Math 508

Problem set 5, due October 11, 2018

Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, the material comes from section 7.2 of **The Way of Analysis**.

Standard problems, whose solutions do not need to be handed in.

1. Suppose that $(a_1, \dots, a_n)$ is a finite sequence, and $(m_1, \dots, m_n)$ is a rearrangement of $(1, \dots, n)$. Prove that
$$\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} a_{m_j}.$$
That is the sum of a finite collection of numbers can be done in any order.
2. Show that if $(a_1, \dots, a_n)$ is a finite sequence, then
$$\left|\sum_{j=1}^{n} a_j\right| \leq \sum_{j=1}^{n} |a_j|.$$

The following problems should be carefully written up and handed in:

1. Give an example of two convergent series $\sum_{j=1}^{\infty} a_j$, $\sum_{j=1}^{\infty} b_j$, such that
$$\sum_{j=1}^{\infty} a_j b_j$$
is divergent. Is this possible if one of the series is absolutely convergent?
2. Let $< a_n >$ be a sequence with $\lim_{n\to\infty} a_n = A$. Show that if $b_n = a_{n+1} - a_n$, then the infinite series
$$S = \sum_{n=1}^{\infty} b_n$$
is convergent. What is S equal to?
3. Let $0 < r < 1$, and let $k \in \mathbb{N}$. Show that the series
$$\sum_{n=1}^{\infty} n^k r^n$$
is absolutely convergent.
4. Suppose that $\sum_{j=1}^{\infty} a_j$, and $\sum_{j=1}^{\infty} b_j$ are absolutely convergent series and let
$$(p_1, p_2, \dots)$$

be *any* enumeration of the complete list of products $a_i b_j$ for $i = 1, \ldots,$ and $j = 1, \ldots$. Show that

$$\sum_{j=1}^{\infty} p_j = \left(\sum_{j=1}^{\infty} a_j \right) \cdot \left(\sum_{j=1}^{\infty} b_j \right).$$

That is, you need to show that $\sum_{j=1}^{\infty} p_j$ converges and gives the value on the right, no matter how the products $< a_i b_j >$ are ordered.

5. Suppose that $\sum_{j=1}^{\infty} a_j$ is conditionally convergent. Use the partial sum formula to show that

$$\sum_{j=1}^{\infty} j^{\frac{1}{j}} a_j$$

is also convergent. Hint: What is $\lim_{j \to \infty} j^{\frac{1}{j}}$?

6. Is the following statement TRUE or FALSE: "If the partial sums of the series $\sum_{j=1}^{\infty} a_j$ are bounded, and $\lim_{j \to \infty} a_j = 0$, then the series converges." Give a proof or counter-example.