Math 508
Problem set 8, due November 6, 2018
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Sections 5.1-5.3 in **The Way of Analysis**.

You should do the following problems, but you do not need to hand in your solutions:

1. Show that if $f(x) = O(|x - x_0|^2)$ as $x \to x_0$, then $f(x) = o(|x - x_0|)$. Give an example to show that the converse is false.
2. Show that if $f(x) = O(|x - x_0|^k)$ and $g(x) = O(|x - x_0|^k)$ as $x \to x_0$, then $f(x) + g(x) = O(|x - x_0|^k)$. Is the same true of "little-o?"

The following problems should be carefully written up and handed in:

1. Show that if $f(x) = O(|x - x_0|^k)$ and $g(x) = o(|x - x_0|^j)$ as $x \to x_0$, then $f(x) \cdot g(x) = o(|x - x_0|^{k+j})$.
2. (a) Show that if $f(x) = O(|x - x_0|^k)$ for a $1 \leq k$, then
$$\frac{f(x)}{x - x_0} = O(|x - x_0|^{k-1}).$$
   (b) Show that if $f(x) = o(|x - x_0|)$, then f is differentiable at x_0. What is $f'(x_0)$?
   (c) If $f(x) = O(|x - x_0|)$, is f always differentiable at x_0? Give a proof or counterexample.
3. Suppose that f is a differentiable function defined on $(-1, 1)$, with $f(0) = 0$. Let
$$g(x) = \frac{f(x)}{x} \text{ for } x \neq 0.$$
Show that g has a continuous extension to $x = 0$. What is $g(0)$? Is g necessarily differentiable at 0? Give a proof or counterexample.
4. If f is a function defined on $\mathbb{R}$ such that
$$|f(x) - f(y)| \leq M|x - y|^\alpha \tag{1}$$
for an $\alpha > 1$, then show that f is constant.
5. Construct a function $f(x)$, defined on $\mathbb{R}$, which is not differentiable anywhere, but so that $f^2(x)$ is differentiable everywhere.

6. Let f be continous on $[0,1]$ and differentiable on $(0,1)$. Assume that the 1-sided derivatives

(2) $$f'_+(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} \text{ and } f'_-(1) = \lim_{x \to 1-} \frac{f(1) - f(x)}{1 - x}$$

both exist. If f assumes a maximum or minimum at an end-point, then what can we say about the 1-sided derivatives at that end-point? Be careful: what you can say depends on which end the max or min occurs at.