Math 508
Problem set 3, due September 25, 2018
Dr. Epstein

This week we will more or less finish Chapter 2, and Chapter 3.1 of **The Way of Analysis**. Your solution to the following problems should be carefully written up and handed in:

1. Let $< r_n >$ be a sequence of positive rational numbers that converges to a rational number r. Suppose that $r_n = \frac{p_n}{q_n}$, in lowest terms. Show that either there exists an N so that $r_n = r$ for $N \leq n$, or the set of numbers $\{q_n\}$ is unbounded.

2. Let $< x_n >$ be a sequence of positive real numbers that converges to x, i.e.,

$$\lim_{n\to\infty} x_n = x.$$

Prove that $0 \leq x$.

3. Let $\{x_k\}$ be a countable set of real numbers, which is bounded from above and set

$$M = \text{l. u. b.}\{x_k\}. \tag{1}$$

Suppose that no element of this set equals the least upper bound. Show that for any $l \in \mathbb{N}$, the set

$$X_l = \{k : M - \frac{1}{l} < x_k\} \tag{2}$$

is infinite.

4. Let x and y be positive real numbers with decimal expansions

$$x = n + \sum_{j=1}^{\infty} \frac{a_j}{10^j}, \quad y = m + \sum_{j=1}^{\infty} \frac{b_j}{10^j}, \tag{3}$$

where $\{a_j\}, \{b_j\} \subset \{0, \dots, 9\}$, and m, n are non-negative integers. For $k \in \mathbb{N}$, let

$$x_k = n + \sum_{j=1}^{k} \frac{a_j}{10^j}, \quad y_k = m + \sum_{j=1}^{k} \frac{b_j}{10^j}. \tag{4}$$

If $l \in \mathbb{N}$, then what is the smallest value of k so that, for any real numbers x, y, we have the estimates:

$$|(x + y) - (x_k + y_k)| < \frac{1}{10^l}; \tag{5}$$

now assuming that $|x|$ and $|y|$ are at most 1, what is the smallest value of k so that

$$|(x \cdot y) - (x_k \cdot y_k)| < \frac{1}{10^l}? \tag{6}$$

The k that works for addition will generally be different from the one that works for multiplication.

5. Let $0 < x$ be a real number. Show that

$$\lim_{n \to \infty} x^{\frac{1}{n}} = 1. \tag{7}$$

Hint: Show that it suffices to do the case that $1 \leq x$, and then use the binomial formula to show that, given $\epsilon > 0$, there exists an n so that

$$x^{\frac{1}{n}} < (1 + \epsilon). \tag{8}$$

Remark: If it was not yet proved in class, you can use the fact that for every positive real number x and $n \in \mathbb{N}$, there exists an nth root, y_n such that $y_n^n = x$.

6. If x is a real number show that there is a Cauchy sequence $< x_j >$ are rational numbers representing x such that $x_j \leq x_{j+1}$ for all $j \in \mathbb{N}$.