Math 508

Problem set 7, due October 30, 2018

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The material on this problem set comes from Chapter 4 in **The Way of Analysis**.

You should do the following problems, but you do not need to hand in your solutions:

1. Suppose that f and g are functions defined in $[0, 1]$ that satisfy a Lipschitz condition: there are constants M_f, M_g so that for all $x, y \in [0, 1]$

$$|f(x) - f(y)| \leq M_f|x - y| \text{ and } |g(x) - g(y)| \leq M_g|x - y|. \tag{1}$$

   Show that $f + g$ satisfies the same kind of estimate, possibly with a different constant.

2. Let $f : D \to E$ be a function, and let $A, B \subset E$. Show that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \text{ and } f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

The following problems should be carefully written up and handed in:

1. Suppose that f is a function defined on $[0, 1]$ that satisfies

$$|f(x) - f(y)| \leq 5|x - y|^{\frac{1}{4}}.$$

   Prove that f is continuous and give an explicit formula for δ as a function of ϵ.

2. Suppose that f is a uniformly continuous function defined on $[0, 1] \cap \mathbb{Q}$. Show that there is a unique continuous function F defined on $[0, 1]$ so that

$$F(x) = f(x) \text{ for every } x \in [0, 1] \cap \mathbb{Q}.$$

   Is this still true if we just assume that f is continuous, but not uniformly continuous. Why or why not?

3. Let $f_1, \ldots, f_n$ be continuous functions defined on $\mathbb{R}$. Show that the set

$$A = \{x : f_1(x) \geq 0, \cdots, f_n(x) \geq 0\} \text{ is closed,}$$

   and the set

$$B = \{x : f_1(x) > 0, \cdots, f_n(x) > 0\} \text{ is open.}$$

4. Let f be a uniformly continuous function defined on $\mathbb{R}$. Show that there are positive constants C, M so that

$$|f(x)| \leq C + M|x|.$$

5. Show that if f and g are uniformly continuous *and* bounded on a domain D, then the function $f \cdot g$ is as well. Give an example to show that this may be false if one of the functions is not bounded.

6. Suppose that $a < b$ are finite and that f is a uniformly continuous function defined on (a, b). Show that $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist, and that f has an extension as a continuous function defined on $[a, b]$.
7. Suppose that f is a continuous function on a compact set K. Show that either there is an $x \in K$ such that $f(x) = 0$, or there is a positive number c so that $|f(x)| \geq c$ for all $x \in K$.