Math 584, Problem set 1 due January 31, 2017
Dr. Epstein

Reading: Read chapters 1 and 2 of the textbook.
You may use MATLAB (or Mathematica, Maple etc.) to help you do these problems. If you do, then please attach the output of the program to your solutions.

1. The three functions
   (a) $\|(x,y)\|_1 = |x| + |y|$,
   (b) $\|(x,y)\|_2 = \sqrt{|x|^2 + |y|^2}$,
   (c) $\|(x,y)\|_\infty = \max\{|x|, |y|\}$,

   define norms on $\mathbb{R}^2$. Draw, on the same graph, the unit circle in each of these norms (that is the set of points of norm equal to 1). Show that in any dimension the unit ball w.r.t $\|\cdot\|_1$ is contained in $\|\cdot\|_2$.

2. Let $A : V \to V$ be a linear transformation and $\|\cdot\|$ a norm on V. The condition number of A can be defined by

$$c_A = \frac{\max_{x:\|x\|=1} \|Ax\|}{\min_{x:\|x\|=1} \|Ax\|}$$

   Using the Euclidean norm, compute the condition numbers of the following matrices.
   (a)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}.$$

   (b)

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{pmatrix}.$$

   (c)

$$C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

   You are free to use MATLAB, etc. Please include your computer output.

3. Do exercise 1.1.3 from the text.

4. Do exercise 1.1.13 of the text.

5. Suppose that we have a system whose state is specified by a pair of numbers (x_1, x_2). We can perform 2 measurements on this system, the input (x_1, x_2) is related to the output, (y_1, y_2) by a linear equation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

However we do not know the values of a, b, c, d.

(a) How many *independent* input-output pairs are needed to determine a, b, c, d?

(b) Write a formula for a, b, c and d in terms of the measurements.

(extra credit) Explain the notion of "independence of measurements" that is relevant to this problem.

6. Do exercises 1.2.1, 1.2.2, and 1.2.7 in the text.

7. In this problem, we model the Earth and Moon as being perfectly spherical.

(a) First suppose that we know the radius of the Earth, and the distance from the Earth to the Moon. Find a formula for the radius of the Moon in terms of the angle θ indicated in the diagram.

(b) Next, assuming that we know *only* the radius of the Earth, explain how to find both the radius of the Moon, and the distance from the Earth to the Moon by making two such measurements, i.e. from different locations on the Earth. Find formulæ in this case as well.

In both cases you may assume that you know your latitude. You are free to travel on the earth and can measure distances traveled.

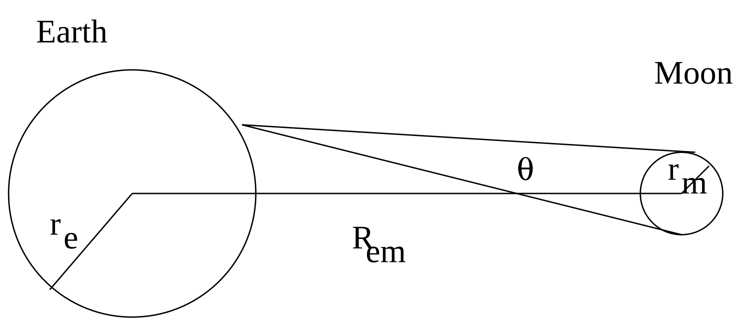


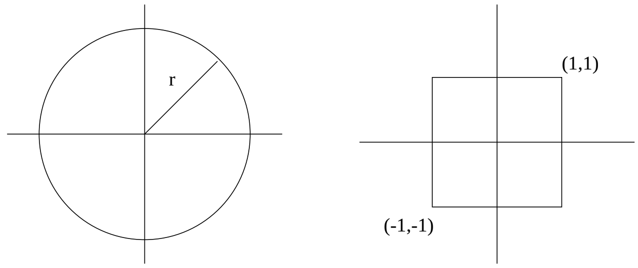
FIGURE 1. Figure for exercise 7

8. Compute the "shadow functions," $h(\theta)$ for

(a) A disk of radius $r > 0$ centered at $(0,0)$.

(b) A square with sides parallel to the x and y axes of length 2, centered at $(0,0)$.

9. Do exercise 1.2.18 in the text.



(a) A disk   (b) A square

FIGURE 2. Figures for exercise 8