

Math 646, Problem set 1 due September 23, 2003

Dr. Epstein

Read Chapter 1 of the notes and Chapter 1 of Hormander. Read sections 2.1, 2.2 and 2.A for next week. Carefully write up solutions to the following problems:

1.

- (a) Deduce Green's formula from the Stokes formula: If u and v are twice differentiable in a bounded region with piecewise \mathcal{C}^1 -boundary Ω , then

$$\int_{\Omega} (v \Delta u - u \Delta v) dA = \int_{\partial \Omega} \left(\frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} \right) ds.$$

Here $\frac{\partial f}{\partial \nu}$ is differentiation with respect to the outer unit normal along $\partial \Omega$.

- (b) Use this formula to show that if u harmonic in Ω , then

$$\int_{\partial \Omega} \frac{\partial u}{\partial \nu} ds = 0.$$

- (c) A function $u \in \mathcal{C}^0(\Omega)$ satisfies the mean value property if, for any point $p \in \Omega$ and $r > 0$ such that $\overline{B_p(r)} \subset \Omega$, we have:

$$u(p) = \frac{1}{2\pi} \int_0^{2\pi} u(p + re^{i\theta}) d\theta.$$

Use Green's formula to show that if $u \in \mathcal{C}^2$, satisfies the mean value property, then $\Delta u = 0$. How about if u is only assumed to be continuous?

2. Prove the "chain rule" for complex derivatives: if $f(z, \bar{z})$, $g(z, \bar{z})$ are complex valued differentiable functions then

$$\partial_z(f \circ g) = f_z \partial_z g + f_{\bar{z}} \partial_z \bar{g}, \quad \partial_{\bar{z}}(f \circ g) = f_z \partial_{\bar{z}} g + f_{\bar{z}} \partial_{\bar{z}} \bar{g}.$$

Use this to show that if $\partial_z f = \partial_z g = 0$ then $f \circ g$ is analytic.

3. Exercise 1 (page 12) in Chapter 1 of the lecture notes.
4. Exercises 3, 4 (page 14) in Chapter 1 of the lecture notes.
5. Let $\Omega \subset \mathbb{C}$ be a bounded domain with smooth boundary. Find necessary conditions on g so that the boundary value problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ \partial_{\bar{z}} u|_{\partial \Omega} = g \end{cases}$$

is solvable. Find necessary and sufficient conditions for $\Omega = D_1$. Hint: First do the case $\Omega = D_1$, and review the conditions for solvability of the Neumann problem.

6. (a) Let $\{z_n = r_n e^{i\theta_n}\}$ be a sequence of points in D_1 . Prove that if

$$\sum_{n=1}^{\infty} (1 - r_n) < \infty,$$

then the infinite product

$$f(z) = \prod_{n=1}^{\infty} \left(\frac{z - z_n}{r_n z - e^{i\theta_n}} \right)$$

converges to define a bounded holomorphic function in D_1 . What are the zeros of $f(z)$? Find a sequence $\{z_n\}$ such that bD_1 is a natural boundary for f .

- (b) (extra credit) If $f(z)$ is a bounded holomorphic function in D_1 vanishing at a sequence of points $\{z_n\}$ then

$$\sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$