Math 508

Problem set 1, due September 11, 2018

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The material covered in this problem set comes from Chapter 1 of Strichartz.

Standard Problems

These problems do not need to be handed in.

1. Show that there is no real number x such that $x^2 = -1$.

2. Prove by induction: If $a_1, \dots, a_n$ are real numbers, then

$$|a_1 + \cdots + a_n| \le |a_1| + \cdots + |a_n|. \tag{1}$$

3. Recall that for a pair of integers (m, n) with $n \neq 0$, we define the equivalence class

$$[(m,n)] = \{(m', n') : \; mn' = m'n\}. \tag{2}$$

Show that for two pairs $(m, n), (p, q)$ either $[(m, n)] = [(p, q)]$ or $[(m, n)] \cap [(p, q)] = \emptyset$.

Problems to Hand In

Your solutions to the following problems should be carefully written up *in English*: Use complete sentences and paragraphs.

1. Let P and Q be statements; explain why the truth of $P \Rightarrow Q$ is equivalent to the truth of $\neg Q \Rightarrow \neg P$.

2. The set theoretic difference $A \setminus B$ is defined by

$$A \setminus B = \{x \in A : \; x \notin B\}. \tag{3}$$

If A, B are subsets of the same "universe," then show that $A \setminus B = A \cap B^c$. Prove the following formulæ for sets:

$$\begin{aligned}
(A \cap B)^c &= A^c \cup B^c, \\
(A \cup B)^c &= A^c \cap B^c, \\
A \setminus (B \cup C) &= (A \setminus B) \cap (A \setminus C), \\
A \setminus (B \cap C) &= (A \setminus B) \cup (A \setminus C).
\end{aligned} \tag{4}$$

3. Remember that a set A is countably infinite if there is a bijective map $f : \mathbb{N} \to A$.

   Use mathematical induction to prove that every *finite* subset of $\mathbb{N}$ has a smallest element. (You may not use the well ordering principle!) Show that every subset of $\mathbb{N}$ is either finite, or countably infinite, and then show that there is no infinite set with cardinality less than that of $\mathbb{N}$.

4. Is the set of all *finite* subsets of $\mathbb{N}$ countable, or uncountable? You must prove your answer.

5. Suppose that $r = \frac{a}{b} < 1$ is a positive rational number. Let

$$a_1 = \min\{q \in \mathbb{N} : \frac{1}{q} \leq \frac{a}{b}\}. \tag{5}$$

   (a) Show that $0 \leq aa_1 - b < a$.

   (b) Recursively, for $1 \leq j$, assume that we have found $\{a_1, \ldots, a_j\}$, (a_1 is define above) and that

$$r - \frac{1}{a_1} - \cdots - \frac{1}{a_j} > 0,$$

   then define

$$a_{j+1} = \min\{q \in \mathbb{N} : \frac{1}{q} \leq \frac{a}{b} - \frac{1}{a_1} - \cdots - \frac{1}{a_j}\}. \tag{6}$$

   Show that $a_{j+1} > a_j$.

   (c) Conclude that every rational number between 0 and 1 can be represented as

$$r = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$$

   where $< a_j >$ is strictly increasing.

   (d) Can you show that this representation is never unique?

6. The notation $\mathbb{Q}[x]$ denotes the set of polynomials in x with rational coefficients.

   (a) Prove that the set $\mathbb{Q}[x]$ is countable.

   (b) A number x that satisfies $p(x) = 0$ for a $p \in \mathbb{Q}[x]$ is called algebraic. How large is the the set of algebraic numbers?