Math 508

Problem set 2, due September 18, 2018

Dr. Epstein

The material covered in class so far comes from Chapter 2 of **The Way of Analysis**. The solutions to the following problems do not need to be handed in:

1. Using the axioms for an ordered field, prove the following inequalities:
   (a) If $x \neq 0$, then $(x^2 + y^2)/x^2 \geq 1$.
   (b) $2xy \leq x^2 + y^2$.
   (c) If $x > 0$, and $0 < y < 1$, then $x < x/y$.
2. Let $x < y$ be real numbers. Prove that the set
$$\{r \in \mathbb{Q} : x < r < y\} \tag{1}$$
   is infinite. It this set countable or uncountable? Explain your answer.

Your solutions to these problems should be written in English: Use complete sentences and paragraphs. Explain your reasoning and why one formula follows from the previous one.

1. Let $< x_j >$ be a Cauchy sequence. For $k \in \mathbb{N}$ define a new sequence
$$y_j = x_{j+k}. \tag{2}$$
   Prove that $< y_j >$ is a Cauchy sequence equivalent to $< x_j >$.
2. Suppose that $< x_n >$ is a sequence of integers that converges to a limit. What can you say about such a sequence and its limit.
3. A real number α is algebraic of degree at most n if there is a polynomial, with rational coefficients, $p(x) = x^n + +a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, such that $p(\alpha) = 0$.
4. Let $< x_j >$ be a Cauchy sequence of rational numbers representing the real number x. Show that $< |x_j| >$ is a Cauchy sequence representing the real number $|x|$.
5. Show that a real number α is algebraic of degree at most n if and only if there are real numbers $\{y_1, \ldots, y_n\}$ such that for every $k \in \mathbb{N}$ there exist rational numbers $\{a_{1,k}, \ldots, a_{n,k}\}$ so that
$$\alpha^k = a_{1,k} y_1 + \cdots + a_{n,k} y_n. \tag{3}$$

Hint: For the forward assertion use the Euclidean algorithm, for the converse use the condition for solvability for an overdetermined linear system. The Euclidean algorithm states that if $p(x) = x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, with $\{a_j\} \subset \mathbb{Q}$, and $h(x)$ is a polynomial, with rational coefficents, of degree greater than n, then there are polynomials $q(x), r(x)$, with rational coefficients, such that $\deg r < n$ and

$$h(x) = q(x)p(x) + r(x). \tag{4}$$

6. Conclude from the previous problem that if x is algebraic of degree m and y is algebraic of degree n, then $x + y$, and $x \cdot y$ are algebraic numbers of degree at most mn. Show that the set of algebraic numbers is an ordered field. Is it complete?