Math 508
Problem set 10, due November 27, 2018
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 6 in **The Way of Analysis**.

You should do the following problems, but you do not need to hand in your solutions:

1. Let g be a continuous function on $[a, b]$ and let
$$f(x) = \int_a^x (x - t)g(t)dt.$$
Prove that f satisfies:
$$f''(x) = g(x) \text{ and } f(a) = f'(a) = 0.$$
2. Suppose that $a < b < c$, and that f is Riemann integrable on $[a, c]$. Show that f is also Riemann integrable on $[a, b]$.

The solutions to the following problems should be carefully written up and handed in.

1. Let $a(x)$, and $b(x)$ be $\mathcal{C}^1$-functions, and $g(x)$ be a $\mathcal{C}^0$-function. Define
$$f(x) = \int_{a(x)}^{b(x)} g(t)dt;$$
show that f is a $\mathcal{C}^1$-function and that
$$f'(x) = b'(x)g(b(x)) - a'(x)g(a(x)).$$
2. Let $w(x)$ be a positive, Riemann integrable function defined on $[a, b]$ and $f(x)$ a continuous function. Show that there is a $\xi \in (a, b)$ so that
$$\int_a^b f(x)w(x)dx = f(\xi)\int_a^b w(x)dx.$$
3. Let f and g be Riemann integrable functions defined on $[a, b]$. Prove that $f \cdot g$ is also Riemann integrable on $[a, b]$.
4. If f is Riemann integrable on $[a, b]$ show that
$$F(x) = \int_a^x f(t)dt$$
is continuous and satisfies a Lipschitz estimate. Give a simple example of a Riemann integable function f so that F is *not* differentiable at every point in (a, b).

5. Let g be a $\mathcal{C}^1$-function on $[a, b]$, which is increasing, and let f be Riemann integrable on $[g(a), g(b)]$. Show that $f \circ g(x) \cdot g'(x)$ is Riemann integrable on $[a, b]$ and that
$$\int_a^b f \circ g(x) g'(x) dx = \int_{g(a)}^{g(b)} f(y) dy.$$
6. Let f be a $\mathcal{C}^2$-function on $[a, b]$ and $x_0 < x_1 < \cdots < x_N = b$, be a partition. The trapezoidal rule approximates the integral over $[x_{j-1}, x_j]$ by
$$\int_{x_{j-1}}^{x_j} f(y) dy \approx \frac{1}{2}(f(x_{j-1}) + f(x_j))(x_j - x_{j-1}).$$
   (a) Show that if $f(x) = ax + b$, then the trapezoidal rule gives the exact answer.
   (b) Show that if $|f''(x)| \leq M_2$, for all $x \in [a, b]$, then
$$\left| \int_{x_{j-1}}^{x_j} f(y) dy - \left[\frac{1}{2}(f(x_{j-1}) + f(x_j))(x_j - x_{j-1}) \right] \right| \leq M_2 \frac{(x_j - x_{j-1})^3}{6}.$$
   (c) If the points $\{x_j\}$ are equally spaced then show that
$$\left| \int_a^b f(x) dx - \left[\frac{f(a) + f(b)}{2} + \sum_{j=1}^{N-1} f(x_j) \right] \left(\frac{b-a}{N} \right) \right| \leq M_2 \frac{(b-a)}{6} \left(\frac{b-a}{N} \right)^2.$$