Math 508
Problem set 6, due October 23, 2018
Dr. Epstein

The material on this problem set comes from Chapter 3 and Section 4.1 in **The Way of Analysis**.

You should do the following problems, but you do not need to hand in your solutions:

1. Let (a, b) and (c, d) be two intervals. What are the possibilities for the intersection, $(a, b) \cap (c, d)$, and the union, $(a, b) \cup (c, d)$?
2. Give an example to show that the infinite union of closed sets need not be closed.
3. Which subsets of $\mathbb{R}$ are both closed and open?
4. Let A be an open set. When is the closure of A a compact set?
5. Find the limit points of the set $\{n + \frac{1}{m} | \, m, n \in \mathbb{N}\}$.
6. Let $A \subset \mathbb{R}$. Show that x is a limit point of A if and only if there is a sequence $< x_n > \subset A$, with $x_n \neq x_m$ for $n \neq m$, such that
$$\lim_{n \to \infty} x_n = x.$$
7. Show that a finite union of compact sets is compact.

The following problems should be carefully written up and handed in:

1. Let A be a subset of $\mathbb{R}$ and let A' be its set of limit points. Show that A' is a closed set.
2. Let U be an open set, show that $U \setminus \{x_1, \ldots, x_n\}$ is always open. Is this true if we remove a countable subset from U?
3. Let A be a subset of $\mathbb{R}$ and define
$$d_A(x) = \inf\{|x - y| : \, y \in A\}.$$
For $\epsilon > 0$ define the set $A_\epsilon = \{x : \, d_A(x) < \epsilon\}$. Prove that A_ϵ is open. What is $\cap_{\epsilon > 0} A_\epsilon$?
4. Let $A \subset \mathbb{R}$ be a compact set with infinitely many points. Show that A has a limit point.
5. Suppose that A and B are compact subsets of $\mathbb{R}$ such that $A \cap B = \emptyset$. Show that there are open sets $U \supset A$ and $V \supset B$ such that $U \cap V = \emptyset$.
6. Let A, B, and C be sets and assume that $f : B \to C$ and $g : A \to B$ are functions. For $a \in A$, the composition $f \circ g(a)$ is defined to be $f(g(a))$. Show that for a set $W \subset C$ the inverse image $(f \circ g)^{-1}(W) = g^{-1}(f^{-1}(W))$.
7. Using the $\epsilon - \delta$ definition of continuity, prove that the function defined on $[0, \infty)$ by $f(x) = \sqrt{x}$ is continuous.