.3.7) to conclude that

$$\partial_{\bar{z}_j}u(z) = f_j(z).$$

Outside of the support of f, u is a holomorphic function. We can write $D = D_1 \times D'$. If z' lies close enough to the boundary of D' then $f(w,z') = 0$ for $w \in D_1$. Thus $u(z)$ vanishes in an open subset of the unbounded component of its domain of holomorphy and therefore must vanish identically in that component.

Using this result we deduce the basic extension result for holomorphic functions due to Hartogs :

Theorem 2.4.8. *Let $\Omega \subset \mathbb{C}^n$, $n > 1$ be an open set and let $K \subset\subset \Omega$ be a compact subset such that $\Omega \setminus K$ is connected. Every $u \in H(\Omega \setminus K)$ has an extension to a $U \in H(\Omega)$.*

Proof. Choose a polydisk D containing Ω in its interior and a function $\psi \in \mathcal{C}^\infty(D)$ with $\psi \leq 1$. We suppose that $\psi = 0$ in a neighborhood of K and the set $C = \{z; \psi(z) < 1\}$ is a relatively compact subset of Ω. The function ψu can be continued to all of Ω and $f = \overline{\partial}(\psi u)$ can be continued by zero to $D \setminus C$. It is clear that f so extended is compactly supported in D and $\overline{\partial}f = 0$. Thus we can apply (2.4.3) to obtain $v \in \mathcal{C}_c^\infty(D)$ such that $\overline{\partial}v = f$. The function $U = \psi u - v \in H(\Omega)$.

Since C is a relatively compact subset of Ω it follows that v vanishes in an open subset of $\Omega \setminus K$. Thus U agrees with u on some open subset of $\Omega \setminus K$, as this set is connected the uniqueness of analytic continuation implies that $U = u$ in all of $\Omega \setminus K$.

We consider two other extension theorems. The first is a straightforward extension of the Riemann removable singularities in one dimension while the second lies somewhere between the Hartogs' and the Riemann theorem.

Riemann removable Singularities Theorem 2.4.9. *Suppose that $\Omega \subset \mathbb{C}^n$ is an open subset and $X \subset \Omega$ is the zero set of a holomorphic function then a bounded function $u \in H(\Omega \setminus X)$ has an extension to a function $U \in H(\Omega)$.*

Proof. As we have not covered all the prerequisites we only prove this theorem in the special case that $X = \Omega \cap \{z_n = 0\}$.

Let $(w', 0) \in X$, choose a number $R > 0$ so that

$$D = \{z;\, |z_i - w_i| \leq R, i = 1, \dots n-1, |z_n| \leq R\} \subset\subset \Omega.$$

Then the function defined by

$$U(z', z_n) = \frac{1}{2\pi\imath} \int\limits_{|\zeta|=R} \frac{u(z', \zeta)d\zeta}{\zeta - z_n} \tag{2.4.10}$$

is holomorphic in D. On the other hand we can apply the single variable removable singularities theorem to $u(z', z_n)$ to conclude that as a function of z_n this has an analytic extension to $z_n = 0$. Therefore $U(z', z_n) = u(z', z_n)$ for $z_n \neq 0$.

The last continuation result is a purely several variables theorem.

Theorem 2.4.11. *Let Ω be an open subset of $\mathbb{C}^n$ and set $Y = \Omega \cap \{z;\, z_{n-1} = z_n = 0\}$ If $u \in H(\Omega \setminus Y)$ then there is a function $U \in H(\Omega)$ extending u.*

Remark. This theorem is not a consequence of (2.4.8) because Y is not a compact subset of Ω and it is not a consequence of (2.4.9) because u is not assumed to be bounded. The previous theorem says, in effect, that for a holomorphic function to be singular on a variety of codimension 1 it must blow up there. The present theorem says that a holomorphic function cannot be singular on a variety of codimension 2.

Proof. As before suppose that $(w'', 0, 0) \in Y$ and that the polydisk with this center and radii all equal to $R > 0$ is a relatively compact subset of Ω. Denote it by D. The function defined by

$$U(z', z_n) = \frac{1}{2\pi\imath} \int\limits_{|\zeta_n|=R} \frac{u(z', \zeta_n)d\zeta_n}{\zeta_n - z_n}, \tag{2.4.12}$$

is holomorphic in D.

If $z_{n-1} \neq 0$ then $u(z'', z_{n-1}, z_n)$ is holomorphic for $|z_n| < R$. Thus it follows from (2.4.12) That

$$U(z) = u(z) \text{ if } z_{n-1} \neq 0. \tag{2.4.13}$$

Since $U(z)$ is holomorphic and $\{z_{n-1} = 0\}$ does not separate the polydisk it follows from (2.4.13) that

$$U \restriction_{D \setminus D \cap Y} = u \restriction_{D \setminus D \cap Y} .$$

This proves the theorem.

The mechanism behind this argument is two pronged. Firstly a holomorphic function is determined by its values on the distinguished boundary. While the putative singular locus does intersect the boundary of any polydisk centered at a point on this locus, it can be arranged to be disjoint from the distinguished boundary. The other basic fact is that a disk, which is of $\mathbb{R}$–codimension two generically does not intersect a subvariety of $\mathbb{R}$–codimension four. A more general statement than (2.4.11) replaces $z_{n-1} = z_n = 0$ with a $\mathbb{C}$–codimension two subvariety of Ω.

2.5 Local solution of the $\overline{\partial}$–equation for p, q–forms

For the sake of completeness we include an argument that the $\overline{\partial}$–equation can be locally solved for p, q–forms. Suppose that $D \subset \mathbb{C}^n$ is an open polydisk and $f \in \mathcal{C}^\infty(D; \Lambda^{p,q})$, $q > 0$, with $\overline{\partial} f = 0$ then we want to find a form $u \in \mathcal{C}^\infty(D; \Lambda^{p,q-1})$ such that

$$\overline{\partial} u = f. \tag{2.5.1}$$

At present we will content ourselves with finding u defined in $D' \subset\subset D$. As contrasted with the previous result we do not assume that f has compact support. To obtain u in all of D we need an Runge type approximation result.

Theorem 2.5.2. *Let $D \subset \mathbb{C}^n$ be an open polydisk and $f \in \mathcal{C}^\infty(D; \Lambda^{p,q})$ with $q > 0$ satisfy $\overline{\partial} f = 0$. If $D' \subset\subset$ is polydisk then we can find a form $u \in \mathcal{C}^\infty(D'; \Lambda^{p,q-1})$ such that*

$$\overline{\partial} u = f. \tag{2.5.3}$$

Proof. The argument is by induction. We assume that if f is independent of $d\bar{z}_{k+1}, \dots, d\bar{z}_n$ then we can solve (2.5.3) in D'. Since $q > 0$ it follows that the case $k = 0$ corresponds to $f = 0$ and therefore the claim is trivially true. If we can verify the claim for $k = n$ then we've proved the theorem. Assume it for k. Suppose that f is of the form

$$f = d\bar{z}_{k+1} \wedge g + h,$$

where g, h are independent of $d\bar{z}_{k+1}, \dots, d\bar{z}_n$. We can write

$$g = \sum_{I,J} g_{IJ} dz^I \wedge d\bar{z}^J,$$

the sum extends over increasing $p, q-1$–multiindices and the entries of J vary between 1 and k. That is f is independent of $d\bar{z}_{k+2}, \dots, d\bar{z}_n$. The hypothesis that $\overline{\partial} f = 0$ is easily seen to imply that

$$\partial_{\bar{z}_j} g_{IJ} = 0, j = k+2, \dots, n. \tag{2.5.4}$$

That is because these are, up to a sign, the coefficients of $dz^I \wedge d\bar{z}^{k+1} \wedge d\bar{z}^j \wedge d\bar{z}^J$ in $\overline{\partial} f$.

Using these facts and the standard Cauchy theorem we can remove the $d\bar{z}^{k=1}$ term in f. Choose a function $\psi \in \mathcal{C}_c^\infty(D)$ such that $\psi = 1$ on D'. We set

$$G_{IJ}(z) = \frac{1}{2\pi\imath} \int \frac{\psi g_{IJ}(z_1, \dots, z_k, \tau, z_{k+2}, \dots, z_n) d\tau \wedge d\overline{\tau}}{\tau - z_k}.$$

The regularity of G follows immediately from the integral formula.

It is clear that

$$\partial_{\bar{z}_j} G_{IJ}(z) = 0, z \in D', j = k+2, \dots, n. \tag{2.5.5}$$

From the Cauchy formula it follows that

$$\partial_{\bar{z}_{k+1}} G_{IJ} = g_{IJ} \text{ in } D'. \tag{2.5.6}$$

We let

$$G = \sum_{I,J} G_{IJ} dz^I \wedge d\bar{z}^J.$$

The formulæ (2.5.5)–(2.5.6) imply that

$$h' = d\bar{z}_{k+1} \wedge g - \overline{\partial} G$$

depends only on $d\bar{z}_1, \dots d\bar{z}_k$. We can therefore apply the inductive hypothesis to $h + h' = f - \overline{\partial} G$. This form is $\overline{\partial}$–closed and only depends on $d\bar{z}_1, \dots, d\bar{z}_k$ thus there exists $v \in \mathcal{C}^\infty(D'; \Lambda^{p,q-1})$ such that

$$\overline{\partial} v = h + h'.$$

Setting $u = v + G$ completes the proof of the theorem.

2.6 Power series and Reinhardt Domains

A important topic in the theory of one complex variable is the study of the convergence properties of power series. There are many criteria, a very simple one is the following: the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges absolutely in the set B defined by the condition

$$z \in B \text{ provided there exists a constant } C \text{ such that } |a_n z^n| \leq C, \, \forall\, n \in \mathbb{N}.$$

A moments thought shows that B is a disk and its diameter is precisely the radius of convergence of the series.

This criterion has a simple generalization to several variables. If

$$\sum_{\alpha} a_\alpha z^\alpha \tag{2.6.1}$$

is a series then we define the set B as those points $z \in \mathbb{C}^n$ such that there exists a constant C such that

$$|a_\alpha z^\alpha| \leq C, \, \forall\, \alpha. \tag{2.6.2}$$

If we denote by D the domain of normal convergence for the series in (2.6.1) then clearly D is contained in the interior of B. In fact it is easy to show that $D = \overset{\circ}{B}$.

Proposition 2.6.3. *The set D is the interior of B.*

Proof. Suppose that $z \in B$ then the series converges normally in the polydisk defined by $|w_i| < |z_i|$. For suppose the w lies in a compact subset of this polydisk, then we can find constants $k_i < 1$ so that

$$|w_i| \leq k_i |z_i|.$$

By assumption there exists a constant C such that

$$|a_\alpha z^\alpha| \leq C \, \forall, \alpha.$$

Therefore

$$\begin{aligned} \sum_{\alpha} |a_\alpha w^\alpha| &\leq \sum_{\alpha} C k^\alpha \\ &= \prod_{i=1}^{n} (1 - k_i)^{-1}. \end{aligned} \tag{2.6.4}$$

From (2.6.4) the conclusion is immediate.

From (2.6.3) it is clear that if $z \in D$ then

$$(\lambda_1 z_1, \ldots, \lambda_n z_n) \in D, \text{ if } |\lambda_i| \leq 1, i = 1, \ldots, n. \tag{2.6.5}$$

A domain that satisfies this condition is called a complete Reinhardt domain. If a domain satisfies (2.6.5) with $|\lambda_i| = 1, i = 1, \ldots, n$ it is called a Reinhardt domain. A little consideration shows that actually the domain of convergence of a power series satisfies a further property.

Suppose that z, w are two points in B then we can rewrite the condition (2.6.2) as follows

$$\begin{aligned} \log|a_\alpha| + |\alpha_1| \log|z_1| + \cdots + |\alpha_n| \log|z_n| \leq \log C \\ \log|a_\alpha| + |\alpha_1| \log|w_1| + \cdots + |\alpha_n| \log|w_n| \leq \log C \end{aligned} \tag{2.6.6}$$

From (2.6.6) it is clear that if ζ is a third point that satisfies

$$\log|\zeta_i| = \lambda \log|z_i| + (1 - \lambda) \log|w_i|, \text{ for some } 0 < \lambda < 1,$$

then $\zeta \in B$ as well. From this we see that the set defined by

$$\mathbb{R}^n \supset D^* = \{(\log|z_1|, \ldots, \log|z_n|); \, z \in D\}$$

is convex. We have proved

Theorem 2.6.7. *If D is the domain of convergence of a power series then the set D^* is an open convex subset of $\mathbb{R}^n$. Furthermore if $(\xi_1, \ldots, \xi_n) \in D^*$ and $\eta_i \leq \xi_i, i = 1, \ldots n$ then $\eta \in D^*$ as well.*

A set which satisfies the hypotheses of (2.6.7) is called a log–convex Reinhardt domain. Now we show that if Ω is a Reinhardt domain containing zero then every function $u \in H(\Omega)$ is represented by a power series that converges normally in Ω.

Theorem 2.6.8. *Suppose that Ω is a bounded Reinhardt domain which contains zero and $u \in H(\Omega)$ then the power series for u about 0 converges normally in Ω.*

Proof. Let

$$\Omega'_\epsilon = \{z \in \Omega;\ d(z, \Omega^c) > \epsilon|z|\}.$$

Let Ω_ϵ be the component of Ω'_ϵ which contains 0. Evidently Ω_ϵ is a Reinhardt domain for each ϵ and since Ω is path connected it is clear that

$$\Omega = \bigcup_{\epsilon > 0} \Omega_\epsilon. \tag{2.6.9}$$

Define the function

$$U_\epsilon(z) = \left(\frac{1}{2\pi\imath}\right)^n \int \cdots \int_{|t_i| = 1+\epsilon} \frac{u(t_1 z_1, \ldots, t_n z_n) dt_1 \ldots dt_n}{(t_1 - 1) \ldots (t_n - 1)}. \tag{2.6.10}$$

If $z \in \Omega_\epsilon$, and $w \in \Omega^c$ then

$$\begin{aligned} |(1+\epsilon)z - w| &\geq |w - z| - \epsilon|z| \\ &\geq d(z, \Omega^c) - \epsilon|z| \\ &> 0. \end{aligned} \tag{2.6.11}$$

The final inequality in (2.6.11) follows because $z \in \Omega_\epsilon$. Since Ω_ϵ is a Reinhardt domain, this shows that the integral in (2.6.10) is defined for $z \in \Omega_\epsilon$; differentiating under the integral sign shows that U_ϵ is analytic.

Using the series expansion for the denominator in (2.6.10) we can expand U_ϵ as a normally convergent series

$$U_\epsilon(z) = \sum_\alpha f_\alpha(z). \tag{2.6.12}$$

If δ is small enough then the polydisk $|z_i| \leq \delta$ is contained in Ω. For z is this polydisk we can use the Cauchy formula to compute that

$$f_\alpha(z) = \frac{\partial_z^\alpha u(0) z^\alpha}{\alpha!}. \tag{2.6.13}$$

Since f_α is holomorphic for each α and Ω_ϵ is connected it follows that (2.6.13) holds in all of Ω_ϵ. Thus

$$U_\epsilon(z) = u(z) \restriction_{\Omega_\epsilon}$$

and the power series for u is normally convergent in Ω_ϵ for all $\epsilon > 0$. The theorem follows from (2.6.9).

It follows from Theorem (2.6.7) that the domain of convergence of a power series is always a complete log–convex Reinhardt domain. It is clear that there is a smallest such domain containing any Reinhardt domain, Ω which contains 0. Call it $\widetilde{\Omega}$. From Theorem (2.6.8) it follows that any function which belongs to $H(\Omega)$ actually extends, via its power series about 0, to a function in $H(\widetilde{\Omega})$. Thus we have another extension theorem in several variables which has no one–variable analogue. Recall that a domain of holomorphy is an open subset $\Omega \subset \mathbb{C}^n$ such that there is a function $u \in H(\Omega)$ which cannot be extended to any open set U which contains Ω as a proper subset. We have proved the following

Theorem 2.6.14. *In order for a Reinhardt domain which contains 0 to be a domain of holomorphy it must be complete and log–convex.*

We shall soon see that this is, in fact, also a sufficient condition.

2.7 Domains of holomorphy and holomorphic convexity

As remarked above a domain of holomorphy is defined in $\mathbb{C}^n$ exactly as in $\mathbb{C}$:

Definition 2.7.1. A domain $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy if there exists a function $u \in H(\Omega)$ such that for any open set V which contains Ω as a proper subset there is no function $U \in H(V)$ with $U\restriction_{\Omega} = u$.

Loosely speaking, the function cannot be extended across any boundary point of Ω.

We saw that any open set in $\mathbb{C}$ is a domain of holomorphy. If $V_i \subset \mathbb{C}$ are open sets then the open subset of $\mathbb{C}^n$ defined by

$$\Omega = \prod_{i=1}^{n} V_i$$

is a domain of holomorphy. Let $f_i \in H(V_i)$ be a function with no holomorphic extension beyond V_i. Defining

$$f(z) = \prod_{i=1}^{n} f_i(z_i),$$

we obtain a function in $H(\Omega)$ which does not extend to any open set properly containing Ω.

As in the case of one variable we can define a notion of holomorphic convexity.

Definition 2.7.2. An open set $\Omega \subset \mathbb{C}^n$ is holomorphically convex if for every compact subset, $K \subset\subset \Omega$, the holomorphic convex hull

$$\widehat{K}_{\Omega} = \{z \in \Omega;\ |f(z)| \leq \sup_K |f| \,\forall\, f \in H(\Omega)\},$$

is a compact subset of Ω.

To study the relationship between these two concepts it is useful to define a notion of distance to the complement of Ω in terms of polydisks. Let $R = (r_1, \dots, r_n)$ be a polyradius then for a point $z \in \Omega$ we define

$$\delta_{\Omega,R} = \sup\{\epsilon;\ D(z;\epsilon R) \subset \Omega\}.$$

For a subset $K \subset \Omega$ we define

$$\delta_{\Omega,R}(K) = \inf_{z \in K} \delta_{\Omega,R}(z).$$

The utility of this concept is illustrated by the following proposition

Proposition 2.7.3. *If K is a compact subset of Ω and $\delta = \delta_{\Omega,R}(K)$, for some polyradius, then any function in $H(\Omega)$ extends to be holomorphic in $D(z;\delta R)$ for any $z \in \widehat{K}_{\Omega}$.*

Proof. Let $0 < \epsilon < \delta$, then the set

$$K_{\epsilon} = \bigcup_{z \in K} D(z;\epsilon R).$$

is a relatively compact subset of Ω. Thus, the Cauchy inequalities imply that for any function $f \in H(\Omega)$ there is a constant M such that

$$|\partial_z^{\alpha} f(z)| \leq M\alpha!(\epsilon R)^{-\alpha}, z \in K. \tag{2.7.4}$$

Since any derivative of f also belongs to $H(\Omega)$ it follows from the definition of holomorphic convex hull that the estimates in (2.7.4) are valid for $z \in \widehat{K}_{\Omega}$. From this it is immediate that the power series for f about a point $z \in \widehat{K}_{\Omega}$ converges normally in $D(z;\delta R)$.

With this lemma we can show that being a domain of holomorphy is equivalent to being a holomorphically convex.

Theorem 2.7.5. *An open set $\Omega \subset \mathbb{C}^n$ is holomorphically convex if and only if it is a domain of holomorphy*

Proof. First we show that if Ω is holomorphically convex then we can find a function which does not extend across any boundary point of Ω. Arguing as in the one dimensional case, choose a nested sequence of compact subsets, K_j such that $\widehat{K}_j = K_j$. Then we choose a sequence of points $\mathcal{A} = \{A_j\}$ such that

(1) The boundary of Ω equals the set of cluster points of $\mathcal{A}$,
(2) The intersections $K_j \cap \mathcal{A}$ are finite.

We can relabel the set $\mathcal{A}$ so that there is a monotone sequence $\{n_j\}$ such that

$$K_j \cap \mathcal{A} = A_1, \dots A_{n_j}.$$

Since $\widehat{K}_j = K_j$, for each $j \in n_j + 1, \dots n_{j+1}$ we can find functions $f_k \in H(\Omega)$ such that $|f_k| < 1$ on K_j but $f_k(A_k) = 1$. By taking sufficiently high powers of these functions, which we also denote by f_k we can arrange that

$$\sup_{z \in K_j} \sum_{k=n_j+1}^{n_{j+1}} |f_k(z)| \leq (j2^j)^{-1}. \tag{2.7.6}$$

If we fix an m then it follow from (2.7.6) that

$$\sum_{j=1}^{\infty} j|f_j(z)|$$

converges uniformly on K_m. Therefore the function defined by

$$F(z) = \prod_{j=1}^{\infty} (1 - f_j(z))^j$$

is nonconstant and belongs to $H(\Omega)$. This function cannot be extended to any polydisk $D(w; R)$ not contained in Ω.

To prove this we observe that F has a zero of order j at A_j. If we could extend F beyond Ω then we could extend it to some polydisk with center $w \in \partial\Omega$. By the construction of the sequence $\mathcal{A}$ we can find a subsequence j_k such that $\lim A_{j_k} = w$. From this it follows easily that for any multiindex α

$$\partial^\alpha F(w) = \lim_{k \to \infty} \partial^\alpha F(A_{j_k}) = 0.$$

This would imply that F is identically zero.

To prove the converse we assume that Ω is a domain of holomorphy but is not holomorphically convex. Let $f \in H(\Omega)$ be a function which cannot be extended across any boundary point. Let K be a compact subset of Ω with a non–compact holomorphic convex hull, $\widehat{K}$. Let R be some polyradius and let $\delta = \delta_{\Omega,R}(K) > 0$. Since $\widehat{K}$ is non–compact we can choose a point $w \in \widehat{K}$ such that $D(w; \delta R)$ is not contained in Ω. According to Proposition (2.7.3) the function f can be extended to $D(w; \delta R)$. But this contradicts the fact that f cannot be extended across any boundary point of Ω.

This turns out to be a useful criterion for deciding if an open set is a domain of holomorphy. The following theorem illustrates why.

Theorem 2.7.7. *In order for an open set Ω to be a domain of holomorphy it is necessary and sufficient that given any discrete sequence of points $\{A_j\} \subset \Omega$ there is a function $f \in H(\Omega)$ such that*

$$\limsup_{k \to \infty} |f(A_k)| = \infty. \tag{2.7.8}$$

Proof. First suppose that such a function can be found for any sequence but that Ω is not holomorphically convex this means that there is a compact subset K such that $\widehat{K}$ is non–compact. We can choose a sequence of points $\{A_j\} \subset \widehat{K}$ such that A_j tend to $\partial\Omega$. Let $f \in H(\Omega)$ satisfy (2.7.8) relative to this sequence. Finally let

$$M = \sup_{z \in K} |f(z)|.$$

Clearly we can find some j_0 such that $|f(A_{j_0})| > M$. but this contradicts the fact that $A_{j_0} \in \widehat{K}$. Thus $\widehat{K}$ must also be a compact set.

To construct a function we argue much as in the previous proof. We can exhaust Ω by a nested sequence of compact subsets K_j which satisfy

$$\widehat{K}_j = K_j. \tag{2.7.9}$$

Possibly after choosing a subsequence we can assume that $A_k \notin K_j, k \geq j$. In light of (2.7.9) we can find a sequence of functions $f_j \in H(\Omega)$ such that

$$\sup_{z \in K_j} |f_j(z)| < 1 \text{ and } |f_j(A_j)| > 1. \tag{2.7.9}$$

By raising this function to a sufficiently high power we can assume that

$$\begin{gathered} \sup_{z \in K_j} |f_j(z)| < 2^{-j} \text{ and} \\ |f_j(A_j)| > j + \sum_{k=1}^{j-1} |f_k(A_j)|. \end{gathered} \tag{2.7.10}$$

It is easy to see that the series

$$f = \sum_{j=1}^{\infty} f_j$$

converges locally uniformly to a function in $H(\Omega)$. Moreover, using the conditions in (2.7.10) we obtain that

$$|f(A_j)| > j - 2^{-j}. \tag{2.7.11}$$

From (2.7.11) it follows easily that

$$\lim_{j \to \infty} |f(A_j)| = \infty.$$

As an application of this theorem we obtain

Theorem 2.7.12. *If $\Omega \subset \mathbb{C}^n$ is linearly convex then it is holomorphically convex and therefore a domain of holomorphy.*

Proof. If $p \in \partial\Omega$ then there is a real valued linear function $l_p(z)$ such that $l_p(p) = 0$ and $l_p(z) > 0$ for $z \in \Omega$. Observe that any real linear function can be written in the form

$$l_p(z) = \text{Re}[z \cdot a + b] \text{ for some } a \in \mathbb{C}^n, b \in \mathbb{C}. \tag{2.7.12}$$

If we set $\lambda_p(z) = z \cdot a + b$ then $\lambda_p(z)$ is a holomorphic function whose real part vanishes at p and is positive in Ω. Thus

$$\Lambda_p(z) = (\lambda_p(z) - \lambda_p(p))^{-1} \in H(\Omega)$$

but

$$\lim_{z \to p} |\Lambda_p(z)| = \infty.$$

From this it is evident that for any compact set, K the holomorphic convex hull $\widehat{K}$ avoids some neighborhood of p. Since $p \in \partial\Omega$ is arbitrary it follows that $\widehat{K}$ avoids some neighborhood and is therefore compact. This proves the theorem.

In fact it is clear that the method used in the proof above give another criterion for holomorphic convexity

Proposition 2.7.13. *Suppose that Ω is an open subset of $\mathbb{C}^n$ such that for every boundary point p there is a function $f \in H(\Omega)$ such that $\operatorname{Re} f(z) < 0$ for every $z \in \Omega$ and*

$$\lim_{z \to p} \operatorname{Re} f(z) = 0$$

then Ω is holomorphically convex and therefore a domain of holomorphy.

This condition comes very close to being a local condition. That is, we can easily give a local condition which will ensure that we can find a holomorphic function defined in some neighborhood of a given boundary point which satisfies the hypotheses of (2.7.13) in that neighborhood. The problem then is to show that we can actually find a global holomorphic function with desired properties. This is what is usually called the Levi problem. More precisely it asks whether there is a local condition which implies that a domain is a domain of holomorphy. Being a domain of holomorphy is clearly a biholomorphically invariant notion. We might try, as a local condition, that some neighborhood of each boundary is biholomorphically equivalent to a linearly convex set. This turns out to be almost correct.

For latter applications we need a somewhat more general version of Proposition (2.7.3):

Theorem 2.7.14. *Let R be a positive polyradius and Ω be an open subset of $\mathbb{C}^n$. Suppose that $K \subset\subset \Omega$ and $f(z) \in H(\Omega)$ satisfies:*

(2.7.15) $$|f(z)| < \delta_{R,\Omega}(z), z \in K.$$

Let $\zeta \in \widehat{K}_\Omega$, if $u \in H(\Omega)$ then the power series of u converges in the polydisk centered at ζ of polyradius $|f(\zeta)|R$.

Proof. The argument is essentially identical to the proof of (2.7.3). Let D denote the polydisk centered at zero with polyradius R. For $w \in K$, the power series for u:

$$u(z) = \sum_\alpha \frac{\partial^\alpha u(w)(z-w)^\alpha}{\alpha!},$$

converges in the polydisk $w + \delta_{\Omega,R}(w)D$. If $t < 1$ it follows from (2.7.15) that the union of the polydisks

$$K_t = \bigcup_{w \in K} w + t|f(w)|D$$

is a relatively compact subset of Ω and therefore we can find a constant M such that $|u(z)| \leq M, z \in K_t$. We therefore have the Cauchy estimates

(2.7.16) $$|\partial^\alpha u(w)| t^{|\alpha|} R^\alpha |f(w)|^{|\alpha|} \leq M\alpha!$$

holding for $w \in K$. Since $\partial^\alpha u(w) f(w)^| \alpha| \in H(\Omega)$ for each α it follows that the estimates in (2.7.16) hold for $w \in \widehat{K}_\Omega$ From this it follows that the series expansion for u converge in $w + t|f(w)|D$ for every $t < 1, w \in \widehat{K}_\Omega$. This proves the theorem.

This theorem has a important corollary

Corollary 2.7.17. *If Ω is a domain of holomorphy, $K \subset\subset \Omega$, R a positive polyradius and $f \in H(\Omega)$ satisfies*

$$|f(z)| \leq \delta_{\Omega,R}(z),\ z \in K,$$

then

$$|f(z)| \leq \delta_{\Omega,R}(z),\ z \in \widehat{K}_\Omega.$$

Proof. If this were not the case then (2.7.14) would imply that we could extend every function in $H(\Omega)$ to some polydisk not entirely contained in Ω. This violates the assumption that Ω is a domain of holomorphy.

This corollary in turn has a corollary which is obtained by using the special functions $f =$constant:

Corollary 2.7.18. *If Ω is a domain of holomorphy then for any polyradius R and any compact set K*

$$\delta_{\Omega,R}(K) = \delta_{\Omega,R}(\widehat{K}_\Omega). \tag{2.7.19}$$

Exercise 2.7.19. The purpose of this exercise is to show that a complete log–convex Reinhardt domain is a domain of holomorphy. We know from Theorem (2.6.14) that this is a necessary condition, we will show that it is also sufficient. The idea is to show that a complete, log–convex is holomorphically convex.

(1) If $K \subset\subset \Omega$ then there is a finite set, $\mathcal{K} \subset \Omega$ such that

$$K \subset \bigcup_{\zeta \in \mathcal{K}} \{z, |z_i| \leq |\zeta_i|,\, i = 1, \ldots n\} \subset\subset \Omega.$$

(2) If α is a multiinidex with $\alpha_1 \ldots \alpha_j \neq 0$ and $\alpha_m = 0, m = j+1, \ldots, n$ then

$$|z_1^{\alpha_1} \ldots z_j^{\alpha_j}| \leq \sup_{\zeta \in \mathcal{K}} \{|\zeta_1^{\alpha_1} \ldots \zeta_j^{\alpha_j}|;\, \zeta \in \mathcal{K}\}.$$

(3) Show that this implies that if $\lambda_i, i = 1, \ldots, j \in [0,1]$ then

$$\sum \lambda_i \log |z_i| \leq \sup_{\zeta \in \mathcal{K}} \sum \lambda_i \log |\zeta_i|.$$

(4) Conclude from this that if Ω is complete, log–convex Reinhardt domain then $\widehat{K}_\Omega \subset\subset \Omega$. hint: show that

$$(\log |z_1|, \ldots, \log |z_j|), z \in \widehat{K}_\Omega$$

is in the convex hull of the $\eta \in \Omega^*$ with

$$\eta_i \leq \log |\zeta_i|, i = 1, \ldots j$$

for some $\zeta \in \mathcal{K}$.

2.8 Pseudoconvexity, the ball versus the polydisc

In this section we will consider a local condition, pseudoconvexity, which is satisfied by any domain of holomorphy. This is geometric condition which is more closely related to the holomorphic geometry of the unit ball than to that of the polydisk. After discussing pseudoconvexity and its relationship to holomorphic convexity we will consider complex analysis on the unit ball.

Recall the definition of sub–harmonic functions in $\mathbb{R}^n$:

Definition 2.2.21. A function $u(x)$ defined in Ω an open subset of $\mathbb{R}^n$ is subharmonic provided

$$\begin{aligned} &u(x) \text{ is bounded from above,} \\ &u(x) \text{ is upper semicontinuous,} \\ &u(x) \leq \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)dy, \text{ whenever } B(x,r) \subset\subset \Omega. \end{aligned} \tag{2.2.23}$$

The assumption that u is upper semicontinuous and bounded from above, implies that the integral in (2.2.23) is well defined.

We need to consider a generalization of subharmonic functions to several variables.

Definition 2.8.1. A function u defined in an open set $\Omega \subset \mathbb{C}^n$ taking values in $[-\infty, \infty)$ is call plurisubharmonic provided

(1) u is upper semicontinuous, that is

$$u(z) \geq \limsup_{w \to z} u(w),$$

(2) for any $w, z \in \mathbb{C}^n$ the function $f(\tau) = u(z + \tau w)$ is subharmonic in the part of $\mathbb{C}$ where it is defined.

The collection of such functions is denoted by $P(\Omega)$.

Briefly, a function is plurisubharmonic if its restriction to any complex line is subharmonic. It is easy to show that a plurisubharmonic function is automatically sub-harmonic, the converse is generally false. Note that if $u \in H(\Omega)$ then $\log |u(z)| \in P(\Omega)$. If $u_1, u_2 \in P(\Omega)$ then so is $au_1 + bu_2$ if a, b are positive, and

$$u(z) = \sup\{u_1(z), u_2(z)\}$$

is also in $P(\Omega)$. Finally is ϕ is a convex non-decreasing function defined on the range of $u \in P(\Omega)$ then $\phi(u) \in P(\Omega)$ as well.

Exercise. Verify these properties of plurisubharmonic functions

As with subharmonic functions, plurisubharmonic functions may not be smooth, however if a plurisubharmonic function has two derivatives then this condition can be described in terms of a differential inequality.

Proposition 2.8.2. *A function $u \in C^2(\Omega)$ is plurisubharmonic if and only if*

$$\sum_{i,j} \partial_{z_i} \partial_{\bar{z}_j} u(w) \xi_i \overline{\xi}_j \geq 0, \; \forall\, w \in \Omega, \; \xi \in \mathbb{C}^n. \tag{2.8.3}$$

Proof. A twice differentiable function $f(\tau, \bar{\tau}), \tau \in \mathbb{C}$ is subharmonic if and only if

$$\Delta f = \frac{1}{4} \partial_\tau \partial_{\bar{\tau}} f \geq 0. \tag{2.8.4}$$

Exercise. Prove this statement.

Let $z, w \in \mathbb{C}^n$ and set $f(\tau) = u(z + \tau w)$ then

$$\Delta f(\tau) = \sum_{i,j} \partial_{z_i} \partial_{\bar{z}_j} u(z + \tau w) w_i \bar{w}_j. \tag{2.8.5}$$

The proposition follows from (2.8.4)–(2.8.5).

Exercise 2.8.6. Suppose that $u \in P(\Omega)$ and suppose that $\phi \in \mathcal{C}_c^\infty(\mathbb{C}^n)$ has support in the ball of radius 1, depends only on $|z_1|, \ldots, |z_n|$ and satisfies

$$\int \phi \, \mathrm{dVol} = 1.$$

(1) The functions

$$u_\epsilon(z) = \int u(z + \epsilon w) \phi(w) \, \mathrm{dVol},$$

are smooth and plurisubharmonic in

$$\Omega_\epsilon = \{z \in \epsilon; \delta(z, \Omega^c) > \epsilon\},$$

(2) $u_\epsilon(z) \geq u_\delta(z)$ if $\epsilon > \delta$,
(3) $\lim_{\epsilon \to 0} u_\epsilon = u$.

We can now establish a connection between domains of holomorphy and plurisubharmonic functions

Theorem 2.8.7. *If Ω is a domain of holomorphy and R is a positive polyradius then $-\log \delta_{\Omega,R}(z)$ is plurisubharmonic and continuous.*

Proof. The continuity is clear from the definition. We use the following characterization of subharmonic functions:

(1) A function $f(\tau)$ is subharmonic in a domain $D \subset \mathbb{C}$ if for every disk $B(w;r) \subset\subset D$ and every harmonic function h defined on $\overline{B}(w;r)$ which satisfies

$$f(w + re^{i\theta}) \leq h(w + re^{i\theta})$$

also satisfies

$$f(w + \rho e^{i\theta}) \leq h(w + \rho e^{i\theta}), 0 \leq \rho < r.$$

Exercise. Prove that a subharmonic function satisfies the maximum principle: If u is a subharmonic function defined in a bounded, connected open set $\Omega \subset \mathbb{C}$ such that, for some $z_0 \in \Omega$,

$$u(z_0) = \sup\{u(z) : z \in \Omega\}$$

then u is constant. Deduce that the comparision with harmonic functions described above characterizes subharmonic functions.

Fix a point $z_0 \in \Omega$ and a $w \in \mathbb{C}^n$. Choose an $r > 0$ so that

$$D = \{(z_0 + \tau w) \in \Omega; |\tau| \leq r\}.$$

Let $f(\tau)$ be an analytic polynomial which satisfies

$$-\log \delta_{\Omega,R}(z_0 + \tau w) \leq \operatorname{Re} f(\tau), \text{ for } |\tau| = r. \tag{2.8.9}$$

Any harmonic function on $|\tau| \leq r$ which satisfies such an estimate can be approximated uniformly by the real parts of holomorphic polynomials. This is so because any harmonic function in a disk is the real part of a holomorphic function and any holomorphic function is uniformly approximated by polynomials. Thus it suffices to consider polynomials.

Let $F(z)$ be a polynomial defined in $\mathbb{C}^n$ which satisfies

$$F(z_0 + \tau w) = f(\tau).$$

The maximum principle implies that the holomorphic convex hull of ∂D, with respect to Ω must contain D and therefore we can apply Corollary (2.7.17) and (2.8.9) to conclude that

$$|e^{-F(z)}| \leq \delta_{\Omega,R}(z), z \in D. \tag{2.8.10}$$

Rewriting this we obtain

$$-\log \delta_{\Omega,R}(z_0 + \tau w) \leq \operatorname{Re} f(\tau).$$

Thus $-\log \delta_{\Omega,R}(z_0 + \tau w)$ is a subharmonic function, where it is defined and thus $-\log \delta_{\Omega,R}(z) \in P(\Omega)$.

The converse of this result is true but it will take some effort to prove.

In analogy with the holomorphic convex hull we define the define a plurisubharmonic convex hull.

Definition 2.8.11. If $K \subset\subset \Omega \subset \mathbb{C}^n$ then we define the plurisubharmonic hull of K relative to Ω by

$$\widehat{K}_{\Omega}^{P} = \{z \in \Omega;\ u(z) \leq \sup_K u \text{ for all } u \in P(\Omega)\}.$$

Definition 2.8.12. An open subset $\Omega \subset \mathbb{C}^n$ is said to be pseudoconvex if for every $K \subset\subset \Omega$ we have

$$\widehat{K}_\Omega^P \subset\subset \Omega.$$

Because $|f(z)|$ is plurisubharmonic for $f \in H(\Omega)$, the $P(\Omega)$–hull of K is contained inside the $H(\Omega)$–hull. From this it is immediate that a holomorphically convex domain is also pseudoconvex.

This turns out to be quite a flexible concept. It has several alternative characterizations .

Theorem 2.8.13. *An open subset $\Omega \subset \mathbb{C}^n$ is pseudoconvex if and only if either*

(2.8.14) *For any positive polyradius R the function $-\log \delta_{\Omega,R}(z)$ is plurisubharmonic*

or

(2.8.15) *There exists a function $u \in P(\Omega)$ such that the sets $\{z;\, u(z) < c\} \subset\subset \Omega$.*

Proof. It is clear that (2.8.14) implies (2.8.15). Furthermore (2.8.15) clearly implies that Ω is pseudoconvex. All that remains is to show that if Ω is pseudoconvex then (2.8.14) holds. Let D denote the polydisk centered at zero with polyradius R. We need to show that $-\log \delta_{\Omega,R}(z)$ is plurisubharmonic. To simplify the notation we denote this function by $-\log \delta$.

Choose $z_0 \in \Omega$, $w \in \mathbb{C}^n$ and an positive number r such that

$$B = \{z_0 + \tau w;\ |\tau| < r\} \subset \Omega$$

and a holomorphic polynomial $f(\tau)$ which satisfies

$$-\log \delta(z_0 + \tau w) \le \operatorname{Re} f(\tau), |\tau| = r. \tag{2.8.16}$$

We need to show that (2.8.16) holds throughout B. We can rewrite this inequality as

$$\delta(z_0 + \tau w) \ge |e^{-f(\tau)}|. \tag{2.8.16'}$$

To extend this inequality to $|\tau| < r$ we let $a \in D$ and consider the mapping for $0 \le \lambda \le 1$

$$\tau \longrightarrow z_0 + \tau w + \lambda a e^{-f(\tau)}.$$

Denote the image of $|\tau| \le r$ by B_λ. If we can show that $B_1 \subset \Omega$ then, since $a \in D$ is arbitrary, (2.8.16') would follow for $|\tau| \le r$. Let

$$\Lambda = \{\lambda \in [0,1];\ B_\lambda \in \Omega\}.$$

It is clear that $0 \in \Lambda$ and furthermore that Λ is an open set. We will show that pseudoconvexity implies that it is also closed. Let

$$K = \{z_0 + \tau w + \lambda a e^{-f(\tau)}; |\tau| = r, \lambda \in [0,1]\}.$$

The inequality (2.8.16') implies that $K \subset \Omega$, it is clearly compact. Let $u \in P(\Omega)$ and suppose that $\lambda \in \Lambda$ then

$$\tau \longrightarrow u(z_0 + \tau w + \lambda a e^{-f(\tau)})$$

is subharmonic in a neighborhood of the disk $|\tau| < r$. Thus we have the inequality

$$u(z_0 + \tau w + \lambda a e^{-f(\tau)}) \le \sup_K u \text{ if } |\tau| < r.$$

Since $u \in P(\Omega)$ is arbitrary this implies that $B_\lambda \subset \widehat{K}_\Omega^P$ for every $\lambda \in \Lambda$. This in turn implies that Λ is closed as $\widehat{K}_\Omega^P$ is assumed to be a relatively compact subset of Ω.

Since $[0,1]$ is connected this implies that $B_1 \subset \Omega$ and therefore

$$z_0 + \tau w + a e^{-f(\tau)} \in \Omega \text{ if } a \in D,\ |\tau| \le r.$$

But this implies that

$$\delta(z_0 + \tau w) \ge |e^{-f(\tau)}|,\ |\tau| \le r$$

which implies that

$$-\log \delta \in P(\Omega).$$

In the proof of the theorem we have made use of families of holomorphic disks. There is a classical criterion for holomorphic convexity in terms of holomorphic disks.

Theorem 2.8.13'. *[Kontinuitätsatz] Let $\{d_\alpha\}_{\alpha\in\mathcal{A}}$ be a family of closed analytic disks contained in Ω. The domain Ω is pseudoconvex if and only if*

$$\bigcup_{\alpha\in\mathcal{A}} \partial d_\alpha \subset\subset \Omega \Rightarrow \bigcup_{\alpha\in\mathcal{A}} d_\alpha \subset\subset \Omega.$$

Proof. We have in fact almost given the proof of this result. First lets assume that Ω is pseudoconvex. Let $\alpha \in \mathcal{A}$ and let $\phi : \overline{D_1} \to \Omega$ be a holomorphic parametrization of the disk d_α. If u is a plurisubharmonic function on Ω then $\phi^*(u)$ is a subharmonic function on D_1 Therefore

$$u(\phi(z)) \leq \sup_{\zeta\in\partial d_\alpha} u(\zeta).$$

This shows that d_α is contained in the p.s.h. hull of ∂d_α and therefore

$$\bigcup_{\alpha\in\mathcal{A}} d_\alpha \subset [\widehat{\bigcup_{\alpha\in\mathcal{A}} \partial d_\alpha}]^P.$$

As Ω is pseudoconvex the hypothesis $\bigcup_{\alpha\in\mathcal{A}} \partial d_\alpha \subset\subset \Omega$ implies that

$$\bigcup_{\alpha\in\mathcal{A}} d_\alpha \subset [\widehat{\bigcup_{\alpha\in\mathcal{A}} \partial d_\alpha}]^P \subset\subset \Omega$$

as well.

The proof in the other direction has already been given in the proof of Theorem 2.8.13. Using that result if suffices to show that $-\log \delta_{R,\Omega}(z)$ is plurisubharmonic. As before we select points $z_0 \in \Omega$, $w \in \mathbb{C}^n$ and a radius $0 < r$ so that $\{z_0 + \tau w : |\tau| \leq r\} \subset \Omega$. Finally choose a holomorphic polynomial $f(\tau)$ such that

$$-\log \delta_{R,\Omega}(z_0 + \tau w) \leq \operatorname{Re} f(\tau) \text{ if } |\tau| = r.$$

To complete the argument we simply use the family of holomorphic disks

$$B_\lambda = \{z_0 + \tau w + \lambda a e^{-f(\tau)} : |\tau| \leq r\},\ \lambda \in [0,1],$$

where as before $a \in D(0;R)$ is arbitrary. Once again we need to show that Λ, the set of parameters such that $B_\lambda \subset \Omega$ equals $[0,1]$. As before we know that $0 \in \Lambda$ and

$$\bigcup_{\lambda\in[0,1]} \partial B_\lambda \subset\subset \Omega.$$

This does **not** immediately imply that $\Lambda = [0,1]$. The argument goes much as before: we observe that the set Λ is open and non-empty. The hypotheses of the Kontinuitätsatz allows us to conclude that the set Λ is closed and therefore equals $[0,1]$.

In Theorem 2.8.13 the function $\delta_{\Omega,R}$ does not have to be defined by a polydisk. In fact the theorem is true if we choose any continuous, non–negative function δ defined on $\mathbb{C}^n$ which satisfies

$$\delta(tz) = |t|\delta(z),\ t \in \mathbb{C} \tag{2.8.17}$$

and define

$$\delta_\Omega(z) = \inf_{w\in\Omega^c} \delta(z - w).$$

For example we could take $\delta(z) = |z|$.

One of the important features of pseudoconvexity is that it is a local property of the boundary:

Theorem 2.8.18. *Let $\Omega \subset \mathbb{C}^n$ be an open set such that to every point in $\overline{\Omega}$ there is an open set ω such that $\omega \cap \Omega$ is pseudoconvex then Ω is also pseudoconvex.*

Proof. We will show that Ω has a plurisubharmonic exhaustion function. For a $z_0 \in \partial\Omega$ choose a neighborhood ω such that $\omega \cap \Omega$ is pseudoconvex. The for some $\delta(z)$ satisfying (2.8.17) the function $-\log \delta_{\omega\cap\Omega}(z)$ is plurisubharmonic. There is a smaller open set $\omega' \subset \omega$ in which we have

$$\delta_\Omega(z) = \delta_{\omega\cap\Omega}(z), z \in \omega'.$$

From this we conclude that $-\log \delta_\Omega(z)$ is itself plurisubharmonic in some neighborhood of $\partial\Omega$. This means that there is a closed subset $F \subset \Omega$ such that $-\log \delta_\Omega$ is plurisubharmonic in $\Omega \setminus F$. It follows from (2.8.17) that we can choose a convex increasing function $\phi(x)$ such that

$$-\log \delta_\Omega(z) < \phi(|z|^2) \text{ for } z \in F. \tag{2.8.19}$$

If we set $u(z) = \sup\{-\log \delta_\Omega(z), \phi(|z|^2)\}$ then $u \in P(\Omega)$ as $u = \phi(|z|^2)$ in a neighborhood of F and the supremum of two plurisubharmonic functions is subharmonic. This function clearly satisfies the condition (2.8.15) and therefore Ω is plurisubharmonic.

We consider various properties of pseudoconvex sets. First we consider the intersection of pseudoconvex sets.

Theorem 2.8.20. *If Ω_1, Ω_2 are pseudoconvex open subsets of $\mathbb{C}^n$ then so is $\Omega_1 \cap \Omega_2$.*

Proof. This follows easily from the properties of the plurisubharmonic hull. Let $K \subset\subset \Omega_1 \cap \Omega_2$. Since

$$P(\Omega_i) \subset P(\Omega_1 \cap \Omega_2), i = 1, 2$$

it follows that

$$\widehat{K}_{\Omega_1\cap\Omega_2} \subset \widehat{K}_{\Omega_1} \cap \widehat{K}_{\Omega_2}. \tag{2.8.21}$$

Since Ω_1 and Ω_2 are pseudoconvex $\widehat{K}_{\Omega_1}$ and $\widehat{K}_{\Omega_2}$ are compact subsets. This implies that each avoids some open neighborhood of $\partial\Omega_1$ or $\partial\Omega_2$ respectively. Since

$$\partial\Omega_1 \cap \Omega_2 = \Omega_1 \cap \partial\Omega_2 \cup \Omega_2 \cap \partial\Omega_1$$

it follows from (2.8.21) that $\widehat{K}_{\Omega_1\cap\Omega_2}$ is a compact subset of $\Omega_1 \cap \Omega_2$.

Note that the same argument shows that the intersection of two domains of holomorphy is a domain of holomorphy. To study unions we need a result about sequences of subharmonic functions.

Lemma 2.8.22. *Suppose that $u_i(z)$ is a decreasing sequence of subharmonic functions in a domain $D \subset \mathbb{C}$. If $\lim u_i(z_0) \neq -\infty$ for some $z_0 \in D$ then*

$$u(z) = \lim_{i\to\infty} u_i(z)$$

if finite almost everywhere and subharmonic.

Proof. To see that $u(z) > -\infty$ for almost every z we use that fact that if $B(z_0, r) \subset\subset D$ then for every i

$$u_i(z_0) \leq \frac{1}{\pi r^2} \iint u_i(z) dxdy. \tag{2.8.23}$$

Since the sequence is decreasing we can apply Lebesgue's monotone convergence theorem to conclude that

$$-\infty < u(z_0) \leq \frac{1}{\pi r^2} \iint u(z) dxdy. \tag{2.8.24}$$

From this it follows that $u(z) > -\infty$ for almost every $z \in B(z_0, r)$. We simply repeat the argument with a new point z_1 near to $\partial B(z_0, r)$ in this way we can show that $u(z) > -\infty$ for almost all $z \in D$. We can also use the Lebesgue theorem to conclude that for every $z \in D$ and $r > 0$ such that $B(z, r) \subset D$

$$u(z) \leq \frac{1}{\pi r^2} \iint\limits_{B(z,r)} u(z) dx dy. \tag{2.8.24}$$

This suffices to conclude that $u(z)$ is subharmonic.

To see this we observe that a function that satisfies (2.8.24) must satisfy the maximum principle. Thus if we take any harmonic function h then $u - h$ also satisfies the integral inequality and therefore the maximum principle. This implies that if $u - h \leq 0$ on $\partial B(w, r)$ then it is also negative in $B(w, r)$. This however was our definition of a subharmonic function.

Exercise 2.8.25**.** Prove that (2.8.24) and the upper semicontinuity of u imply that u satisfies the maximum principle.

Using the lemma we can study unions of pseudoconvex sets.

Theorem 2.8.26**.** *Suppose that Ω_i, $i = 1, \ldots$ is an increasing sequence of pseudoconvex domains then*

$$\Omega = \bigcup_{i=1}^{\infty} \Omega_i$$

is pseudoconvex.

Proof. We use the characterization given by (2.8.14). Let R be a fixed polyradius. The sequence of functions $-\log \delta_{\Omega_i,R}(z)$ is decreasing for each fixed z. Evidently the limit is $-\log \delta_{\Omega,r}(z)$. In virtue of (2.8.14) each function in the sequence is plurisubharmonic and therefore by (2.8.22) the limit is as well.

We will now consider an important special case. We suppose that Ω has a C^2 boundary. This means that there is a function ρ twice differentiable in some neighborhood of Ω such that

$$\Omega = \{z; \, \rho(z) < 0\}$$

and furthermore

$$d\rho(z) \neq 0 \text{ for } z \in \partial\Omega.$$

Such a function is called a defining function for Ω. We have the following description of a pseudoconvex domain in terms of a defining function. We will only prove an important special case.

Theorem 2.8.27**.** *A domain with C^2–boundary is pseudoconvex if and only if there is a defining function, ρ such that*

$$\partial\overline{\partial}\rho(z)(X, \overline{X}) \geq 0, \text{ for all } z \in \partial\Omega, \text{ for which } \partial\rho(X) = 0. \tag{2.8.28}$$

Proof. One direction is obvious. Let

$$\rho(z) = -d(z, \Omega^c), z \in \Omega, \rho(z) = d(z, \Omega), z \in \Omega^c.$$

Using the implicit function theorem one can show that ρ is C^2 in some neighborhood of $\partial\Omega$. At such point we can compute $-\partial\overline{\partial} \log \rho$:

$$-\partial_{z_i}\partial_{\bar{z}_j} \log \rho = \sum_{i,j} [-\frac{\partial_{z_i}\partial_{\bar{z}_j}\rho}{\rho} + \frac{\partial_{z_i}\rho}{\rho}\frac{\partial_{\bar{z}_j}\rho}{\rho}]. \tag{2.8.29}$$

If $\partial\rho(X) = 0$ then the second term in (2.8.29) vanishes and the condition that $-\log \rho$ be plurisubharmonic reduces to the positivity of the first term. This evidently persists as we approach the boundary.

We will not prove this result in full generality but instead use a somewhat stronger hypotheses: we assume that $\partial\overline{\partial}\rho > 0$ on $\ker \partial\rho$ at $\partial\Omega$. We also assume that Ω is bounded. To show that Ω is pseudoconvex we apply (2.8.15) and produce a plurisubharmonic defining function. Clearly $-\log \rho$ is plurisubharmonic in a neighborhood of the boundary and blows up as we approach the boundary. To correct it in the interior we simply add a multiple of $|z|^2$. Thus we have a plurisubharmonic exhaustion function of the form

$$-\log r + M|z|^2,$$

which proves the theorem in this case.

Exercise 2.8.30. Let Ω be a domain with a C^2-boundary and let $\rho(z) = -\operatorname{dist}(z, \Omega^c)$. Show that if there is a point $z_0 \in \partial\Omega$ and a vector w such that

$$\sum_{i,j} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(z_0) w_i \bar{w}_j < 0 \text{ where } \sum_i \frac{\partial \rho}{\partial z_j}(z_0) w_j = 0$$

then there is a holomorphic map

$$\phi : D_1 \to \mathbb{C}^n$$

so that

$$\phi(0) = z_0 \text{ and } \phi(D_1 \setminus \{0\}) \subset \Omega.$$

Use the existence of this disk to conclude that Ω cannot be pseudoconvex.

If a C^2 domain is strictly pseudoconvex then is possible to find a holomorphic change of coordinates, in a neighborhood of each boundary point, so that, near the given point the boundary is linearly convex in the new coordinates. This is a consequence of Taylor's formula. With out loss of generality we can assume that the point of interest is $0 \in \partial\Omega$. Taylor's formula, in complex notation states

$$\rho(z) = \operatorname{Re} \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(0) z_j + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + \operatorname{Re}\left[\sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0) z_i z_j\right] + o(|z|^2). \tag{2.8.31}$$

Again, by a complex linear change of coordinates, it can be assumed that

$$\frac{\partial \rho}{\partial z_1}(0) = 1, \quad \frac{\partial \rho}{\partial z_j}(0) = 0 \text{ for } j = 2, \ldots, n.$$

We continue to denote these coordinates by $(z_1, \ldots, z_n)$. We define a local holomorphic change of variables by setting

$$w_1 = z_1 + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial z_j}(0) z_i z_j, \quad w_j = z_j, \quad j = 2, \ldots, n.$$

This is clearly a locally invertible map. In these coordinates we see that

$$\rho(w) = \operatorname{Re} w_1 + \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \partial \bar{z}_j}(0) w_i \bar{w}_j + o(|w|^2).$$

If ρ is strictly plurisubharmonic at 0 then the real Hessian, in the w-coordinates of ρ at 0 is positive definite. This show that the hypersurface $\{\rho(w) = 0\}$ is locally strictly convex near to $w = 0$. If ρ is just plurisubharmonic (not strictly) then this a holomorphic change of variables making $\{\rho^{-1}(0)\}$ locally convex may not exist. Kohn and Nirenberg gave an example showing that this is the case.

A C^2–domain which has a defining function ρ which satisfies

$$\partial\bar{\partial}\rho(z) > 0 \text{ on } \ker \partial\rho \text{ at } \partial\Omega$$

is called strictly pseudoconvex. Consider the defining functions for Ω given by

$$\phi_t = e^{t\rho} - 1, t > 0.$$

If we compute $\partial\bar{\partial}\phi_t$ we see that

$$\partial\bar{\partial}\phi_t = te^{t\rho}(\partial\bar{\partial}\rho + t\partial\rho\bar{\partial}\rho).$$

Evidently if t is sufficiently large then $\partial\bar{\partial}\phi_t > 0$ at $\partial\Omega$ with no further restriction. Using techniques similar to those we have been using one can show that any pseudoconvex domain can be exhausted by smooth strictly pseudoconvex subdomains. These domains have much simpler analytic properties than general pseudoconvex domains and also have a very rich geometric structure which we will discuss. The canonical example is the unit ball

$$\mathbb{CB}^n = \{z \in \mathbb{C}^n; \, |z|^2 - 1 < 0\}.$$

It serves as the model domain for the study of strictly pseudoconvex domains in much the same way as the polydisk served as a model for a general pseudoconvex domain.

2.9 CR-structures and the Lewy extension theorem

Let $\Omega \subset \mathbb{C}$ be an open set with a smooth boundary. The question as to whether a function $f \in C^0(\Omega)$ is the boundary value of a function $u \in H(\Omega)$ is studied using a non–local, pseudodifferential operator defined on $\partial\Omega$. If we donte this operator by P_Ω then f is the boundary value of a holomorphic function if and only if $P_\Omega f = 0$. In more dimensions the situation is quite different. The condition that a function defined on $\partial\Omega$ be the boundary value of a holomorphic function defined in Ω is given by local, i.e. differential operators.

Recall that a complex structure on Ω is determined by a subbundle of the complexified tangent space, $T\Omega \otimes \mathbb{C}$, denoted by $T^{1,0}\Omega$. It satisfies two conditions; let $T^{0,1}\Omega = \overline{T^{1,0}\Omega}$ then

(2.9.1)
$$T^{1,0}\Omega \cap T^{0,1}\Omega = \{0\},$$
$$\text{If } X, Y \text{ are sections of } T^{1,0}\Omega \text{ then so is } [X, Y].$$

In this formulation a function $u \in \mathcal{C}^1(\Omega)$ is holomorphic if

(2.9.2)
$$\bar{Z}u = 0 \text{ for all sections,} \bar{Z} \text{ of } T^{0,1}\Omega.$$

From (2.9.2) we can easily derive necessary conditions for a function $f \in \mathcal{C}^1(\Omega)$ to be the boundary value of a holomorphic function. In some cases these turn out to also be sufficient. Let M denote $\partial\Omega$ and J denote the almost complex structure underlying the complex structure. Suppose that $\bar{Z}$ is a section of $T^{0,1}\Omega$ with the property that $\operatorname{Re}\bar{Z}$ and $\operatorname{Im}\bar{Z}$ are tangent to M. Under this assumption it is immediately clear that the if $u \in \mathcal{C}^1(\overline{\Omega})$ then $\bar{Z}u \restriction_M$ is determined by $u \restriction_M$. Thus we see that in order for $f = u \restriction_\Omega$ to be the boundary value of a holomorphic function it is necessary that

(2.9.3)
$$\bar{Z}f = 0 \text{ for all sections of } T^{0,1}\Omega \text{ which are tangent to } M.$$

These are called the tangential Cauchy–Riemann equations. If u is defined in a neighborhood of $\partial\Omega$ then an equivalent condition is given by

(2.9.3')
$$\overline{\partial}u \wedge \overline{\partial}\rho = 0 \text{ at } \partial\Omega$$

here ρ is a defining function for Ω.

Exercise. Prove that (2.9.3) and (2.9.3') are equivalent

The next order of business is therefore to understand $T^{0,1}\Omega \cap TM \otimes \Omega$. This is a simple exercise in linear algebra. Since any invariant subspace of J is even dimensional it follows easily that JTM is not contained in TM and therefore

$$T\Omega \restriction_M = TM + JTM.$$

From this formula and the fact that $\dim(V + W) = \dim V + \dim W - \dim(V \cap W)$ it follows that

(2.9.4)
$$\dim TM \cap JTM = 2n - 2.$$

From (2.9.4) we immediately deduce that

(2.9.5)
$$\dim_{\mathbb{C}} TM \otimes \mathbb{C} \cap T^{0,1}\Omega \restriction_M = n - 1.$$

We denote the subspace

$$T^{0,1}M = TM \otimes \mathbb{C} \cap T^{0,1}\Omega \restriction_M .$$

Its conjugate is $T^{1,0}M$, clearly

(2.9.6)
$$T^{1,0}M \cap T^{0,1}M = \{0\}.$$

If X, Y are two sections of $T^{0,1}M$ then it is easy to see that they can be extended to a neighborhood of M in Ω as sections of $T^{0,1}\Omega$. From this observation and (2.9.1) it follows that

(2.9.7)
$$\text{If } X, Y \text{ are sections of } T^{0,1}M \text{ then so is } [X, Y].$$

Thus we see that the complex structure on Ω induces a structure on a codimension 1, real submanifold. This structure is called a CR–structure. It is clear from the construction above, of an induced CR–structure, that a CR–structure can be defined intrinsically on an odd dimensional manifold. Abstractly we have

Definition 2.9.8**.** Let M be a $2n-1$–dimensional manifold and let $T^{0,1}M$ be a $n-1$–dimensional subbundle of $TM\otimes\mathbb{C}$ which satisfies the non–degeneracy condition (2.9.6) and the formal integrability condition (2.9.7) then we say that $T^{0,1}M$ defines a CR–structure on M.

Underlying the CR–structure is a real hyperplane field spanned by $T^{0,1}M+T^{1,0}M$. Locally a hyperplane field is defined as the kernel of a one form. Denote such a one form by θ. This one form allows us to study the degree to which the formal integrability condition is true integrability in the sense of Frobenius. Put differently, the condition (2.9.7) does not imply that $\ker\theta$ is tangent to a foliation of M. This would require knowing that $\theta[X,\overline{Y}]=0$ for X,Y sections of $T^{0,1}M$. The formula of Cartan states that

$$\theta[X,\overline{Y}]=X\theta(\overline{Y})-\overline{Y}\theta(X)-d\theta(X,\overline{Y}). \tag{2.9.9}$$

Thus we see that true integrability properties of $\ker\theta$ are determined by $d\theta$.

For analytic purposes it is essential to know whether or not the manifold M contains any holomorphic submanifolds. If this were the case then we could find a vector field $\bar{Z}\in T^{0,1}M$ such that $[Z,\bar{Z}]=\alpha Z+\beta\bar{Z}$ and therefore

$$\theta([Z,\bar{Z}])=d\theta(Z,\bar{Z})=0. \tag{2.9.10}$$

If on the other hand the hermitian pairing defined on $T^{0,1}M$ by

$$L(\bar{Z},\bar{Z})=id\theta(Z,\bar{Z}). \tag{2.9.11}$$

is definite then M has no holomorphic submanifolds. The hermitian form defined in (2.9.11) is called the Levi form. It is defined up to a conformal factor. We can replace θ with a non–vanishing multiple $f\theta$; the Levi forms are related by:

$$d(f\theta)\restriction_{T^{1,0}M+T^{0,1}M}=fd\theta\restriction_{T^{1,0}M+T^{0,1}M}. \tag{2.9.12}$$

The complex structure defines an orientation on $\ker\theta$ thus if M is oriented then we can pick a definite sign for the conformal class of θ and therefore the signature of Levi form is well defined. We denote the signature by (m,n,p) if L is positive on a m–dimensional subspace, negative on an n–dimensional subspace and degenerate on a p–dimensional subspace. When we can choose a definite sign for the conformal class of θ we will say that M has a CR–orientation.

Definition 2.9.13**.** Suppose that M is an CR–oriented manifold such that the Levi form is positive definite then we say the structure is strictly pseudoconvex.

Now we return to the case of a domain $\Omega\subset\mathbb{C}^n$. Let ρ be a defining function for $\partial\Omega$, this implies that

$$\Omega=\{z;\rho(z)<0\}$$

with $d\rho(z)\neq 0$, $z\in\partial\Omega$. Since $d\rho=\partial\rho+\overline{\partial}\rho$ it follows that

$$-\partial\rho\restriction_{\partial\Omega}=\overline{\partial}\rho\restriction_{\partial\Omega}. \tag{2.9.14}$$

On the other hand a vector, $X\in T_z\Omega, z\in\partial\Omega$ is tangent to $\partial\Omega$ if and only if $d\rho(z)(X)=0$. Thus if $\bar{Z}\in T^{0,1}_z\Omega$, $z\in\partial\Omega$ then it is tangent to $\partial\Omega$ if anf only if $\overline{\partial}\rho(z)(\bar{Z})=0$. This proves the following geometric lemma.

Lemma 2.9.15**.** *Suppose that $\Omega\subset\mathbb{C}^n$ has a $\mathcal{C}^1$–defining function ρ then*

$$T^{0,1}_z\partial\Omega=\ker\overline{\partial}\rho\restriction_{T^{0,1}_z\Omega}.$$

We let

$$\theta=-ij^*\overline{\partial}\rho, \tag{2.9.16}$$

here $j : \partial\Omega \hookrightarrow \Omega$ is the inclusion of the via a biholomorphic map. Evidently a CR–structure defined in this way has a CR–orientation: it is fixed by the hypothesis that a defining function is negative in Ω. Suppose that F is a biholomorphic map defined in a neighborhood, $U \subset \overline{\Omega}$ of $q \in \partial\Omega$. Suppose that it carries the interior of Ω locally onto the exterior. We suppose that ρ is defined in a neighborhood of $\partial\Omega$.

Since F is biholomorphic it follows that

$$F^*\partial\overline{\partial}\rho = \partial\overline{\partial}F^*\rho. \tag{2.9.17}$$

Since $-F^*\rho$ is also a defining function for $\partial\Omega$ near to q it follows from (2.9.17) that the signature of the Levi form satisfies

$$(m(F(q)), n(F(q)), p(F(q))) = (n(q), m(q), p(q)).$$

From this we deduce the proposition

Proposition 2.9.18. *If Ω is pseudoconvex and the Levi form has at least one positive eigenvalue at each point then there is no biholomorphic map carrying the $\partial\Omega$ to itself and mapping the interior of Ω onto the exterior.*

This already shows that there is a significant difference between the biholomorphic equivalence problem for one and several variables.

Now we return to the problem of holomorphic extension. As we shall see it is essentially a local problem in $\mathbb{C}^n$. Let ρ be a C^4 function defined in a neighborhood of $0 \in \mathbb{C}^n$ suppose that $\rho(0) = 0$ and that $d\rho(0) \neq 0$. Thus there is open set $U \subset \mathbb{C}^n$ such that $M = U \cap \{z;\ \rho(z) = 0\}$ is a smooth hypersurface. The next result shows the interaction between the problem of holomorphic extension and the signature of the Levi form.

Lewy Extension Theorem 2.9.19. *Let ρ be as above, assume that there exists a vector $Z \in T^{1,0}\mathbb{C}^n$ such that*

$$\partial\rho(0)(Z) = 0 \text{ and } \partial\overline{\partial}\rho(0)(Z, \bar{Z}) < 0.$$

Then there exists a neighborhood $\omega \subset U$ of 0 such that for every function $v \in C^4(\omega)$ which satisfies the tangential Cauchy Riemann equations,

$$\overline{\partial}v \wedge \overline{\partial}\rho \restriction_{\partial\Omega} = 0,$$

there is a function $V \in C^1(\omega)$ such that

$$v = V \text{ along } \rho = 0 \text{ and } \overline{\partial}V = 0 \text{ in } \omega_+, \tag{2.9.20}$$

where

$$\omega_+ = \{z \in \omega;\ \rho(z) \geq 0\}. \tag{2.9.21}$$

Proof. We will reduce this to the case considered in Theorem (2.4.8). First we change variables to obtain a somewhat simpler form for the defining function. The hypotheses imply that after a linear change of coordinates we can assume that

$$\begin{aligned} \rho(z, \bar{z}) = x_n + &\sum_{i,j=1}^{n} \partial_{z_i}\partial_{\bar{z}_j}\rho(0) z_i \bar{z}_j + \\ &\operatorname{Re} \sum_{i,j=1}^{n} \partial_{z_i}\partial_{z_j}\rho(0) z_i z_j + O(|z|^3). \end{aligned} \tag{2.9.22}$$

We make the holomorphic change of variables

$$z'_j = z_j, j = 1, \ldots, n-1,\ z'_n = z_n + \sum_{i,j=1}^{n} \partial_{z_i}\partial_{z_j}\rho(0) z_i z_j.$$

In terms of the new coordinates the defining function assumes the simpler form:

$$\rho = \operatorname{Re} z_n' + \sum_{i,j=1}^{n} \partial_{z_i}\partial_{\bar z_j}\rho(0) z_i'\bar z_j' + O(|z|^3). \tag{2.9.23}$$

To simplify notation we drop the primes and use A_{ij} to denote the matrix of the hermitian form in (2.9.23).

The hypothesis implies that the form

$$\sum_{i,j=1}^{n-1} A_{ij} z_i \bar z_j$$

is not positive definite. By a linear change of coordinates we may achieve that $A_{11} < 0$. Therefore, since

$$\rho(z_1, 0, \dots, 0) = A_{11}|z_1|^2 + O(|z_1|^3),$$

we can choose a $\delta > 0$ and then an $\epsilon > 0$ so that

$$\partial_{z_1}\partial_{\bar z_1}\rho(z) < 0 \text{ if } z \in \omega = \{z \in V; |z_1| < \delta, |z_2| + \cdots + |z_n| < \epsilon\}$$

and $\rho(z) < 0$ on the part of $\partial\omega$ where $|z_1| = \delta$. If we fix $z_2, \dots, z_n$ with $|z_2| + \cdots + |z_n| < \epsilon$ then the set of z_1 with $|z_1| \le \delta$ where $\rho(z) < 0$ is a connected set. This is so because ρ is negative where $|z_1| = \delta$ thus if there were two components then one would necessarily be compact and therefore ρ would have a local minimum in that component. This violates the hypothesis on $\partial_{z_1}\partial_{\bar z_1}\rho$ in ω. The point to this construction is that the boundaries of the disks contained in ω defined by $z_2, \dots, z_n$ constant do not intersect the hypersurface $\rho = 0$. This will allow us to use the Cauchy integral to solve the $\overline{\partial}$–equation.

Now we will construct a function V which agrees with v where $\rho = 0$ and is holomorphic in ω_+. First we construct a function V_0 in $C^2(\omega)$ such that

$$\overline{\partial} V_0 = O(\rho^2) \text{ along } \rho = 0. \tag{2.9.24}$$

By assumption

$$\overline{\partial} v = h_0 \overline{\partial}\rho + \rho h_1,$$

where $h_0 \in C^3(\omega)$ and $h_1 \in C^2(\omega; \Lambda^{0,1})$. Hence

$$\overline{\partial}(v - h_0\rho) = \rho(h_1 - \overline{\partial} h_0) = \rho h_2 \text{ where } h_2 \in C^2(\omega; \Lambda^{0,1}).$$

Since $\overline{\partial}(\rho h_2) = \overline{\partial}\rho \wedge h_2 + \rho\overline{\partial} h_2 = 0$ we have that $\overline{\partial}\rho \wedge h_2 = 0$ where $\rho = 0$. Thus we can write

$$h_2 = h_3 \overline{\partial}\rho + \rho h_4$$

where $h_3 \in C^2(\omega)$ and $h_4 \in C^1(\omega; \Lambda^{0,1})$. We set

$$V_0 = v - h_0\rho - \frac{1}{2} h_3 \rho^2,$$

to obtain that

$$\overline{\partial} V_0 = \rho^2 (h_4 - \frac{1}{2}\overline{\partial} h_3). \tag{2.9.25}$$

Note that at this point we could take $h_5 = (h_4 - \frac{1}{2}\overline{\partial} h_3)$ the condition $\overline{\partial}\rho^2(h_4 - \frac{1}{2}\overline{\partial} h_3) = 0$ implies that

$$h_5 = h_6 \overline{\partial}\rho + \rho h_7.$$

If we set $V_1 = V_0 - \frac{1}{3}\rho^3 h_6$ then $\overline{\partial} V_1 = O(\rho^3)$. Evidently if the data is sufficiently differentiable we can continue this indefinitely and obtain a function V' such that $\overline{\partial} V'$ vanishes to infinite order along $\rho = 0$.

To complete this argument V_0 suffices. Define a $(0,1)$–from in ω by

$$\alpha = \begin{cases} \overline{\partial} V_0 & \text{in } \omega_+ \\ 0 & \text{in } \omega \setminus \omega_+. \end{cases}$$

In virtue of (2.9.25) it is clear that $\alpha \in C^1(\omega; \Lambda^{0,1})$and $\overline{\partial}\alpha = 0$. By the construction of ω, $\alpha = 0$ in a neighborhood of $\omega \cap \{|z_1| = \delta\}$. If

$$\alpha = \sum_{j=1}^{n} \alpha_j d\bar{z}_j,$$

then we set

$$f(z_1, z') = \frac{1}{2\pi\imath} \iint\limits_{|z_1| \leq \delta} \frac{\alpha_1(\tau, z') d\tau \wedge d\overline{\tau}}{(\tau - z_1).} \tag{2.9.26}$$

Clearly $f \in C^1(\omega)$. Since the integrand in (2.9.26) is compactly supported the integrability conditions and the complex version of Stokes' theorem imply that

$$\overline{\partial} f = \alpha \text{ in } \omega$$

and furthermore f vanishes in $\omega \setminus \omega_+$. This is because f is holomorphic in $\omega \setminus \omega_+$ and vanishes in an open subset. To see this we observe that $\{(z_1, 0, \dots, 0) : |z_1| \leq \delta\}$ lies in $\{\rho^{-1}((-\infty, 0])\}$. Along this disk $\nabla\rho = (1 + O(\epsilon))\partial_{x_n}$. Hence disks of the form $\{(z_1, z') : |z_1| \leq \delta\}$ with $\operatorname{Re} z_n < 0$ and the remaining coordinates $(z_2, \dots, z_{n-1})$ sufficiently small are contained in $\{\rho^{-1}((-\infty, 0))\}$ and therefore $\alpha_1 = 0$ on these disks. The argument given above showed that $\omega \setminus \omega_+$ is connected therefore f must vanish in the whole set. Since f is differentiable it follows that $f = 0$ along $\rho = 0$ consequently if we set

$$V = V_0 - f$$

then

$$\overline{\partial} V = 0 \text{ in } \omega_+ \text{ and } V \restriction_{\rho=0} = v \restriction_{\rho=0} .$$

This completes the proof of the theorem.

This theorem has several important corollaries

Corollary 2.9.27. *If ρ is plurisubharmonic along $\rho = 0$ and the Levi form has at least one positive eigenvalue then any function u defined on $\rho = 0$ which satisfies the tangential Cauchy Riemann equations has an extension to a neighborhood of $\rho < 0$ as a holomorphic function. Furthermore the extension is local.*

This shows quite clearly that the property of being a boundary value of a holomorphic function is very different in one and several variables.

As another corollary we see that if the Levi form of $\{\rho = 0\}$ has eigenvalues of both signs then the theorem implies that any function which satisfies the tangential Cauchy Riemann equations extends to be holomorphic in a full neighborhood of $\rho = 0$. This is in sharp contrast to the case of a pseudoconvex domain where there exist holomorphic functions which are smooth up to the boundary but do not extend.

Another case to consider is when the Levi form is identically zero. In this case the boundary is foliated by complex manifolds. For example the real hyperplane in $\mathbb{C}^n$ given by $y_n = 0$. The tangential CR–equations are the

$$\partial_{\bar{z}_j} u = 0, j = 1, \dots, n-1.$$

Clearly any function of x_n satisfies these equations but in general has no extension as a holomorphic function.

We close this section with two more corollaries.

Corollary 2.9.28**.** *Suppose that $\Omega \subset \mathbb{C}^n$ is a connected bounded domain with a C^3 boundary. Then any solution $u \in C^3(\Omega)$ of the tangential Cauchy Riemann equations has an extension to Ω as a holomorphic function.*

Proof. Let u be a function on $\partial\Omega$ satisfying the tangential CR–equations. Using the argument use in the proof of the Lewy extension theorem we can construct a function V_0 in $C^2(\overline{\Omega})$ such that

$$\overline{\partial} V_0 = O(\rho^2), \quad V_) \restriction_{b\Omega} = u.$$

As before we define

$$\alpha(z) = \begin{cases} \overline{\partial} V_0 \text{ for } z \in \Omega, \\ 0 \text{ for } z \in \Omega^c. \end{cases}$$

This $(0,1)$-form is continuously differentiable, closed and compactly supported. Using the Dolbeault lemma we obtain a compactly supported C^1-function f such that

$$\overline{\partial} f = \alpha, \quad f \restriction_{\Omega^c} = 0.$$

Setting $U = V_0 - f$ gives the desired extension of u. Evidently $\overline{\partial} U = 0$ in Ω and $U \restriction_{b\Omega} = u$.

Note that if we have a solution to $\overline{\partial}_b u = 0$ on the entire boundary of domain then we do not need a convexity hypothesis to obtain a holomorphic extension. This result, known as Bochner's theorem is really an extension of the Hartogs extension theorem.

Corollary 2.9.29**.** *Suppose that ρ satisfies the hypotheses of* (2.9.19) *and v satisfies the tangential Cauchy Riemann equations, then the extension of v as a holomorphic function to ω_+ is unique.*

Proof. The Lewy extension result provides a local procedure for extending v to ω_+. In the normal form given for ρ it is clear that $\partial_{z_1}\rho(z_1, z')$ does not vanish along the locus $\rho = 0$ away from $z_1 = 0$. Thus for an open set of z' the boundary of the set of z_1 such that $\rho(z_1, z') > 0$ is a single smooth circle. Thus we can apply the Cauchy integral formula to obtain a representation of the extension of v to ω_+ in terms of its values on $\rho = 0$. This formula is valid in open subset of ω_+ and thus establishes the uniqueness of the continuation.

This result has as a corollary the fact that if a holomorphic function vanishes on an open subset of a hypersurface with nondegenerate Levi form then it is, in fact, identically zero.

2.10 The Weierstraß preparation theorem

This is the last theorem in the local theory of holomorphic functions which we will consider. It provides a local description of the zero set of a holomorphic function. This result is essential in the local study of the intersections of analytic varieties and the study of the local ring structure of germs of holomorphic functions. It is the higher dimensional analogue of the one dimensional result to the effect that the behavior of a holomorphic function near z_0 is determined by the order of vanishing of $f(z) - f(z_0)$. That is there is a unique integer n and a holomorphic function $v(z)$ such that

$$f(z) = (z - z_0)^n v(z), \quad v(z_0) \neq 0.$$

The function, v is a 'unit' in the ring of germs of holomorphic functions at z_0, thus we have a unique factorization in this ring.

The simplest holomorphic functions are polynomials, next simpler might be functions of the form

$$P(w, z) = \sum_{i=0}^{k} a_i(z) w^i \tag{2.10.1}$$

where $a_i(z), i = 0, \ldots, k$ are holomorphic near to zero. If we suppose that $a_k(0) \neq 0$ but $a_i(0) = 0, i = 1, \ldots, k$ then the zero set of P has a simple local description as a branched cover of $\mathbb{C}^n$. More generally if $F(w,z) = q(w,z)P(w,z)$ with $q(0,0) \neq 0$ then the zero set of F is also a branched cover. Moreover q is a unit and we could try proving a unique factorization theorem by studying polynomials with analytic coefficients. In fact every holomorphic function has a local representation of this sort.

Suppose that F is holomorphic in some neighborhood of $0 \in \mathbb{C}^{n+1}$ and not identically zero. From this we conclude that there is some direction $Z \in \mathbb{C}^{n+1}$ such that the function of one variable $f(w) = F(wZ)$ is not identically zero. Let k be the non–negative integer such that

$$f(w) = w^k g(w), \quad g(0) \neq 0. \tag{2.10.2}$$

We introduce coordinates (z, w) into $\mathbb{C}^{n+1}$ so that wZ corresponds to $z = 0$.

Weierstraß Preparation Theorem 2.10.3. *Suppose that $F(w,z)$ is holomorphic in a neighborhood of $(0,0) \in \mathbb{C}^{n+1}$ and satisfies* (2.10.2) *then there exists a analytic polynomial*

$$P(w,z) = w^k + \sum_{i=1}^{k-1} a_i(z) w^i,$$

and a holomorphic function $q(w,z)$ defined in a neighborhood of $0 \in \mathbb{C}^{n+1}$ such that $q(0,0) \neq 0$ and

$$F(w,z) = q(w,z)P(w,z). \tag{2.10.4}$$

This theorem is a special case of another result called the Weierstraß division theorem. This provides a generalization of the 'Euclidean' algorithm for dividing polynomials to the context of holomorphic function in several variables.

The Weierstraß Division Theorem 2.10.5. *Let $F(w,z)$ and k be as above and $G(w,z)$ an arbitrary holomorphic function defined in a neighborhood of $(0,0)$ then there exist unique holomorphic functions $q(w,z)$ and $r(w,z)$ such that*

$$G(w,z) = q(w,z)F(w,z) + r(w,z)$$

where

$$r(w,z) = \sum_{i=0}^{k-1} a_i(z) w^i.$$

(2.10.5)$\Longrightarrow$(2.10.3). To deduce (2.10.3) from (2.10.5) we simply let $G(w,z) = w^k$, then

$$w^k = q(w,0)F(w,0) + r(w,0). \tag{2.10.6}$$

From the hypotheses on F and the form of r it is clear that $q(0,0) \neq 0$ thus

$$F(w,z) = \frac{1}{q(w,z)}(w^k - r(w,z)).$$

This proves (2.10.3).

Uniqueness in (2.10.5). This follows from a simple application of Rouchè's theorem in one complex variable. Suppose that

$$G(w,z) = q_1 F + r_1 = q_2 F + r_2, \tag{2.10.7}$$

then

$$r_1 - r_2 = F(q_2 - q_1). \tag{2.10.8}$$

Since $F(w,0) = w^k g(w)$ it follows that for z sufficiently near to 0 that $F(w,z)$ has at least k zeros in some small disk about 0. On the other hand $r_1 - r_2$ is a polynomial of degree at most $k-1$ in w. Therefore (2.10.8) implies that $r_1 - r_2$ is identically zero.

The construction of q, r uses a slightly indirect argument. We define

$$P_k(w,\lambda) = w^k + \sum_{i=1}^{k-1} \lambda_i w^i.$$

This is a holomorphic function in $\mathbb{C} \times \mathbb{C}^k$. We have a second division theorem from which we can deduce the first:

Polynomial Division Theorem 2.10.9. *Let $G(w,z)$ be a holomorphic function defined in a neighborhood of 0 then there exist holomorphic functions $q(w,z,\lambda)$ and $r(w,z,\lambda)$ in a neighborhood of 0 in $\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^k$ such that*

$$G(w,z)=q(w,z,\lambda)P_k(w,\lambda)+r(w,z,\lambda),$$

where

$$r(w,z,\lambda)=\sum_{i=0}^{k-1}r^i(z,\lambda)w^i.$$

(2.10.9)$\Longrightarrow$(2.10.5). We apply the polynomial division theorem to F,G to obtain

(2.10.10) $$F=q_FP_k+r_F \quad G=q_GP_k+r_G.$$

We can deduce a few facts about q_F and r_F:

(2.10.11) $$q_F(0)\neq 0 \text{ and } r_F^i(0,0)=0, i=0\dots,k-1.$$

To prove these statements we let $z=0,\lambda=0$ in (2.10.10),

(2.10.12) $$w^kg(w)=F(w,0)=q_F(w,0,0)w^k+\sum_{i=0}^{k-1}r_F^i(0,0)w^i.$$

The statements follows by comparing coefficients of powers of w in (2.10.12). Now we set $f_i(\lambda)=r_F(0,\lambda)$, we claim that

$$\det\partial_{\lambda_j}f_i(0)\neq 0.$$

To prove this we set $z=0$ in (2.10.10) and differentiate with respect to λ_j, evaluating at $\lambda=0$ we obtain

$$0=\partial_{\lambda_j}q_F(w,0,0)w^k+w^j+\sum_{i=0}^{k-1}\partial_{\lambda_j}f_i(0)w^i.$$

By comparing coefficients of w^i we deduce that

(2.10.13) $$\partial_{\lambda_j}f_i(0)=0,\text{ if } i>j \text{ and } \partial_{\lambda_i}f_i(0)=-q_F(0,0).$$

From (2.10.13) it follows that

(2.10.14) $$\det\partial_{\lambda_j}f_i(0)=(-q_F(0,0))^k.$$

In light of (2.10.11) and (2.10.14) we can apply the holomorphic implicit function theorem to find holomorphic functions $\theta_i(z), i=0,\dots,k-1$, defined in some neighborhood of zero, such that

(2.10.15) $$\theta_i(0)=0, \quad r_F^i(z,\theta(z))=0, i=0,\dots,k-1.$$

Substituting from (2.10.15) into (2.10.8) we derive that

(2.10.16) $$F(w,z)=q_F(w,z,\theta(z))P_k(w,\theta(z)).$$

Since $\theta(0)=0$ and $q_F(0,0,0)\neq 0$ we can write

$$G(w,z)=q(w,z)F(w,z)+r(w,z)$$

where

(2.10.17) $$q(w,z)=\frac{q_G(w,z,\theta(z))}{q_F(w,z,\theta(Z))}, \quad r(w,z)=\sum_{i=0}^{k-1}r_G^i(z,\theta(z))w^i.$$

This completes the deduction of (2.10.5) from (2.10.8).

All that remains is to prove (2.10.8).

Proof of (2.10.8). This follows from the Cauchy integral formula. We can write

$$G(w, z) = \frac{1}{2\pi\imath} \int_{\gamma} \frac{G(\eta, z) d\eta}{(\eta - w)} \tag{2.10.18}$$

where γ is some contour enclosing w. By comparing coefficients it follows that

$$P_k(w, \lambda) - P_k(\eta, \lambda) = (\eta - w) \sum_{i=0}^{k-1} s_i(\eta, \lambda) w^i$$

where $s_i(\eta, \lambda)$ are analytic functions. Dividing we obtain

$$\frac{P_k(w, \lambda)}{\eta - w} - \frac{P_k(\eta, \lambda)}{\eta - w} = \sum_{i=0}^{k-1} s_i(\eta, \lambda) w^i. \tag{2.10.19}$$

As $P_k(\eta, 0) = \eta^k$, we can find a contour, γ along which $P_k(\eta, \lambda)$ does not vanish for sufficiently small λ. Integrating along this contour we obtain

$$G(w, z) = \frac{1}{2\pi\imath} \int_{\gamma} \frac{G(\eta, z) d\eta}{(\eta - w)} \frac{P_k(\eta, \lambda)}{P_k(\eta, \lambda)}. \tag{2.10.20}$$

Using (2.10.19) in (2.10.20) we get

$$\begin{aligned} G(w, z) =& \frac{1}{2\pi\imath} \int_{\gamma} \frac{G(\eta, z) d\eta}{(\eta - w)} \frac{P_k(w, \lambda)}{P_k(\eta, \lambda)} - \\ & \sum_{i=0}^{k-1} \left(\frac{1}{2\pi\imath} \int_{\gamma} G(\eta, z) s_i(\eta, \lambda) d\eta\right) w^i. \end{aligned} \tag{2.10.21}$$

The theorem follows by setting

$$q(w, z, \lambda) = \frac{1}{2\pi\imath} \int_{\gamma} \frac{G(\eta, z) d\eta}{(\eta - w)} \frac{1}{P_k(\eta, \lambda)} \text{ and}$$

$$r(w, z, \lambda) = -\sum_{i=0}^{k-1} \left(\frac{1}{2\pi\imath} \int_{\gamma} G(\eta, z) s_i(\eta, \lambda) d\eta\right) w^i.$$

Exercise 2.10.22**.**

(1) Show that if F and G are polynomials as in (2.10.1) then the function q in (2.10.5) is as well.

(2) Show that if F is a polynomial as in (2.10.1) and has a factorization of the form $F = g_1 g_2$ where g_1, g_2 are holomorphic, then they can be taken to be polynomials.

(3) Using the previous part and the theorem from commutative algebra that if a ring R is a unique factorization domain, then so is $R[z]$, show that the holomorphic functions defined in a neighborhood of $0 \in \mathbb{C}^{n+1}$ is a unique factorization domain. Note that the units are functions which do not vanish at 0.

2.11 The Bergman kernel function

Attached to a domain in $\mathbb{C}^n$ or more generally any complex manifold is the so called Bergman kernel function. Let Ω be a domain in $\mathbb{C}^n$, for each $w \in \Omega$ we define a linear functional on $H^2(\Omega)$ by setting

$$l_w(f) = f(w).$$

It follows from Corollary 2.A.6 that these functionals are all bounded. Since $H^2(\Omega)$ is a Hilbert space the Riesz representation theorem implies that, for each $w \in \Omega$ there is a unique element $h_w \in H^2(\Omega)$ such that

$$l_w(f) = \int_\Omega f(z)\overline{h_w(z)} dz \wedge d\bar{z}.$$

Exercise 2.11.1**.** Prove that

$$\|h_w\|_{L^2} = \sup_{f \in H^2(\Omega) \setminus \{0\}} \frac{|f(w)|}{\|f\|_{L^2}}.$$

Let $\{f_j(z)\}$ be an orthonormal basis for $H^2(\Omega)$. For each w we have an expansion

$$h_w = \sum_{j=1}^\infty a_j f_j$$

where the coefficients are given by

$$a_j =< h_w, f_j >= \overline{f_j(w)}.$$

Hence we see that

$$h_w(z) = \sum_{j=1}^\infty f_j(z)\overline{f_j(w)}$$

and therefore

$$\|h_w\|_{L^2}^2 = \sum_{j=1}^\infty |f_j(w)|^2 < \infty.$$

It follows from the Cauchy-Schwartz inequality that the sum defining $h_w(z)$ converges locally uniformly and in fact any derivative does as well.

We let

$$B_\Omega(z, w) = \sum_{j=1}^\infty f_j(z)\overline{f_j(w)}.$$

This is the Schwartz kernel of the Bergman projection operator: if $f \in L^2(\Omega)$ then

$$B_\Omega f = \int_\Omega B_\Omega(z, w) f(w) dw \wedge d\bar{w} \in H^2(\Omega).$$

If $f \in H^2(\Omega)$ then $B_\Omega f = f$; since $B_\Omega(z, w) = \overline{B_\Omega(w, z)}$ the operator defined by B_Ω is an orthogonal projection. From these observations, or the construction above it is clear that $B_\Omega(z, w)$ does not depend on the choice of orthonormal basis for $H^2(\Omega)$. We let

$$k_\Omega(z) = B_\Omega(z, z),$$

this is called the Bergman kernel function. It has many remarkable properties. Using Exercise 2.11.1 we obtain a variational characterization of the $k_\Omega(z)$.

Exercise 2.11.1'**.** Prove that
$$\sqrt{k_\Omega(z)} = \sup_{f \in H^2(\Omega)\setminus\{0\}} \frac{|f(z)|}{\|f\|_{L^2}}.$$
Show that if $\Omega_1 \subset \Omega_2$ and $z \in \Omega_1$ then
$$k_{\Omega_1}(z) \geq k_{\Omega_2}(z).$$
This gives a practical way to approximate the Bergman kernel function on domains with smooth strictly pseudoconvex boundaries.

To see certain of these properties more clearly it is better to rephrase the construction in terms of holomorphic $(n,0)$-forms. On a domain in $\mathbb{C}^n$ a holomorphic $(n,0)$-form, α has a representation in the form
$$\alpha = f dz_1 \wedge \cdots \wedge dz_n.$$
What makes $(n,0)$-forms special is that they have a canonically defined inner product which does not require a choice of metric. This is because $\alpha \wedge \overline{\alpha}$ is an (n,n)-form which can be integrated over Ω. Indeed it is this point of view which generalizes to the manifold context. We can simply repeat our discussion above replacing the orthonormal basis $\{f_j\}$ for $H^2(\Omega)$ by an orthonormal basis
$$\alpha_j = f_j dz \text{ where } dz = dz_1 \wedge \cdots \wedge dz_n$$
for $H^2(\Omega; \Lambda^{n,0})$. The Bergman projector is now
$$B_\Omega(z,w) = \sum \alpha_j(z) \wedge \overline{\alpha_j(w)}.$$
The reason for this shift in point of view is the following observation. Suppose that Ω_1 and Ω_2 are bounded domains and
$$F : \Omega_1 \longrightarrow \Omega_2$$
is a biholomorphic mapping. If $\{\beta_j\}$ is an orthonormal basis for $H^2(\Omega_2; \Lambda^{n,0})$ then $\{F^*(\beta_j)\}$ is an orthonormal basis for $H^2(\Omega_1; \Lambda^{n,0})$. This is because for $\beta, \gamma \in H^2(\Omega_2; \Lambda^{n,0})$ the change of variable formula for integrals implies that
$$\int_{\Omega_2} \beta \wedge \overline{\gamma} = \int_{\Omega_1} F^*(\beta) \wedge \overline{F^*(\gamma)}.$$
From this we obtain the theorem

Theorem 2.11.2**.** *If Ω_1 and Ω_2 are domains in $\mathbb{C}^n$ and $F : \Omega_1 \to \Omega_2$ is a biholomorphic map then*
$$F_2^*(B_{\Omega_2}) = B_{\Omega_1}.$$
Here $F_2(z,w) = (F(z), F(w))$.

Let $F'(z)$ denote the Jacobian of the map F and $J_F(z) = \det F'(z)$. As a corollary of this theorem we get the transformation formula for the Bergman kernel function

Corollary 2.11.3**.** *Under the hypotheses of Theorem* 2.11.2 *we see that*

$$k_{\Omega_1}(z) = k_{\Omega_2}(F(z))|J_F(z)|^2. \tag{2.11.4}$$

This observation motivates the following construction. We define a $(1,1)$ form κ_Ω by setting
$$\kappa_\Omega = \partial\overline{\partial}\log(k_\Omega).$$
Because $\partial\overline{\partial}\log|f|^2 = 0$ for a non-vanishing holomorphic function, we obtain the following invariance property for κ_Ω.

Proposition 2.11.5. *Under the hypotheses of Theorem* 2.11.2

$$F^*(\kappa_{\Omega_2}) = \kappa_{\Omega_1}.$$

The $(1,1)$-form κ_Ω defines a pseudometric on Ω, called the Bergman pseudometric. For a domain in $\mathbb{C}^n$ it is easy to show that this pseudometric is actually a metric at every point, that is $\log k_\Omega$ is a strictly plurisubharmonic function at every point of Ω.

Exercise 2.11.6. Prove this statement.

Thus we see that the Bergman kernel function defines a bi-holomorphically invariant metric on a domain in $\mathbb{C}^n$. Note that this immediately implies the following theorem:

Theorem 2.11.7. *Let Ω be a connected domain in $\mathbb{C}^n$ and let G_Ω be the group of bi-holomorphic maps of Ω onto itself then G_Ω is a Lie group and the identity component is finite dimensional.*

Proof. We will take for granted that the identity component, G^0_Ω is a Lie group and concentrate on proving that it is finite dimensional. Fix a point $x \in \Omega$ then the condition $g \sim h$ if $g \cdot x = h \cdot x$ defines an equivalence relation on G^0_Ω. Let $\Omega_x = \{g \cdot x : x \in G^0_\Omega\}$. In the process of showing that G^0_Ω is Lie group one shows that Ω_x is a smooth manifold and that $G^0_\Omega \to \Omega_x$ is a fibration, indeed a principal bundle with fiber G_x, the subgroup of G^0_Ω consisting of the elements that fix x (the stabilizer of x). The equivalence classes are of the form $g \cdot G_x$. It therefore suffices to show that G_x is finite dimensional. If $g \in G_x$ then $g(x) = x$ and $dg : T_x\Omega \to T_x\Omega$ is an isometry of the metric $\kappa_\Omega(x)$. Thus we have a faithful representation of $G_x$in $U(n)$ thus showing that it is finite dimensional.

Suppose that $f(z, \bar{z})$ is a smooth function defined on Ω and $z = h(w)$ is a holomorphic mapping then we see that

$$\frac{\partial^2 h^*(f)}{\partial w_i \partial \bar{w}_j} = \sum_{p,q=1}^{n} \frac{\partial^2 f}{\partial z_p \partial \bar{z}_q} \circ h \frac{\partial h_p}{\partial w_i} \overline{\frac{\partial h_q}{\partial \bar{w}_j}}.$$

If J_h denotes the Jacobian of the mapping h then

$$\det \frac{\partial^2 h^*(f)}{\partial w_i \partial \bar{w}_j} = h^* \left(\det \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right) |\det J_h|^2. \tag{2.11.8}$$

Arguing as before we see that this implies that

$$\partial\overline{\partial} \log \det \partial\overline{\partial} h^*(f) = h^*(\partial\overline{\partial} \log \det \partial\overline{\partial} f). \tag{2.11.9}$$

If we fix holomorphic coordinates $(z_1, \ldots, z_n)$ in a domain Ω then

$$\kappa_\Omega = \sum_{i,j} \frac{\partial^2 \log k_\Omega}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j.$$

From (2.11.9) it follows that

$$R_\Omega = \partial\overline{\partial} \log \det \frac{\partial^2 \log k_\Omega}{\partial z_i \partial \bar{z}_j}$$

is well defined and does not depend on the choice of holomorphic of coordinates. This tensor is called the Ricci tensor.

If we have two domains and a biholomorphic map $h : \Omega_1 \to \Omega_2$ then combining this observation with (2.11.9) and Proposition 2.11.5 we see that

$$h^*(R_{\Omega_2}) = R_{\Omega_1}.$$

This can combined with the fact that $F^*(\kappa_{\Omega_2}) = \kappa_{\Omega_1}$ to obtain a scalar function which is invariant under biholomorphic mappings. If $\kappa_\Omega = g_{ij} dz_i \wedge d\bar{z}_j$ and $R_\Omega = R_{ij} dz_i \wedge d\bar{z}_j$ then we define

$$r_\Omega = \sum_{i,j=1}^{n} g^{ij} R_{ij},$$

where g^{ij} is the inverse matrix to g_{ij}. As both κ_Ω and R_Ω are $\partial\overline{\partial}$ of something it is clear that r_Ω is a scalar (in the tensor calculus) which is invariant under biholomorphic maps, if $h : \Omega_1 \to \Omega_2$ is a biholomorphic map then

$$r_{\Omega_1}(z) = r_{\Omega_2}(h(z)).$$

This gives a simple way to prove that the polydisk and the unit ball in $\mathbb{C}^n$ are biholomorphically inequivalent. Recall that the group of Möbius transformations acts transitively on the unit disk in $\mathbb{C}$,

$$z \longrightarrow e^{i\theta} \frac{z-\alpha}{1-\bar{\alpha}z},$$

here $\alpha \in D_1$. We denote this group $\mathrm{Aut}(D_1)$. The product of n-copies of $\mathrm{Aut}(D_1)$ acts transitively on D_1^n,

$$(z_1, \ldots, z_n) \longrightarrow \left(e^{i\theta_1} \frac{z_1-\alpha_1}{1-\bar{\alpha}_1 z_1}, \ldots, e^{i\theta_n} \frac{z_n-\alpha_n}{1-\bar{\alpha}_n z_n} \right),$$

here $(\alpha_1, \ldots, \alpha_n) \in D_1^n$.

If $F : B_n \to D_1^n$ were a biholomorphic equivalence then there would be no loss in generality in assuming that $F(0) = 0$. By computing the Bergman kernel functions for the ball and the polydisk we will see that $F^*(r_{D_1^n})$ cannot equal r_{B_n} which will in turn imply that such an F cannot exist. The Bergman kernel functions are given by the following formulæ

$$\begin{aligned} k_{D_1^n}(z) =& \frac{1}{\pi^n} \prod_{j=1}^n \frac{1}{(1-|z_j|^2)^2}, \\ k_{B_n}(z) =& \frac{n!}{\pi^n} \frac{1}{(1-|z|^2)^{n+1}}. \end{aligned}$$

Exercise 2.11.10**.** By using the fact that both the polydisk and the ball are complete, log-convex Reinhardt domains prove these formulæ.

Theorem 2.11.11**.** *If $n \geq 2$ then the unit ball and polydisk in $\mathbb{C}^n$ are biholomorphically inequivalent.*

Proof. Suppose there is a biholomorphic map $F : B_n \to D_1^n$; as remarked above we can assume that $F(0) = 0$. A computation shows that $r_{B_n}(0) = n(n+1)$ and $r_{D_1^n}(0) = 2n$. From this we conclude that F cannot exist.

Exercise 2.11.12**.** Prove these formulæ for $r_{D_1^n}(0)$ and $r_{B_n}(0)$. Show that $r_{D_1^n}(x) = r_{D_1^n}(0)$ for all $x \in D_1^n$.

Exercise 2.11.13**.** Let $A_{r,1} = \{z \in \mathbb{C} : r < |z| < 1\}$. Compute the Bergman kernel, metric and scalar curvature for each A_{r1}, with $0 \leq r < 1$. Be careful when $r = 0$!

2.12 Additional exercises

Exercise 2.12.1**.** Suppose that $\Omega \subset \mathbb{R}^n$ is a connected open set and f is a subharmonic function defined on Ω which is not identically $-\infty$. We will show that the set $\mathcal{P} = \{x \in \Omega : f(x) = -\infty\}$ has Lebesgue measure zero and that f is integrable over compact subsets of Ω. Hint: Use the mean value property of subharmonic functions.

(1) Let $x \in U \subset \Omega$ provided f is integrable over a neighborhood of x. Show that U is an open set and $U^c \subset \mathcal{P}$.
(2) Show that if $x \in U^c$ then $f = -\infty$ in a ball centered at x.
(3) Conclude that $U^c = \emptyset$ and therefore $\mathcal{P}$ has measure zero.

Exercise 2.12.2**.** Let $\Omega = \{(z_1, z_2) : |z_1| < |z_2| < 1\}$, the Hartogs triangle. Prove that there is no bounded plurisubharmonic function u on Ω which tends to 0 as z tends to $\partial\Omega$. Hint: You need to prove the following lemma

2.12.2' Lemma**.** *If u is a subharmonic function in $B_1(0) \setminus \{0\}$ then u extends to be subharmonic in $B_1(0)$.*

Exercise 2.12.3. Suppose that $\Omega \subset \mathbb{C}^n$ has a smooth boundary which is strongly pseudoconvex in a neighborhood of p. Suppose that $\varphi : D_1 \to \mathbb{C}^n$ is an analytic disk with image d and that $d \subset \Omega^c$, then

$$\liminf_{d \ni z \to p} \frac{\operatorname{dist}(z, \partial\Omega)}{|z-p|^2} > 0.$$

Exercise 2.12.14. Let $\Omega \subset \mathbb{C}^n$ for an $n \geq 3$. Show that Ω is pseudoconvex if and only if its intersection with every complex hyperplane is either empty or pseudoconvex. Why is the case $n = 2$ different?

3.1 The unit ball

The unit disk is a model for the complete, simply connected Riemannian manifold with constant negative curvature. The metric can be written in the form

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}. \tag{3.1.1}$$

We can identify the tangent space to the unit disk with the vectors of type $1,0$. When represented relative to the basis $\partial_z, \partial_{\bar{z}}$ a real vector takes the form

$$X = \alpha \partial_z + \overline{\alpha} \partial_{\bar{z}}. \tag{3.1.2}$$

From (3.1.2) it is clear that the map

$$X \longrightarrow \alpha \partial_z = Z_X$$

defines an isomorphism of $T\mathbb{CB}^1$ with $T^{1,0}\mathbb{CB}^1$ as a real vector space.

The $(1,1)$–form,

$$\omega = -\partial\overline{\partial}\log(1-|z|^2)$$

defines a hermitian pairing on $T^{1,0}$ by

$$h(W, Z) = \omega(W, \overline{Z}).$$

A simple calculation shows that

$$ds^2(X, Y) = 4 \operatorname{Re} h(Z_X, Z_Y).$$

Thus we have a relation between the strictly plurisubharmonic defining function and the hyperbolic geometry of the unit disk.

Using this connection we can easily show that the metric is invariant under all biholomorphic self maps of the unit disk. The Schwarz lemma implies that all such mappings are of the form:

$$w = \gamma z = \frac{az + b}{\overline{b}z + \overline{a}},\ |a|^2 - |b|^2 = 1.$$

An elementary computation establishes that

$$\gamma^*(1-|z|^2) = \frac{(1-|z|^2)}{(\overline{b}z + \overline{a})(b\bar{z} + a)}. \tag{3.1.3}$$

Because γ is a holomorphic map,

$$\gamma^* \partial\overline{\partial}\log(1-|z|^2) = \partial\overline{\partial}\gamma^*\log(1-|z|^2) = \partial\overline{\partial}\log(1-|z|^2), \tag{3.1.4}$$

from which the claim follows. Thus the biholomorphic self maps are isometries relative to the constant curvature metric and $\mathbb{CB}^1$ is a homogeneous space. It is isomorphic to

$$\mathbb{CB}^1 = SU(1,1)/U(1). \tag{3.1.5}$$

A simple count of dimensions shows that all orientation preserving isometries are of this form. Note that the element $-\operatorname{Id}$ acts trivially on the disk, so the true automorphism group is $PSU(1,1) = SU(1,1)/\{\operatorname{Id}, -\operatorname{Id}\}$.

For the purposes of generalization there is a different model which is better suited to computation. To define the projective model we consider $\mathbb{C}^2$ with the hermitian inner product

$$< X, Y >= X_1\overline{Y}_1 - X_2\overline{Y}_2.$$

A simple calculation verifies that if $A \in SU(1,1)$ then $< AX, AY >=< X, Y >$. Of course the metric induced by $\operatorname{Re} < \cdot, \cdot >$ has signature $(2,2)$ however if we restrict to the hypersurface given by

$$H = \{X; < X, X >= -1\}$$

then we get a metric of signature $(2,1)$. The unit circle acts on H via $X \longrightarrow e^{i\theta}X$. This action commutes with the action of $SU(1,1)$ on H and therefore allows us to define a metric on the quotient $H/U(1)$ by identifying the tangent space of the quotient with the orthocomplement of the vector field which generates the $U(1)$ action. Since the $U(1)$–action commutes with the action of $SU(1,1)$ this group acts as isometries of the quotient space. It is sometimes useful to have a second representation, we set

$$N = \{X :< X, X >< 0\} \text{ then } H/U(1) = N/\mathbb{C}^*.$$

If we use z_1, z_2 as local coordinates for $\mathbb{C}^2$ then the projection

$$\pi(z_1, z_2) = \frac{z_1}{z_2}$$

carries N onto $\mathbb{CB}^1$. In this representation it is a simple matter to deduce (3.1.3) and therefore (3.1.4). Suppose that $A \in \operatorname{SU}(1,1)$ then the action of A on $\mathbb{CB}^1$ is defined as follows,

$$A \cdot z = \frac{(A\begin{pmatrix} z \\ 1 \end{pmatrix})_1}{(A\begin{pmatrix} z \\ 1 \end{pmatrix})_2}. \tag{3.1.6}$$

By definition $A^*(|z_1|^2 - |z_2|^2) = |z_1|^2 - |z_2|^2$ thus setting $\rho = |z|^2 - 1$ we have

$$A^*(\rho) = \frac{\rho}{|(A\begin{pmatrix} z \\ 1 \end{pmatrix})_2|^2}. \tag{3.1.7}$$

The important point being that $A^*\rho/\rho$ is the squared modulus of a holomorphic function.

This construction easily generalizes to n–dimensions. Our goal is to find a metric on the unit ball in $\mathbb{CB}^n$ which is invariant under a very large group of holomorphic transformations. As before we consider $\mathbb{C}^{n+1}$ with the lorentz metric

$$< X, Y >= X_1\overline{Y}_1 + \cdots + X_n\overline{Y}_n - X_0\overline{Y_0}.$$

The lie group $SU(n,1)$ is defined as those matrices of determinant 1 such that

$$< AX, AY >=< X, Y >, \text{ for all } X, Y \in \mathbb{C}^{n+1}.$$

As before we let

$$H = \{X \in \mathbb{C}^{n+1}; < X, X >= -1\}, \quad N = \{X \in \mathbb{C}^{n+1}; < X, X >< 0\}.$$

Evidently H is invariant under the action of $SU(n,1)$ and also the action of $U(1)$ defined by $X \longrightarrow e^{i\theta}X$.

We define a projection of N into $\mathbb{C}^n$ by

$$\pi(X) = (\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}).$$

One easily sees that the image of H is exactly the interior of the unit ball. The map is not one to one as X and $e^{i\theta}X$ have the same projection, but this is easily seen to be the only possibility. Therefore

$$\mathbb{CB}^n \simeq H/U(1).$$

To obtain a representation as a homogeneous space we observe that

$$H/U(1) \simeq SU(n,1)/U(n). \tag{3.1.8}$$

Since $SU(n,1)$ acts transitively on $H/U(1)$ to prove (3.1.8) we need only compute the stabilizer of a single fiber in. This is simplest for the equivalence class $[(0,\dots,0,e^{i\theta})]$ The stabilizer is easily seen to have the form

$$\begin{bmatrix} & & & 0 \\ & e^{-i\theta}SU(n) & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & e^{in\theta} \end{bmatrix}.$$

This is simply $U(n) \hookrightarrow SU(n,1)$ as asserted.

The action of $SU(n,1)$ on $\mathbb{CB}^n$ is defined as before by

$$A\cdot z = \pi(A\pi^{-1}z) = \left(\frac{(A\begin{pmatrix} z \\ 1 \end{pmatrix})_1}{(A\begin{pmatrix} z \\ 1 \end{pmatrix})_0}, \dots, \frac{(A\begin{pmatrix} z \\ 1 \end{pmatrix})_n}{(A\begin{pmatrix} z \\ 1 \end{pmatrix})_0} \right).$$

We let $\rho = |z|^2 - 1 =< (z,1),(z,1) >$, using this formula and the invariance of the lorentz inner product one easily derives that

$$A^*(\rho) = \frac{\rho}{|(A\begin{pmatrix} z \\ 1 \end{pmatrix})_0|^2}. \tag{3.1.9}$$

As before the ratio $A^*\rho/\rho$ is the squared modulus of a holomorphic function.

Therefore the invariant metric on $\mathbb{CB}^n$ is given, as before, by

$$g = -\partial\overline{\partial}\log\rho$$

To see that it is invariant we need to show that $A^*g = g$ for all A in $SU(n,1)$. This follows immediately from (3.1.9) because the denominator is the squared modulus of a holomorphic function. This metric is called the Bergman metric. One can show that it agrees with the metric induced by π from the inner product $< \cdot, \cdot >$ on H, restricted to the orthogonal trajectories of the action of $U(1)$.

Using the group invariance one can show that if z, w are two points in $\mathbb{CB}^n$ then

$$\cosh\frac{1}{2}d(z,w) = \frac{|1-(z,w)|}{[(1-|z|^2)(1-|w|^2)]^{\frac{1}{2}}}. \tag{3.1.10}$$

Here $(\cdot,\cdot)$ is the standard hermitian inner product on $\mathbb{C}^n$.

Exercise 3.1.11**.**

(1) Prove that the function defined on the right hand side of (3.1.10) is invariant under the action of $SU(n,1)$.
(2) Prove that the real lines through 0 are geodesics of the Bergman metric.
(3) Prove the formula (3.1.10), hint: show that it suffices to consider $z = 0, w = (\zeta, 0, \dots, 0)$ and then compare (3.1.10) with the result of integrating the Bergman metric.
(4) Prove that the "Bergman" metric defined in this section is the same as that defined in section 2.11.

The group $SU(n,1)$ therefore acts as isometries of $\mathbb{CB}^n$ with the Bergman metric, however the action is not effective. The center of the group

$$Z_n = \{\begin{bmatrix} e^{im\omega}\,\mathrm{Id}_n & 0 \\ 0 & e^{im\omega} \end{bmatrix}, m \in \mathbb{Z}, \omega = \frac{2\pi}{n+1}\},$$

acts trivially. A dimension count shows that all biholomorphic isometries of the Bergman metric arise from elements of $SU(n,1)$. One can also show that all orientation preserving isometries of the Bergman metric are necessarily biholomorphic maps, thus

$$\mathrm{Isom}(\mathbb{CB}^n) = SU(n,1)/Z_n.$$

Since this metric is **the** Bergman it follows from Proposition 2.11.5 that all bi-holomorphic self maps of ball are isometries of this metric. From this we conclude that the group of biholomorphic self maps of the unit ball is isomorphic to $SU(n,1)/Z_n$.

Exercise 3.1.12**.**

(1) Prove that any isometry of the Bergman metric is a biholomorphic mapping. hint: By composing with elements of $SU(n,1)/Z_n$ one can reduce consideration to an isometry fixing the origin and consider the tangent map only at 0.
(2) Prove that the volume form of the Bergman metric is given by

$$\mathrm{dVol} = \frac{c_n\,\mathrm{dV_{euclid}}}{(1-|z|^2)^{n+1}}, \tag{3.1.13}$$

where c_n is a dimensional constant.

Exercise 3.1.14**.** Show that if $A \in SU(n,1)$ then one of the following is true:

(1) $A(z) = z$ for some $z \in \mathbb{B}_1^n$,
(2) $A(z_i) = z_i$ for exactly two points $z_1, z_2 \in \partial\mathbb{B}_1^n$ but A fixes no interior point,
(3) $A(z) = z$ for exactly one point on $\partial\mathbb{B}_1^n$, but A fixes no interior point.

What can you say about the eigenvalues of A is each case. Do the case $n = 1$ first.

3.2 Analysis on the unit ball

In this section we consider two analytic problems on the unit ball in $\mathbb{C}^n$: the first is constructing the resolvent kernel for the Laplace operator, the second is the construction of the Bergman projector. To keep technical difficulties to a minimum we only construct the resolvent for the Laplace operator acting on functions. Similar considerations lead to an analogous construction for the Laplace operator on p,q–forms.

As remarked in the introduction the space of $n,0$–forms has a canonical bilinear pairing which induces an L^2–structure. Bounded holomorphic $n,0$–forms belong to $L^2(\mathbb{CB}^n; \Lambda^{n,0})$, we denote the space of all L^2 holomorphic $n,0$ forms by $\mathcal{H}^2(\mathbb{CB}^n)$. The orthogonal projection from $L^2(\mathbb{CB}^n; \Lambda^{n,0})$ onto $\mathcal{H}^2(\mathbb{CB}^n)$ is called the Bergman projector. It can be represented by a kernel of the form

$$B(z,w)dz \wedge d\bar{w};$$

the action is then given by

$$\mathcal{B}\omega(z) = \int\limits_{\mathbb{CB}^n} \omega \wedge B(z,w) dz \wedge d\bar{w}.$$

Much of the analysis of holomorphic functions in several variables can be reduced to an analysis of the Bergman kernel function. Once one has solved the $\overline{\partial}$–Neumann problem one can easily construct the Bergman kernel. On the unit ball it is easy to construct the Bergman kernel directly. We conclude this section with that computation as motivation for what comes in the later sections.

To study the resolvent kernel for the Laplacian we employ the group invariance of the Laplace operator. Recall that the Bergman metric on the unit ball is given by

$$g_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{1-|z|^2} + \frac{z_{\bar{\imath}} z_j}{(1-|z|^2)^2}.$$

We can easily show that if $\gamma \in \mathrm{Isom}(()\mathbb{CB}^n)$ and $f \in \mathcal{C}_c^\infty(\mathbb{CB}^n)$ then

$$\gamma^*(\varDelta_{\mathrm{B}} f) = \varDelta_{\mathrm{B}} \gamma^* f. \tag{3.2.1}$$

The resolvent kernel is defined by the distributional equation

$$(\varDelta_{\mathrm{B}} - \lambda) R(\lambda) = \delta_\Delta \tag{3.2.2}$$

and decay properties for $\lambda \notin \operatorname{spec} \varDelta_{\mathrm{B}}$.

Suppose that we could find a fundamental solution with pole located at $0 \in \mathbb{CB}^n$, i.e. a solution to

$$(\varDelta_{\mathrm{B}} - \lambda) F_0 = \delta_0, \tag{3.2.3}$$

which is square integrable near to $|z| = 1$ if $\lambda \notin \operatorname{spec} \varDelta_{\mathrm{B}}$. Let γ carry p to 0 then (3.2.1) implies that

$$F_p(z) = F_0(\gamma \cdot z)$$

satisfies (3.2.3) with the singularity moved to p. Since the resolvent kernel is unique we do not expect the solution we obtain to (3.2.3) to depend on the 'angle', for otherwise we could obtain different functions F_p by choosing different different group elements γ with $\gamma \cdot p = 0$.

In fact if $\gamma \in \mathrm{Isom}(()\mathbb{CB}^n)$ fixes 0 then $\gamma^* F_0$ is another solution to (3.2.3). By averaging over the stabilizer of $0 \simeq U(n)$ we obtain a solution to (3.2.3) which also satisfies

$$\gamma^* F_0 = F_0, \text{ for all } \gamma \in U(n). \tag{3.2.4}$$

It is easy to see that any function which satisfies (3.2.4) is of the form

$$F_0(z) = f(r^2), \text{ with } r^2 = |z|^2. \tag{3.2.5}$$

An elementary calculation shows that with $\tau = r^2$,

$$\varDelta_{\mathrm{B}} F_0 = \tau(1-\tau)^2 (f_{\tau\tau} + \left[\frac{n}{\tau} + \frac{n-1}{1-\tau}\right] f_\tau). \tag{3.2.6}$$

Exercise 3.2.7. Prove (3.2.6), hint: for functions in $\mathcal{C}_c^\infty$ the Bergman laplacian is defined by

$$\int\limits_{\mathbb{CB}^n} (\varDelta_{\mathrm{B}} f)\overline{g} \,\mathrm{dVol} = \int\limits_{\mathbb{CB}^n} < \overline{\partial} f, \overline{\partial} g > \mathrm{dVol}\,. \tag{3.2.7}$$

If we reparametrize the eigenvalue by

$$\lambda = s(n-s)$$

then the equation for the radial fundamental solution is a classical P-Riemann equation:

$$\mathcal{P}\left\{\begin{array}{ccccc} 0 & \infty & 1 & \\ 0 & 0 & s & ;\tau \\ -n & 0 & n-s & \end{array}\right\}. \tag{3.2.8}$$

If you are unfamiliar with this notation I suggest that you consult Modern Analysis by Whittaker and Watson. Briefly it states that the equation has regular singular points at $0, 1, \infty$ with indicial roots $(0, -n)$ at 0, $(0, 0)$ at ∞ and $(s, n-s)$ at 1.

The reason that we introduce the parameter s is so that the indicial roots are both analytic functions of the parameter, this in turn produces a solution which depends analytically on the parameter. The parameter s defines a two fold cover of the energy parameter λ. The half plane $\operatorname{Re} s > \frac{1}{2}n$ is the 'physical' half plane, the spectrum corresponds to $\operatorname{Re} s = \frac{1}{2}n$. The half plane $\operatorname{Re} s < \frac{1}{2}n$ is the non–physical half plane and it has interpretations in term of scattering theory.

In any case the solution of (3.2.8) which we need can be expressed in term of classical functions by

$$r(\tau; s) = c_n \frac{\Gamma(s)^2}{\Gamma(2s-n+1)} (\tau - 1)^s {}_2F_1(s, s; 2s-n+1; 1-\tau). \tag{3.2.9}$$

Using well known facts about these functions we can show that for $\operatorname{Re} s > \frac{1}{2}n$, $r(|z|^2; s)$ is square integrable, near to $|z| = 1$, with respect to the Bergman metric. It has a singularity at 0 such that

$$\int_{\mathbb{CB}^n} r(|z|^2; s)(\Delta_{\mathrm{B}} - s(n-s)) f(z) \, \mathrm{dVol} = f(0), \tag{3.2.10}$$

for $f \in \mathcal{C}_c^\infty(\mathbb{CB}^n)$.

The parameter τ has a more invariant interpretation, it is given by

$$\tau = 1 - [\cosh \frac{1}{2} d(z, 0)]^{-2}. \tag{3.2.10}$$

From this it follows easily that

$$R(z, w; s) = r(1 - [\cosh \frac{1}{2} d(z, w)]^{-2}; s). \tag{3.2.11}$$

In (3.1.13) we showed that

$$\cosh \frac{1}{2} d(z, w) = \frac{|1 - (z, w)|}{[(1 - |z|^2)(1 - |w|^2)]^{\frac{1}{2}}}.$$

We set

$$\iota(z, w) = \frac{|1 - (z, w)|}{[(1 - |z|^2)(1 - |w|^2)]^{\frac{1}{2}}},$$

then

$$R(z, w; s) = \iota(z, w)^{-2s} G_n(\iota(z, w)^{-2}; s), \tag{3.2.12}$$

where $G_n(t, s)$ is analytic for $t \in [0, 1)$ and has a pole at $t = 1$. The main conclusion we draw from (3.2.12) is that if we understand the singularities of the function $\iota(z, w)$ on $\mathbb{CB}^n \times \mathbb{CB}^n$ then it is relatively simple matter to understand the singularities of the resolvent kernel. As a function of s, $G_n(\cdot; s)$ is analytic in $\operatorname{Re} s > 0$ with simple poles at $-\mathbb{N}_0$.

To conclude this section we construct the Bergman kernel function. As before we make use of the group invariance. By definition, the Bergman projector acts like the identity on holomorphic $n,0$–forms. More explicitly, if fdz is holomorphic then

$$f(z)dz = \int\limits_{\mathbb{CB}^n} f(w)dw \wedge B(z,w)d\bar{w} \wedge dz. \tag{3.2.13}$$

Since the components of a holomorphic $n,0$ form are also harmonic with respect to the euclidean Laplacian it follows from the mean value theorem that

$$f(0)dz = c_n \int\limits_{\mathbb{CB}^n} f(w)dw \wedge d\bar{w} \wedge dz. \tag{3.2.14}$$

Suppose that $\gamma \in \text{Isom}(()\mathbb{CB}^n)$ carries 0 to z then

$$\gamma^{-1\,*}(\gamma^*(fdz)(0)) = f(z)dz. \tag{3.2.15}$$

By combining (3.2.14) and (3.2.15) we obtain

$$\begin{aligned} f(z)dz &= c_n \int\limits_{\mathbb{CB}^n} \gamma^*(f(w)dw) \wedge d\bar{w} \wedge \gamma^{-1\,*}dz \\ &= c_n \int\limits_{\mathbb{CB}^n} f(w)dw \wedge \gamma^{-1\,*}(d\bar{w} \wedge dz). \end{aligned} \tag{3.2.16}$$

To complete the construction all we need is to compute the Jacobian of γ^{-1}. This we leave as an exercise, the answer is

$$\gamma^{-1\,*}(d\bar{w} \wedge dz) = \frac{d\bar{w} \wedge dz}{(1-(z,w))^{n+1}}, \tag{3.2.17}$$

where

$$(z,w) = \sum_{i=1}^{n} z_i \bar{w}_i.$$

The formula we finally obtain is that

$$\mathcal{B}(fdz) = c_n \int\limits_{\mathbb{CB}^n} \frac{f(w)dw \wedge d\bar{w} \wedge dz}{(1-(z,w))^{n+1}}. \tag{3.2.18}$$

How do we know this is the correct kernel? It is uniquely determined by three properties: the range of $\mathcal{B}$ is $\mathcal{H}^2$, it is hermitian symmetric and $\mathcal{B}^2 = \mathcal{B}$. This is equivalent to the statement that it is an *orthogonal* projection onto $\mathcal{H}^2$. The hermitian symmetry is apparent from the formula. Since $B(z,w)$ depends holomorphically on z it follows that $\mathcal{B}f(z)$ is a holomorphic function of z. Furthermore one can adapt the for going argument to show that:

$$B(z,w) = \int\limits_{\mathbb{CB}^n} B(z,x) \wedge B(x,w).$$

Thus we have proved the following theorem:

Theorem 3.2.19. *The Bergman projector for the the unit ball in $\mathbb{C}^n$ is given by the following kernel*

$$B(z,w) = \frac{d\bar{w} \wedge dz}{(1-(z,w))^{n+1}}.$$

Once again we see that understanding the singularities of the Bergman kernel is reduced to studying the singularities of a very simple function $(1-(z,w))^{-1}$. Clearly this function is smooth away from the intersection of the diagonal in $\mathbb{CB}^n \times \mathbb{CB}^n$ with the boundary. On the other hand it seems to have a very similar sort of a singularity as that which arose in the kernel of the resolvent for the Bergman laplacian. In the next lecture we discuss a method for analyzing such singularities. It amounts essentially to introducing polar coordinates.

References

The bulk of the material in §2.1–§2.9 comes form *An Introduction to Complex Analysis in Several Variables* by Lars Hörmander. Additional material was taken from *Introduction to Holomorphic Functions of Several Variables, I,II* by Robert C. Gunning. Section 2.10 follows *Stable Mapping and Their Singularities* by M. Golubitsky and V. Guillemin. Many of the exercises are taken from *Function Theory of Several Complex Variables* by S. Krantz.

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