Math 508
Problem set 9, due November 13, 2018
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Sections 5.3-5.5 in **The Way of Analysis**.

You should do the following problems, but you do not need to hand in your solutions:

1. Suppose that f is a 1-1 continuous function defined on an interval (a, b). Show that f is either strictly increasing, or strictly decreasing.
2. Define the function x_+ by
$$x_+ = \begin{cases} x \text{ if } 0 \leq x, \\ 0 \text{ if } 0 > x. \end{cases}$$
If k is an integer greater than 1 then show that x_+^k is a differentiable function on $\mathbb{R}$. What about the $k = 1$ case?
3. Show that a polynomial of positive, even order has either a global maximum, or a global minimum, but never both.
4. If f is $\mathcal{C}^2((a, b))$, then for $x \in (a, b)$ we have that
$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x).$$

The following problems should be carefully written up and handed in. In these problems we let $T_n(f, x_0, x)$ denote the nth order Taylor polynomial of f at x_0
$$T_n(f, x_0, x) = \sum_{j=0}^{n} \frac{f^{[j]}(x_0)}{j!}(x - x_0)^j.$$

1. Show that, for $a < b$, the function $(x-a)_+^2(b-x)_+^2$ is a continuously differentiable function, which is non-zero exactly on the interval (a, b). Now prove that if $A \subset \mathbb{R}$ is a closed set, then there is a differentiable function that vanishes exactly on A. Hint: What is A^c?
2. For $n \in \mathbb{Z} \setminus \{0\}$ show that the functions $f_n : (0, \infty) \to (0, \infty)$ defined by $f_n(x) = x^n$ are invertible, and that the inverse functions, $g_n(x)$, are differentiable. Using the inverse function theorem, give the formula for $g_n'(x)$.
3. Suppose that $f \in \mathcal{C}^n((a, b))$, and $T_n(f, x_0, x)$ is its nth order Taylor polynomial at x_0. Show that the first derivative of $T_n(f, x_0, x)$ with respect to x is the $(n-1)$st order Taylor polynomial of f' at x_0, that is:
$$\frac{d}{dx} T_n(f, x_0, x) = T_{n-1}(f', x_0, x).$$

4. Let f and g belong to $\mathcal{C}^n((a,b))$, and let $T_n(f,x_0,x), T_n(g,x_0,x)$ be their nth order Taylor polynomials.
   (a) Prove that the nth order Taylor polynomial of $f+g$ is $T_n(f,x_0,x)+T_n(g,x_0,x)$.
   (b) Prove that the nth order Taylor polynomial of $f\cdot g$ is obtained from the product of polynomials, $T_n(f,x_0,x)\cdot T_n(g,x_0,x)$, by dropping all terms of degree greater than n.
   (c) Prove that the nth order Taylor polynomial of $1/(1+x)$ at 0 is
$$T_n(f,0,x)=\sum_{j=0}^{n}(-1)^j x^j.$$
5. Let $f \in \mathcal{C}^n((a,b))$.
   (a) If f has $n+1$ distinct zeros in (a,b), then prove that $f^{[n]}$ has a least one zero in this interval.
   (b) Show that if $f^{[n]}$ never vanishes in (a,b), then f has at most n zeros, counted with multiplicity, in this interval.
   (c) Prove that a polynomial of degree n has at most n zeros.
6. Suppose that $f \in \mathcal{C}^n((a,b))$ and its nth order Taylor polynomial, $T_n(f,x_0,x)$ is the same function for every $x_0 \in (a,b)$. What can you say about $f(x)$?